Hopf Bifurcation in Two Inhibitors on one Activator in a Reaction-Diffusion System\textsuperscript{1}

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Abstract. We consider a three-component reaction-diffusion system with two inhibitors and one activator. We analyze the existence of the radially symmetric solutions and the occurrence of Hopf bifurcation in the interfacial problem as the bifurcation parameters vary.

Mathematics Subject Classifications: 35R35, 35B32, 35B25, 35K55, 35K57, 58J55

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1. Introduction

A three-component reaction-diffusion system with one activator and two inhibitors is proposed as an extension of a phenomenological model of the planar dc gas-discharge system with semiconductor electrode [1, 3, 15, 13].

We consider a three-component reaction-diffusion system with one activator $u$ and two inhibitors $v$ and $w$ ([7]):

\begin{equation}
\begin{aligned}
\varepsilon \sigma u_t &= \varepsilon^2 \nabla^2 u - u + H(u - a(w)) - v, \\
v_t &= \nabla^2 v + \mu u - v, \\
bw_t &= d \nabla^2 w + u + v - w - s_0, \quad t > 0, \quad x \in \mathbb{R}^n,
\end{aligned}
\end{equation}

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with

\[ a(w) = \frac{1}{2} (1 + \tanh(\alpha w + a_0)), \]

where \( \varepsilon, \sigma, \mu, b, d, s_0, \alpha \) and \( a_0 \) are positive constants.

A free boundary problem of (1) with (2) for the case \( \varepsilon = 0 \) will be studied in this paper. Suppose that there is only one \((n-1)\)-dimensional hypersurface \( \eta(t) \) which is a single closed curve given in the domain in such a way that \( \mathbb{R}^n = \Omega_1(t) \cup \eta(t) \cup \Omega_0(t) \), where \( \Omega_1(t) = \{ x \in \mathbb{R}^n : u(x,t) > a(w) \} \) and \( \Omega_0(t) = \{ x \in \mathbb{R}^n : u(x,t) < a(w) \} \). The velocity of an interface \( \eta(t) \) is given by (see [9, 11]):

\[
\frac{d \eta(t)}{dt} \cdot \nu = \frac{1}{\sigma} C(v_i), \ x \in \eta(t),
\]

where \( \nu \) is the outward normal vector on \( \eta(t) \), \( v_i \) is the value of \( v \) on the interface \( \eta(t) \), and the velocity of the interface \( C \) is a continuously differentiable function defined on an interval \( I := (-a(w), 1 - a(w)) \) and the velocity of the interface can be normalized by ([2, 8, 9])

\[
C(v; a(w)) = \frac{1 - 2v - 2a(w)}{\sqrt{(v + a(w))(1 - a(w) - v)}}.
\]

An analysis of the dynamics of this process has been shown (see for example [2, 10, 12]) to lead a free boundary problem consisting of the initial-boundary value problem

\[
\begin{align*}
  v_t &= \nabla^2 v - (\mu + 1)v + \mu, \quad t > 0, \ x \in \Omega_1(t) \\
  v_t &= \nabla^2 v - (\mu + 1)v, \quad t > 0, \ x \in \Omega_0(t) \\
  v(\eta(t) - 0, t) &= v(\eta(t) + 0, t) \\
  \frac{d}{d\nu} v(\eta(t) - 0, t) &= \frac{d}{d\nu} v(\eta(t) + 0, t) \\
  \lim_{|x| \to \infty} v(x, t) &= 0
\end{align*}
\]

and

\[
\begin{align*}
  w_t &= \nabla^2 w - w + 1 - s_0, \quad t > 0, \ x \in \Omega_1(t) \\
  w_t &= \nabla^2 w - w - s_0, \quad t > 0, \ x \in \Omega_0(t) \\
  w(\eta(t) - 0, t) &= w(\eta(t) + 0, t) \\
  \frac{d}{d\nu} w(\eta(t) - 0, t) &= \frac{d}{d\nu} w(\eta(t) + 0, t) \\
  \lim_{|x| \to \infty} w(x, t) &= -s_0.
\end{align*}
\]

We note that if the limits \( b \to 0 \) and \( d \to \infty \), then \( w \) is a time-dependent but spatially independence variables, given by

\[
w(t) = \frac{1}{|\Omega|} \int_{\mathbb{R}^n} (u + v) d\mathbf{x} - s_0 = |\Omega_1(t)| - s_0,
\]
where \(|\Omega_1(t)|\) is a measure of \(\Omega_1(t)\). Then the system (1) with (2) is reduced to the following with a global inhibitory coupling term (6):

\[
\begin{aligned}
\sigma \varepsilon u_t &= \varepsilon^2 \nabla^2 u - u + H(u - a(t)) - v, \\
v_t &= \nabla^2 v + \mu u - v, \\
a(t) &= \frac{1}{2} \left( 1 + \tanh(\alpha(|\Omega_1(t)| - s_0) + a_0). \right.
\end{aligned}
\]

