Exact Defensive Alliances in Graphs

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Abstract

A nonempty set $S \subseteq V$ is a defensive $k$-alliance in $G = (V, E)$, $k \in [-\Delta, \Delta] \cup \mathbb{Z}$, if for every $v \in S$, $d_S(v) \geq d_S^+(v) + k$. A defensive $k$-alliance $S$ is called exact, if $S$ is defensive $k$-alliance but is no defensive $(k+1)$-alliance in $G$. In this paper we study the mathematical properties of exact defensive $k$-alliances in graphs. In particular, we obtain several bounds for defensive $k$-alliance of a graph. Furthermore, we characterize the exact defensive alliances in graph join $G_1 \uplus G_2$ in terms of $G_1, G_2$.

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1 Exact Defensive Alliance

The study of mathematical properties of alliances in graphs was first introduced by P. Kristiansen, S. M. Hedetniemi and S. T. Hedetniemi [2]. They proposed different types of alliances namely defensive alliances [4, 6, 7], offensive alliances [1, 5] and dual alliances or powerful alliances [3]. This paper study mathematical properties of exact defensive alliances.

We begin by stating the used terminology. Throughout this article, $G = (V, E)$ denotes a simple graph of order $|V| = n$ and size $|E| = m$. We denote two adjacent vertices $u$ and $v$ by $u \sim v$. For a nonempty set $X \subseteq V$, and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors $v$ has in $X$, i.e., $N_X(v) :=$
\{u \in X : u \sim v\}; and the degree of \(v\) in \(X\) will be denoted by \(\delta_X(v) = |N_X(v)|\). We denote the degree of a vertex \(v_1 \in V\) by \(\delta(v_1)\) (or by \(\delta_1\) for short) and the minimum degree of \(G\) will be denoted by \(\delta\) and the maximum degree by \(\Delta\). The subgraph induced by \(S \subseteq V\) will be denoted by \(\langle S \rangle\) and the complement of the set \(S\) in \(V\) will be denoted by \(\overline{S}\).

A nonempty set \(S \subseteq V\) is a defensive \(k\)-alliance in \(\Gamma = (V,E)\), \(k \in [-\delta_1,\delta_1] \cap \mathbb{Z}\), if for every \(v \in S\),

\[
\delta_S(v) \geq \delta_{\overline{S}}(v) + k. \tag{1}
\]

A vertex \(v \in S\) is said to be \(k\)-satisfied by the set \(S\), if (1) holds. Notice that (1) is equivalent to

\[
\delta(v) \geq 2\delta_{\overline{S}}(v) + k \quad \text{and} \quad 2\delta_S(v) \geq \delta(v) + k. \tag{2}
\]

We denote \(K := [-\delta_1,\delta_1] \cap \mathbb{Z}\). In some graph \(\Gamma\), there are some values of \(k \in K\), such that do not exist defensive \(k\)-alliances in \(\Gamma\). For instance, for \(k \geq 2\) in star graph \(S_n\), do no exist defensive \(k\)-alliances; besides, \(V(\Gamma)\) is a defensive \(\delta_n\)-alliance in \(\Gamma\). Notice that for any \(S\) there exists some \(k \in K\) such that it is a defensive \(k\)-alliance in \(\Gamma\).

We denote by

\[
k_S := \max\{k \in K : S \text{ is a defensive } k\text{-alliance}\}. \tag{3}
\]

We say that \(k_S\) is the exact index of alliance of \(S\), or also, \(S\) is an exact defensive \(k_S\)-alliance in \(\Gamma\).

**Proposition 1.1.** Let \(\Gamma\) be a graph and let \(S \subseteq V\). The following statements are equivalent

1. \(k\) is the exact exact of alliance of \(S\).

2. \(S\) is a defensive \(k\)-alliance in \(\Gamma\) with one vertex \(v \in S\) such that \(\delta_S(v) = \delta_{\overline{S}}(v) + k\).

3. \(S\) is a defensive \(k\)-alliance but is not defensive \((k+1)\)-alliance in \(\Gamma\).

**Proof.** (1) \(\implies\) (2) Seeking for a contradiction assume that for all \(v \in S\) we have \(\delta_S(v) > \delta_{\overline{S}}(v) + k\), then we obtain \(\delta_S(v) \geq \delta_{\overline{S}}(v) + (k+1)\). This is the contradiction we were looking for, since \(k\) is a maximum; so, there is \(v \in S\) such that \(\delta_S(v) = \delta_{\overline{S}}(v) + k\).