Letting \(\varepsilon = 0\) in (7), then the free boundary problem is given by

\[
\begin{aligned}
v_t &= \nabla^2 v - (\mu + 1)v + \mu H(\eta(t) - x), \quad t > 0, \ x \in \mathbb{R}^n \\
\sigma \eta'(t) &= C(v(\eta); a) = \frac{1 - 2v(\eta) - 2a}{\sqrt{(v(\eta) + a)(1 - a - v(\eta))}}.
\end{aligned}
\]

The Hopf bifurcation of (8) is investigated in [5].

The organization of the paper is as follows. In section 2, a change of variables is given which regularizes problem (4) and (5) in such a way that results from the theory of nonlinear evolution equations can be applied. In this way, we obtain enough regularity of the solution for an analysis of the bifurcation. In section 3, we show the existence of radially symmetric localized equilibrium solutions for (4) and (5) and obtain the linearization of problem (4) and (5). In the last section we show the existence of the periodic solutions and the bifurcation of the interface problem as the parameter \(\sigma\) varies in two and three dimensions.

2. The Regularized system

We look for an existence problem of radially symmetric equilibrium solutions of (4) and (5) with \(|x| = r\), where the center and the interface are located at the origin and \(r = \eta\), respectively. The problem is given by :

\[
\begin{aligned}
v_t &= \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r} \frac{\partial v}{\partial r} - (\mu + 1)v + \mu, \quad t > 0, \ r \in \Omega_1(t), \\
v_t &= \frac{\partial^2 v}{\partial r^2} + \frac{n-1}{r} \frac{\partial v}{\partial r} - (\mu + 1)v, \quad t > 0, \ r \in \Omega_0(t), \\
\frac{\partial w}{\partial r}(0, t) &= 0, \ \lim_{r \to \infty} v(r, t) = 0, \quad t > 0,
\end{aligned}
\]

\[
\begin{aligned}
w_t &= \frac{\partial^2 w}{\partial r^2} + \frac{n-1}{r} \frac{\partial w}{\partial r} - w + 1 - s_0, \quad t > 0, \ r \in \Omega_1(t), \\
w_t &= \frac{\partial^2 w}{\partial r^2} + \frac{n-1}{r} \frac{\partial w}{\partial r} - w - s_0, \quad t > 0, \ r \in \Omega_0(t), \\
\frac{\partial w}{\partial r}(0, t) &= 0, \ \lim_{r \to \infty} w(r, t) = -s_0, \quad t > 0,
\end{aligned}
\]

\[
\sigma \eta'(t) = C(v(\eta); a(w(\eta))), \quad t > 0, \quad \eta(0) = \eta_0,
\]

where \(\Omega_1(t) = \{ r : 0 < r < \eta(t) \}\) and \(\Omega_0(t) = \{ r : \eta(t) < r < \infty \}\).

Let \(q(r, t) = w(r, t) + s_0\). As a first step we obtain more regularity for the solution by semigroup methods, considering \(A := -\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + \mu + 1\) and \(A_0 := -\frac{\partial^2}{\partial r^2} + \frac{n-1}{r} \frac{\partial}{\partial r} + 1\) as densely defined operators \(A : D(A) \subset X \to X\) where \(D(A) = \{ v \in H^{2,2}(0, \infty) \} :\)
\[ \frac{\partial v}{\partial t}(0, t) = 0, \lim_{r \to \infty} v(r, t) = 0 \] and \( A_0 : D(A_0) \subset X \to X \) where \( D(A_0) = \{ w \in H^{2,2}((0, \infty)) : \frac{\partial w}{\partial r}(0, t) = 0, \lim_{r \to \infty} q(r, t) = 0 \} \) and \( X := L_2((0, \infty)) \) with norm \( \| \cdot \|_2 \).

The abstract evolution system of (9) is given by:

\[ \begin{cases} 
  v_t + Av = \mu H(\eta(t) - r), \\
  q_t + A_0q = H(\eta(t) - r), \\
  \sigma q'(t) = C(v(\eta); a(q(\eta) - s_0)), \quad \eta(0) = \eta_0.
\end{cases} \]  

We define \( g : [0, \infty) \times [0, \infty) \to \mathbb{R} \),

\[ g(r, \eta) := A^{-1}(\mu H(\eta - \cdot)(r)) = \mu \int_0^\infty G(r, y) H(\eta - y) \, dy \]

and \( \gamma : [0, \infty) \to \mathbb{R} \),

\[ \gamma(\eta) := g(\eta, \eta). \]

Here \( G : [0, \infty)^2 \to \mathbb{R} \) is a Green’s function of \( A \) satisfying the boundary conditions:

\[
G(r, z) = \begin{cases} \frac{1}{\sqrt{1+\mu}} \cosh(r \sqrt{1+\mu}) e^{-\sqrt{1+\mu}z}, & 0 < r < z \\ \frac{1}{\sqrt{1+\mu}} e^{-\sqrt{1+\mu}r} \cosh(z \sqrt{1+\mu}), & z < r \end{cases} (n = 1),
\]

\[
G(r, z) = \begin{cases} zK_0(z \sqrt{1+\mu}) I_0(r \sqrt{1+\mu}), & 0 < r < z \\ zI_0(z \sqrt{1+\mu}) K_0(r \sqrt{1+\mu}), & z < r \end{cases} (n = 2),
\]

where \( I_0 \) and \( K_0 \) are modified Bessel functions and

\[
G(r, z) = \begin{cases} ze^{-z \sqrt{1+\mu}} \frac{\sinh(r \sqrt{1+\mu})}{r \sqrt{1+\mu}}, & 0 < r < z \\ z \sinh(z \sqrt{1+\mu}) \frac{e^{-r \sqrt{1+\mu}}}{r \sqrt{1+\mu}}, & z < r \end{cases} (n = 3).
\]

We define \( \hat{g} : [0, \infty) \times [0, \infty) \to \mathbb{R} \),

\[ \hat{g}(r, \eta) := A_0^{-1}(H(\eta - \cdot)(r)) = \int_0^\infty \hat{G}(r, y) H(\eta - y) \, dy \]

and \( \hat{\gamma} : [0, \infty) \to \mathbb{R} \),

\[ \hat{\gamma}(\eta) := \hat{g}(\eta, \eta). \]

Here \( \hat{G} : [0, \infty)^2 \to \mathbb{R} \) is a Green’s function of \( A_0 \) satisfying the boundary conditions (10):

\[
\hat{G}(r, z) = \begin{cases} \cosh r e^{-z}, & 0 < r < z \\ e^{-r} \cosh z, & z < r \end{cases} (n = 1),
\]

\[
\hat{G}(r, z) = \begin{cases} zK_0(z) I_0(r), & 0 < r < z \\ zI_0(z) K_0(r), & z < r \end{cases} (n = 2),
\]
where $I_0$ and $K_0$ are modified Bessel functions and
\[
\hat{G}(r, z) = \begin{cases} 
    z e^{-z} \sinh \frac{r}{z}, & 0 < r < z \\
    z \sinh z \frac{r}{z}, & z < r \quad (n = 3).
\end{cases}
\]

We note that $\gamma'(\eta) > 0$, $\gamma''(\eta) - \mu G(\eta, \eta) < 0$, $\gamma'(\eta) > 0$, $\gamma'(\eta) - \hat{G}(\eta, \eta) < 0$ for all $\eta > 0$. Applying the transformation $u(t)(r) = v(r, t) - g(r, \eta(t))$ and $z(t)(r) = q(r, t) - \hat{g}(r, \eta(t))$, we obtain an equivalent abstract evolution equation of (9):
\[
\begin{cases}
\frac{d}{dt}(u, z, \eta) + \tilde{A}(u, z, \eta) = F(u, z, \eta) \\
(u, z, \eta)(0) = (u_0, z_0, \eta_0),
\end{cases}
\]
where $\tilde{A}$ is a $3 \times 3$ matrix defined on $D(\tilde{A}) = D(A) \times D(A_0) \times \mathbb{C}$ and given by
\[
\left( \begin{array}{ccc}
    A & 0 & 0 \\
    0 & A_0 & 0 \\
    0 & 0 & 0
\end{array} \right).
\]
The nonlinear forcing term $F$ defined on the set $S := \{(u, z, \eta) \in C^1(0, \infty) \times C^1(0, \infty) \times (0, \infty) : u(\eta) + \gamma(\eta) \in I, z(\eta) + \hat{\gamma}(\eta) \in I \}_{\text{open}} \subset C^1(\mathbb{R}) \times C^1(\mathbb{R}) \times \mathbb{R}$ as
\[
F(u, \eta, w) = \begin{pmatrix}
    -f_3(u, z, \eta) \cdot f_1(\eta) \\
    -f_3(u, z, \eta) \cdot f_2(\eta) \\
    f_3(u, z, \eta)
\end{pmatrix},
\]
where $f_1 : (0, \infty) \to X$, $f_1(\eta)(r) := \mu G(r, \eta)$, $f_2 : (0, \infty) \to X$, $f_2(\eta)(r) := \hat{G}(r, \eta)$ and $f_3 : S \to \mathbb{R}$, $f_3(u, z, \eta) := \frac{1}{\sigma} \mathcal{C}(u(\eta); a(z(\eta) + \hat{\gamma}(\eta) - s_0))$. The velocity of $\eta$ is written by
\[
C(u(\eta); a(z(\eta) + \hat{\gamma}(\eta) - s_0)) = C(\chi(u, z, \eta)) = \frac{1 - 2\chi(u, z, \eta)}{\sqrt{\chi(u, z, \eta)(1 - \chi(u, z, \eta))}}
\]
where $\chi(u, z, \eta) = u(\eta) + \gamma(\eta) + a(z(\eta) + \hat{\gamma}(\eta) - s_0)$.