(2) \(\implies\) (3) Since \(\exists v \in S : \delta_S(v) = \delta_{\overline{S}}(v) + k\), we have that \(S\) is not a defensive \((k+1)\)-alliance in \(\Gamma\).

(3) \(\implies\) (1) It is easily seen that \(k = k_S\).
Remark 1.2. The exact index of alliance of $S$ in $\Gamma$ is $k_S = \min_{v \in S} \{\delta_S(v) - \delta_{\overline{S}}(v)\}$.

Proposition 1.3. Let $\Gamma$ be a graph with any vertex has odd (respectively, even) degree, then $\Gamma$ don’t have exact defensive $k$-alliance when $k$ is even (respectively, odd).

Proof. Consider $S \subset V$ an exact defensive $k$-alliance. By Proposition 1.1, we have $\exists v \in S$ such that $2\delta_S(v) = \delta_{\Gamma}(v) + k$. This finish the proof since $\delta_{\Gamma}(v) + k$ is even. \hfill $\square$

Lemma 1.4. Let $G$ be a graph and let $S \subset V(G)$. If $S$ is an exact defensive $k$-alliance in $G$, then it is a defensive $r$-alliance in $G$ for all $r = -\Delta_G, \ldots, k$.

Proof. Since $S$ is an exact defensive $k$-alliance in $G$, we have $\delta_S(v) \geq \delta_{\overline{S}}(v) + k$ for all $v \in S$ and $\delta_S(w) = \delta_{\overline{S}}(w) + k$ for some $w \in S$. Hence, obviously we have $\delta_S(v) \geq \delta_{\overline{S}}(v) + k \geq \delta_{\overline{S}}(v) + r$ for every $v \in S$ with $r \leq k$ (i.e., $r = -\Delta_G, \ldots, k$). \hfill $\square$

The following results should be useful in order to obtain defensive alliance $S \subset V$ in $\Gamma$ with exact index of alliance $k$.

Proposition 1.5. Let $\Gamma$ be a graph and let $S \subset V$ be a defensive $k$-alliance in $\Gamma$. Then

$$|S| \geq k + 1. \quad (4)$$

Theorem 1.6. Let $\Gamma$ be a graph and let $S \subset V$ be an exact defensive $k$-alliance in $\Gamma$. Then

$$\left\lceil \frac{\delta_n + k + 2}{2} \right\rceil \leq |S| \leq \left\lfloor \frac{2n - \delta_n + k}{2} \right\rfloor. \quad (5)$$

Proof. On the one hand, since $|S| - 1 \geq \delta_S(v)$ and $\delta_\Gamma(v) \geq \delta_n$ for all $v \in S$, we have

$$2(|S| - 1) \geq 2\delta_S(v) \geq \delta_\Gamma(v) + k \geq \delta_n + k, \quad \forall v \in S,$$

hence, we deduce the first inequality.

On the other hand, from (2) we have $\delta_\Gamma(v) - k \geq 2\delta_{\overline{S}}(v)$ for all $v \in S$ and the equality holds for some $w \in S$, i.e., $\delta_\Gamma(w) - k = 2\delta_{\overline{S}}(w)$. Then, since $\delta_n \leq \delta_\Gamma(w)$ and $\delta_{\overline{S}}(w) = |N_V(w) \setminus S| \leq |V \setminus S| = n - |S|$, we obtain $\delta_n - k \leq 2(n - |S|)$. \hfill $\square$
2 Exact defensive alliances in graph join.

The graph join $G_1 \sqcup G_2$ of two graphs is their graph union with all the edges that connect the vertices of the first graph $G_1$ with the vertices of the second graph $G_2$. It is a commutative operation.