**Lemma 2.1.** The functions $f_1 : (0, \infty) \to X$, $f_2 : (0, \infty) \to X$, $f_3 : S \to \mathbb{R}$ and $F : S \to X \times X \times \mathbb{R}$ are continuously differentiable with derivatives given by
\[
f_1'(\eta) = \frac{\partial G}{\partial \eta}(\cdot, \eta)
\]
\[
f_2'(\eta) = \frac{\partial \hat{G}}{\partial \eta}(\cdot, \eta)
\]
\[
Df_3(u, z, \eta)(\tilde{u}, \tilde{z}, \tilde{\eta}) = C(\chi(u, z, \eta))(u'(\eta)\tilde{\eta} + \tilde{u}(\eta) + \chi(\eta)\tilde{\eta})
\]
\[
+ a'(z(\eta) + \hat{\gamma}(\eta) - s_0) \cdot (z'(\eta)\tilde{\eta} + \tilde{z}(\eta) + \hat{\gamma}'(\eta)\tilde{\eta})
\]
\[
DF(u, z, \eta)(\tilde{u}, \tilde{z}, \tilde{\eta}) = f_3(u, z, \eta) \cdot (f_1'(\eta), f_2'(\eta), 0) \tilde{\eta} + Df_3(u, z, \eta)(\tilde{u}, \tilde{z}, \tilde{\eta}) \cdot (f_1(\eta), f_2(\eta), 1),
\]
where $\tilde{u}, \tilde{z}, \tilde{\eta}$ are modified Bessel functions and $\chi(u, z, \eta) = u(\eta) + \gamma(\eta) + a(z(\eta) + \hat{\gamma}(\eta) - s_0)$.
where $C_x = \frac{\partial C}{\partial x}$.

The well posedness of solutions was shown in [5] applying the semigroup theory using domains of fractional powers $\theta \in (3/4, 1]$ of $A$ and $\tilde{A}$ ([6]). Moreover, they obtained that $F : S \cap D(\tilde{A}^\theta) \to X \times X \times \mathbb{R}$ is a continuously differentiable function where $D(\tilde{A}^\theta) = D(A^\theta) \times D(A_0^\theta) \times \mathbb{R}$.

3. Radially symmetric equilibrium solutions and linearization

The steady states are solutions of the following problem:

$$
\begin{align*}
Au^* &= -\mu G(\cdot, \eta^*)C(u^*(\eta^*) + \gamma(\eta^*); a(z(\eta^* + \hat{\gamma}(\eta^*) - s_0)) \\
A_0z^* &= -\tilde{G}(\cdot, \eta^*)C(u^*(\eta^*) + \gamma(\eta^*); a(z(\eta^* + \hat{\gamma}(\eta^*) - s_0)) \\
0 &= C(u^*(\eta^*) + \gamma(\eta^*); a(z(\eta^* + \hat{\gamma}(\eta^*) - s_0)) \\
u^*(0) &= u^*(\infty) \\
z^*(0) &= z^*(\infty)
\end{align*}
$$

(13)

for $(u^*, z^*, \eta^*) \in D(\tilde{A}) \cap S$.

**Theorem 3.1.** Suppose that $\alpha s_0 > a_0$ and $\frac{\mu}{1+\mu} + \tanh(\alpha(\frac{1}{2} - s_0)) + a_0 > 0$. Then the stationary problem of (11) has the only stationary solution $(u^*, z^*, \eta^*)$ for all $\sigma \neq 0$ with $u^* = 0 = z^*$. The linearization of $F$ at $(0, 0, \eta^*)$ is

$$
DF(0, 0, \eta^*)(\hat{u}, \hat{z}, \hat{\eta}) = \begin{pmatrix}
\frac{1}{\sigma} \mu G(\cdot, \eta^*) (\hat{u}(\eta^*) + \gamma'(\eta^*)\hat{\eta} + a'(\hat{\gamma}(\eta^*) - s_0)) \cdot (\hat{z}(\eta^*) + \hat{\gamma}'(\eta^*)\hat{\eta}) \\
\frac{1}{\sigma} \tilde{G}(\cdot, \eta^*) (\hat{u}(\eta^*) + \gamma'(\eta^*)\hat{\eta} + a'(\hat{\gamma}(\eta^*) - s_0)) \cdot (\hat{z}(\eta^*) + \hat{\gamma}'(\eta^*)\hat{\eta}) \\
-\frac{4}{\sigma} (\hat{u}(\eta^*) + \gamma'(\eta^*)\hat{\eta} + a'(\hat{\gamma}(\eta^*) - s_0)) \cdot (\hat{z}(\eta^*) + \hat{\gamma}'(\eta^*)\hat{\eta})
\end{pmatrix}.
$$

The triple $(0, 0, \eta^*)$ corresponds to a unique steady state $(v^*, w^*, \eta^*)$ of (9) for $\sigma \neq 0$ with $v^*(r) = g(r, \eta^*)$ and $w^*(r) = \hat{g}(r, \eta^*) - s_0$.

**Proof:** From the system (13), $\eta^*$ and $w^*$ are solutions of the following equations

$$
u^* = 0, \quad z^* = 0 \quad \text{and} \quad C(0, 0, \eta^*) = 0.
$$

(14)

We only check the existence of $\eta^*$ of (12) and (14) and thus we let

$$
\Gamma(\eta) := \frac{1}{2} - \gamma(\eta) - a(\hat{\gamma}(\eta) - s_0) = -\gamma(\eta) - \frac{1}{2} \tanh(\alpha(\hat{\gamma}(\eta) - s_0) + a_0).
$$

Since $\gamma'(\eta) > 0$ and $\hat{\gamma}'(\eta) > 0$, we have $\Gamma'(\eta) < 0$. Therefore if $\Gamma(0) > 0$ and $\Gamma(\infty) < 0$ then there is a unique $\eta^* \in (0, \infty)$. Since $\gamma(0) = 0 = \hat{\gamma}(0), \Gamma(0) > 0$ if $\alpha s_0 - a_0 > 0$. We have $\frac{\mu}{2(1+\mu)} = \frac{\mu}{2(1+\mu)}$ and $\hat{\gamma}(\infty) = \frac{1}{2}$, and so $\Gamma(\infty) < 0$ if $\frac{\mu}{2(1+\mu)} + \tanh(\alpha(\frac{1}{2} - s_0)) + a_0 > 0$.

The formula for $DF(0, 0, \eta^*)$ follows from Lemma 2.1, the relation $C'(0, 0, \eta^*) = -4$. The corresponding steady state $(v^*, w^*, \eta^*)$ for (9) is obtained using the transformation and Theorem 2.1 in [4].
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Definition 3.2. Suppose that \( \alpha s_0 > a_0 \) and \( \frac{\mu}{1+\mu} > \tanh(\alpha(-\frac{1}{2}+s_0)-a_0) \). We define (for \( 1 \geq \theta > 3/4 \)) the operator \( B \) is a linear operator from \( D(\tilde{A}^0) \) to \( D(\tilde{A}) \) as

\[
B := \frac{\sigma}{\tau} DF(0,0,\eta^\ast).
\]

We then define \((0,0,\eta^\ast)\) to be a Hopf point for (11) if there exists an \( \epsilon_0 > 0 \) and a \( C^1 \)-curve

\[
(-\epsilon_0 + \tau^\ast, \tau^\ast + \epsilon_0) \mapsto (\lambda(\tau), \phi(\tau)) \in \mathbb{C} \times D(\tilde{A})\mathbb{C}
\]

\((YC \) denotes the complexification of the real space \( Y \)) of eigendata for \(-\tilde{A} + \tau B\) such that

(i) \((-\tilde{A} + \tau B)(\phi(\tau)) = \lambda(\tau)\phi(\tau), \quad (-\tilde{A} + \tau B)(\tilde{\phi}(\tau)) = \tilde{\lambda}(\tau)\tilde{\phi}(\tau)\);

(ii) \(\lambda(\tau^\ast) = i\beta \) with \( \beta > 0 \);

(iii) \(\text{Re}(\lambda) \neq 0 \) for all \( \lambda \) in the spectrum of \((-\tilde{A} + \tau^\ast B) \setminus \{ \pm i\beta \}\);

(iv) \(\text{Re}(\lambda(\tau^\ast)) \neq 0 \) (transversality).

where \( \tau = 4/\sigma \).