**Proposition 2.1.** Let $G_1, G_2$ be two graphs with orders $n_1$ and $n_2$ respectively. Consider $G = G_1 \sqcup G_2$ and $S \subset V(G_1)$. Then we have the following statements

a) $S$ is an exact defensive $k$-alliance in $G$ if and only if this is an exact defensive $(n_2 + k)$-alliance in $G_1$.

b) If $G_1$ is $\Delta_{G_1}$-regular, then $S$ is a defensive $k$-alliance in $G$ if and only if

$$\delta_S \geq \frac{\Delta_{G_1} + n_2 + k}{2}.$$

c) If $\delta_{G_1} \geq n_2 + k$, then $V(G_1)$ is a defensive $k$-alliance in $G$.

d) If $\Delta_{G_1} < n_2 + k$, then there not exists defensive $k$-alliance $S$ in $G$.

e) If $S$ is an exact defensive $k$-alliance in $G$, then

$$|S| \geq \delta_{G_1} + n_2 + k + 1.$$

**Proof.** a) In the one hand, if $S$ is an exact defensive $k$-alliance in $G$, then we have $\delta_S(v) \geq \delta_{\overline{G}}(v) + k$ for all $v \in S$ and the equality is hold at some $w \in S$. Hence, since $\delta_{\overline{G}}(v) = \delta_{\overline{G}_{G_1}}(v) + n_2$ for all $v \in S$, we have that $S$ is an exact defensive $(n_2 + k)$-alliance in $G_1$. In the other hand, if $n_2 + k$ is the exact index of alliance of $S$ in $G_1$, then we have that $k$ is the exact index of alliance of $S$ in $G$, since $\delta_{\overline{G}}(v) = \delta_{\overline{G}_{G_1}}(v) + n_2$ for all $v \in S$.

b) If $S$ is a defensive $k$-alliance in $G$, then we have

$$\delta_S(v) \geq \delta_{\overline{G}}(v) + k = \Delta_{G_1} - \delta_S(v) + n_2 + k \quad \forall v \in S,$$

i.e., $2\delta_S(v) \geq \Delta_{G_1} + n_2 + k$ for all $v \in S$. Besides, if $\delta_S \geq (\Delta_{G_1} + n_2 + k)/2$, then we have $2\delta_S(v) \geq 2\delta_S \geq \Delta_{G_1} + n_2 + k = \delta_G(v) + k$ for all $v \in S$. Then, $S$ is a defensive $k$-alliance in $G$.

c) We have $\delta_{\overline{G}_{G_1}}(v) = n_2$ for all $v \in V(G_1)$. Then, since $\delta_{G_1} \geq n_2 + k$, we have $\delta_{G_1}(v) \geq \delta_{G_1} \geq n_2 + k = \delta_{\overline{G}_{G_1}}(v) + k$ for all $v \in V(G_1)$. 

d) Seeking for a contradiction assume that $S$ is a defensive $k$-alliance in $G$, then we have $\delta_S(v) \geq \delta_{S_G}(v) + k \geq n_2 + k$ for all $v \in S$. Therefore, we have

$$\Delta_{G_1} \geq \delta_{S}(v) \geq \delta_{S_G}(v) + k \geq n_2 + k > \Delta_{G_1}, \quad \forall v \in S.$$ 

In fact, it is a contradiction we were looking for.

e) We have that $2\delta_{S}(v) \geq \delta_{G}(v) + k \geq \delta_{G_1}(v) + n_2 + k$ for all $v \in S$. So, since $\delta_{G_1} \leq \delta_{G_1}(v)$ and $\delta_S(v) \leq |S| - 1$ for all $v \in S$ we have $|S| - 1 \geq \delta_{G_1} + n_2 + k$.

**Corollary 2.2.** If $S \subset V(G_1)$ is a defensive 0-alliance in $G_1 \uplus G_2$, then $|V(G_2)| \leq \Delta_{G_1}$.

**Theorem 2.3.** Let $G_1, G_2$ be two graphs with order $n_1$ and $n_2$, respectively. Consider $S = S_1 \cup S_2$ such that $\emptyset \neq S_1 \subset V(G_1)$ and $\emptyset \neq S_2 \subset V(G_2)$. Denote by $k_i$ the exact index of alliance of $S_i$ in $G_i$ for $i = 1, 2$. Then $S$ is an exact defensive $k$-alliance in $G_1 \uplus G_2$ with $k = \min\{k_1 - n_2 + 2|S_2|, k_2 - n_1 + 2|S_1|\}$.

Furthermore, if $S$ is a defensive $t$-alliance in $G_1 \uplus G_2$, then $|S| \geq t + [(n_1 + n_2 - k_1 - k_2)/2]$.

**Proof.** In the one hand, for every $v \in S_1$ we have

$$\delta_{S_1}(v) \geq \delta_{S_1}(v) + k_1, \quad \delta_{S_1}(v) + |S_2| \geq \delta_{S_1}(v) + k_1 + |S_2|$$

and $\delta_{S}(v) \geq \delta_{S}(v) + k_1 + 2|S_2| - n_2$.