4. Hopf Bifurcation Analysis

We shall show that there is a Hopf bifurcation from the curve \( \sigma \mapsto (0,0,\eta^\ast) \) of radially symmetric stationary solution. The linearized eigenvalue problem of (11) is

\[
-\tilde{A}(u,z,\eta) + \tau B(u,z,\eta) = \lambda I_3(u,z,\eta),
\]

where \( I_3 \) is an 3 by 3 identity matrix. This is equivalent to

\[
\begin{cases}
(A + \lambda)u = \tau \mu G(\cdot, \eta^\ast)(u(\eta^\ast) + \gamma'(\eta^\ast)\eta + a'(\hat{\gamma}(\eta^\ast) - s_0) \cdot (z(\eta^\ast) + \hat{\gamma}'(\eta^\ast)\eta)) \\
(A_0 + \lambda)z = \tau \hat{G}(\cdot, \eta^\ast)(u(\eta^\ast) + \gamma'(\eta^\ast)\eta + a'(\hat{\gamma}(\eta^\ast) - s_0) \cdot (z(\eta^\ast) + \hat{\gamma}'(\eta^\ast)\eta)) \\
\lambda \eta = -(u(\eta^\ast) + \gamma'(\eta^\ast)\eta + a'(\hat{\gamma}(\eta^\ast) - s_0) \cdot (z(\eta^\ast) + \hat{\gamma}'(\eta^\ast)\eta))
\end{cases}
\]

(15)

Our main theorem is stated as below:

Theorem 4.1. Suppose that \( \alpha s_0 > a_0 \) and \( \frac{\mu}{1+\mu} > \tanh(\alpha(-\frac{1}{2}+s_0)-a_0) \), the problem (11), and (9), has a unique stationary solution \((u^\ast, z^\ast, \eta^\ast)\) where \( u^\ast = 0 = z^\ast \) and \((v^\ast, w^\ast, \eta^\ast)\), respectively for all \( \tau > 0 \). Then there exists a unique \( \tau^\ast \) such that the linearization \(-\tilde{A} + \tau^\ast B\) has a purely imaginary pair of eigenvalues \( \beta \). The point \((0,0,\eta^\ast, \tau^\ast)\) is then a Hopf point for (11) and there exists a \( C^0 \)-curve of nontrivial periodic orbits for (11), and (9), bifurcating from \((0,0,\eta^\ast, \tau^\ast)\), and \((v^\ast, w^\ast, \eta^\ast, \tau^\ast)\), respectively.

We shall show the following three theorems that verify the above theorem. The next theorem show that the steady state is the only Hopf point.

Theorem 4.2. For \( \tau^\ast \in \mathbb{R} \setminus \{0\}\), the operator \(-\tilde{A} + \tau^\ast B\) has a unique pair of purely imaginary eigenvalues \( \{ \pm i\beta \}\). Then the point \((0,0,\eta^\ast, \tau^\ast)\) satisfies the conditions (i),(ii) and (iii) in Definition 3.2.
Proof: We assume without loss of generality that \( \beta > 0 \), and \( \Phi^* \) is the (normalized) eigenfunction of \( -\tilde{A} + \tau^*B \) with eigenvalue \( i\beta \). We have to show that \( (\Phi^*, i\beta) \) can be extended to a \( C^1 \)-curve \( \tau \mapsto (\Phi(\tau), \lambda(\tau)) \) of eigendata for \( -\tilde{A} + \tau B \) with \( \text{Re}(\lambda(\tau^*)) \neq 0 \).

For this let \( \Phi^* := (\psi_0, z_0, \eta_0) \in D(\tilde{A}) \). We see that \( \eta_0 \neq 0 \), for otherwise, by (15), \( (A + i\beta)\psi_0 = i\beta \mu G(\cdot, \eta^*) \eta_0 = 0 \) and \( (A_0 + i\beta)z_0 = i\beta \tilde{G}(\cdot, \eta^*) \eta_0 = 0 \), which are not possible because \( A \) and \( A_0 \) are symmetric. So without loss of generality, let \( \eta_0 = 1 \). Define

\[
E : D(A)_C \times D(A_0)_C \times \mathbb{C} \times \mathbb{R} \longrightarrow X_C \times X_C \times \mathbb{C},
\]

\[
E(u, z, \lambda, \tau) := \begin{pmatrix}
(A + \lambda)u - \tau \mu G(\cdot, \eta^*)(u(\eta^*) + \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0) \cdot (z(\eta^*) + \hat{\gamma}'(\eta^*))) \\
(A_0 + \lambda)z - \tau \hat{G}(\cdot, \eta^*)(u(\eta^*) + \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0) \cdot (z(\eta^*) + \hat{\gamma}'(\eta^*))) \\
\lambda + \tau (u(\eta^*) + \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0) \cdot (z(\eta^*) + \hat{\gamma}'(\eta^*)))
\end{pmatrix}
\]

The equation \( E(u, z, \lambda, \tau) = 0 \) is equivalent to \( \lambda \) being an eigenvalue of \( -\tilde{A} + \tau B \) with eigenfunction \( (u, z, 1) \). By (15), we have \( E(\psi_0, z_0, i\beta, \tau^*) = 0 \), which is equivalent to

\[
\begin{align*}
(A + i\beta)\psi_0 &= -i\beta \mu G(\cdot, \eta^*) \\
(A_0 + i\beta)z_0 &= -i\beta \hat{G}(\cdot, \eta^*) \\
i\beta &= -\tau^* (\psi_0(\eta^*) + \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0) \cdot (z(\eta^*) + \hat{\gamma}'(\eta^*))).
\end{align*}
\]

To apply the implicit function theorem to \( E \), we have to check that \( E \) is in \( C^1 \) and that

\[
D_{(u, z, \lambda)}E(\psi_0, z_0, i\beta, \tau^*) \in L(D(A)_C \times D(A_0)_C \times X_C \times X_C \times \mathbb{C})
\]

is an isomorphism. In addition, the mapping

\[
D_{(u, z, \lambda)}E(\psi_0, z_0, i\beta, \tau^*)(\hat{u}, \hat{z}, \hat{\lambda})
\]

\[
= \begin{pmatrix}
(A + i\beta)\hat{u} + \hat{\lambda}\psi_0 - \tau^* \mu G(\cdot, \eta^*)(\hat{u}(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0) \cdot \hat{z}(\eta^*)) \\
(A_0 + i\beta)\hat{z} + \hat{\lambda}z_0 - \tau^* \hat{G}(\cdot, \eta^*)(\hat{u}(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0) \cdot \hat{z}(\eta^*)) \\
\hat{\lambda} + \tau^* (\hat{u}(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0) \cdot \hat{z}(\eta^*))
\end{pmatrix}
\]

is a compact perturbation of the mapping

\[
(\hat{u}, \hat{z}, \hat{\lambda}) \mapsto \left( (A + i\beta)\hat{u}, (A_0 + i\beta)\hat{z}, \hat{\lambda} \right)
\]

which is invertible. In order to verify (17), it suffices to show that the system

\[
D_{(u, z, \lambda)}E(\psi_0, z_0, i\beta, \tau^*)(\hat{u}, \hat{z}, \hat{\lambda}) = 0
\]
which are
\[
\begin{align*}
(A + i\beta)\dot{u} + \lambda \psi_0 &= \tau^* \mu G(\cdot, \eta^*) (\dot{u}(\eta^*) + \alpha'(\dot{\gamma}(\eta^*) - s_0) \cdot \dot{\gamma}(\eta^*)) \\
(A_0 + i\beta)\dot{z} + \lambda z_0 &= \tau^* \hat{G}(\cdot, \eta^*) (\dot{u}(\eta^*) + \alpha'(\dot{\gamma}(\eta^*) - s_0) \cdot \dot{\gamma}(\eta^*)) \\
\hat{\lambda} &= -\tau^* (\dot{u}(\eta^*) + \alpha'(\dot{\gamma}(\eta^*) - s_0) \cdot \dot{\gamma}(\eta^*))
\end{align*}
\] (18)

necessarily implies that \( \dot{u} = 0 \), \( \dot{z} = 0 \) and \( \hat{\lambda} = 0 \). We define \( \phi := \psi_0 + \mu G(\cdot, \eta^*) \) and \( \xi = z_0 + \hat{G}(\cdot, \eta^*) \) then the first equation of (18) is given by
\[
(A + i\beta)\phi = 0
\] (19)
\[
(A_0 + i\beta)\xi = 0
\] (20)
\[
\hat{\lambda} = -\tau^* (\xi(\eta^*) + \alpha'(\xi(\eta^*) - s_0) \cdot \xi(\eta^*)).
\] (21)

Since \( (v, q, \eta, \lambda) = (\phi, \xi, 1, i\beta) \) solves (15), \( \phi \) is a solution to the equation
\[
(A + i\beta)\phi = \mu \delta_{n^*},
\] (22)
\[
(A_0 + i\beta)\xi = \bar{\delta}_{n^*},
\] (23)
\[
-\frac{id}{\tau^*} = \phi(\nu^*) - \mu G(\xi^*, \eta^*) + \gamma(\eta^*) + \alpha'(\dot{\gamma}(\eta^*) - s_0) \cdot (\xi(\eta^*) + \hat{G}(\eta^*, \eta^*) + \dot{\gamma}'(\eta^*)).
\] (24)

We multiply (22) by \( r^{n-1} \bar{\phi} \) and (23) by \( \mu a'(\dot{\gamma}(\eta^*) - s_0) r^{n-1} \bar{\xi} \) and integrate the resulting equation to obtain
\[
\int (A + i\beta) \phi r^{n-1} \bar{\phi} + \mu a'(\dot{\gamma}(\eta^*) - s_0) \int (A_0 + i\beta) \xi r^{n-1} \bar{\xi} = \mu(\eta^*)^{n-1} (\bar{\phi}(\eta^*) + \mu a'(\dot{\gamma}(\eta^*) - s_0) \bar{\xi}(\eta^*)).
\] (25)