Furthermore, for every $w \in S_2$ we have

$$\delta_{S_2}(w) \geq \delta_{S_2}(w) + k_2, \quad \delta_{S_2}(w) + |S_1| \geq \delta_{S_2}(w) + k_2 + |S_1|$$

and $\delta_{S}(w) \geq \delta_{S}(w) + k_2 + 2|S_1| - n_1$.

Since, $k_1$ and $k_2$ are exact indices of defensive alliances, then $k$ is the exact index of defensive alliance of $S$ in $G$.

In the other hand, Lemma 1.4 gives $k_S^{b(G_1 \uplus G_2)} \geq t$. Hence, we have

$$k_1 - n_2 + 2|S_2| \geq t, \quad k_2 - n_1 + 2|S_1| \geq t \quad \Rightarrow \quad |S| \geq t + (n_1 + n_2 - k_1 - k_2)/2.$$
Proof. We have $\delta_S(v) \geq \delta_S^-(v) + k$ for all $v \in S$, and the equality holds for some $w \in S$. Then, since $S_1 \subset S$ and $\delta_S(v) = \delta_{S_1}(v) + |S_2|$ for every $v \in S_1$, we have

$$\delta_S(v) \geq \delta_S^-(v) + k \iff \delta_{S_1}(v) \geq \delta_{S_1}^-(v) + n_2 - 2|S_2| + k, \quad \forall v \in S_1$$

By symmetry we have the same result by $S_2$.

Besides, $w \in S_1$ or $w \in S_2$ (without loss of generality we can assume that $w \in S_1$); thus, it is a simple matter to $S_1$ is an exact defensive $(k + n_2 - 2|S_2|)$-alliance in $G_1$. \hfill $\square$

**Corollary 2.5.** Let $G_1, G_2$ be two graphs. Let $S \subset V(G_1 \uplus G_2)$ such that $\emptyset \neq S_1 := S \cap V(G_1)$ and $\emptyset \neq S_2 := S \cap V(G_2)$. If $S$ is an exact defensive $k$-alliance in $G_1 \uplus G_2$, then

$$|S_1| \leq \frac{k + \Delta_G}{2} \quad \text{or} \quad |S_2| \leq \frac{k + \Delta_G}{2}.$$ 

**Proof.** By Theorem 2.4 we have that at least one of $S_1, S_2$ is an exact defensive alliance, hence, $k + n_2 - 2|S_2| \geq -\Delta_{G_1}$ or $k + n_1 - 2|S_1| \geq -\Delta_{G_2}$. Without loss of generality we can assume that it is $S_1$. Therefore, since $\Delta_G = \max\{n_2 + \Delta_{G_1}, n_1 + \Delta_{G_2}\}$ we obtain $2|S_1| \leq k + n_2 + \Delta_{G_1} \leq k + \Delta_G$. \hfill $\square$

**Theorem 2.6.** Let $G_1, G_2$ be two graphs with order $n_1$ and $n_2$, respectively. Let $S \subset V(G_1 \uplus G_2)$ such that $\emptyset \neq S_1 := S \cap V(G_1)$, $\emptyset \neq S_2 := S \cap V(G_2)$. If $S$ is a defensive $k$-alliance in $G_1 \uplus G_2$, then

$$|S| \geq \frac{k + \sqrt{k^2 + 4n}}{2}. \quad (6)$$

**Proof.** We have $\delta_S(v) \geq \delta_S^-(v) + k$ for all $v \in S$, therefore, doing the summation over $v \in S$ we obtain

$$\sum_{v \in S} \delta_S(v) \geq \sum_{v \in S} \delta_S^-(v) + |S| \cdot k.$$ 

Let $m_S$ be the number of edges of $\langle S \rangle$. So, we have

$$\sum_{v \in S} \delta_S(v) = 2m_S \leq 2 \left( \frac{|S|}{2} \right).$$

Besides, we have domination of $S$, since $\emptyset \neq S_1$ and $\emptyset \neq S_2$ (i.e., $N_{G_1 \uplus G_2}(S) = V(G_1 \uplus G_2)$). Therefore, we obtain

$$|S|^2 - |S| \geq n_1 - |S_1| + n_2 - |S_2| + |S| \cdot k, \quad \iff \quad |S|^2 - k|S| - n \geq 0.$$ 

Finally, positivity of $|S|$ allow to obtain the result. \hfill $\square$
References


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