The imaginary part of the above equation is given by
\[
\beta \left( \int r^{n-1} \phi \bar{\phi} + \mu a'(\dot{\gamma}(\eta^*) - s_0) \int r^{n-1} \xi \bar{\xi} \right) = -\mu(\eta^*)^{n-1} (\text{Im}\phi(\eta^*) + \mu a'(\dot{\gamma}(\eta^*) - s_0) \text{Im}\xi(\eta^*))
\] and so,
\[
||r^{(n-1)/2}\phi||^2 + ||r^{(n-1)/2}\xi||^2 = \mu(\eta^*)^{n-1} \frac{1}{\tau^*}, \tag{26}
\]

where \( ||r^{(n-1)/2}\phi||^2 = \int r^{n-1} \phi \bar{\phi} dr \).

We multiply (19) by \( r^{n-1} \phi \) and (20) by \( \mu a'(\dot{\gamma}(\eta^*) - s_0) r^{n-1} \bar{\xi} \) and integrate the resulting equation to obtain
\[
0 = \mu(\eta^*)^{n-1} (\dot{u}(\eta^*) + \mu a'(\dot{\gamma}(\eta^*) - s_0) \dot{z}(\eta^*)) + 2i\beta \left( \int r^{n-1} \dot{\phi} + \mu a'(\dot{\gamma}(\eta^*) - s_0) \int r^{n-1} \dot{z} \bar{\xi} \right) + \hat{\lambda} (||r^{(n-1)/2}\phi||^2 + ||r^{(n-1)/2}\xi||^2).
\]

Applying (21) and (26), we have
\[
0 = 2i\beta \left( \int r^{n-1} \dot{\phi} + \mu a'(\dot{\gamma}(\eta^*) - s_0) \int r^{n-1} \dot{z} \bar{\xi} \right). \tag{27}
\]
We multiply (19) by $r^{n-1} \bar{u}$ and (20) by $\mu a'(\hat{\gamma}(\eta^*) - s_0) r^{n-1} \bar{z}$ and integrate the resulting equation to obtain

$$0 = ||A^{1/2} r^{(n-1)/2} \bar{u}||^2 + \mu a'(\hat{\gamma}(\eta^*) - s_0)||A^{1/2} r^{(n-1)/2} \bar{z}||^2$$

$$+ i \beta \left(||r^{(n-1)/2} \bar{u}||^2 + \mu a'(\hat{\gamma}(\eta^*) - s_0)||r^{(n-1)/2} \bar{z}||^2\right) + \lambda \left(\int r^{n-1} \phi \, \bar{u} + \mu a'(\hat{\gamma}(\eta^*) - s_0) \int r^{n-1} \xi \, \bar{z}\right).$$

Applying (27) and then the real and the imaginary parts of the resulting equation is

$$\left\{ \begin{array}{l}
||A^{1/2} r^{(n-1)/2} \bar{u}||^2 + \mu a'(\hat{\gamma}(\eta^*) - s_0) \cdot ||A^{1/2} r^{(n-1)/2} \bar{z}||^2 = 0 \\
||r^{(n-1)/2} \bar{u}||^2 + \mu a'(\hat{\gamma}(\eta^*) - s_0) \cdot ||r^{(n-1)/2} \bar{z}||^2 = 0.
\end{array} \right.$$ 

From the second equation, we have

$$||r^{(n-1)/2} \bar{u}||^2 = 0 \text{ and } ||r^{(n-1)/2} \bar{z}||^2 = 0$$

since $a'(\hat{\gamma}(\eta^*) - s_0) > 0$. Hence $\bar{u} = 0$ and $\bar{z} = 0$. Therefore, we should have $\lambda = 0$. \qed

**Theorem 4.3.** Under the same condition as in Definition 3.2, $(0, 0, \eta^*, \tau^*)$ satisfies the transversality condition. Hence this is a Hopf point for (11).

**Proof:** By implicit differentiation of $E(\psi_0(\tau), z(\tau), \lambda(\tau), \tau) = 0$,

$$D_{(u,z,\lambda)} E(\psi_0, z_0, i\beta, \tau^*)(\psi'_0(\tau^*), z'(\tau^*), \lambda'(\tau^*))$$

$$= \left\{ \begin{array}{l}
\mu G(\eta^*, \eta^*) (\psi_0(\eta^*) + \gamma'(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) \cdot (z_0(\eta^*) + \hat{\gamma}'(\eta^*)) \\
\hat{G}(\eta^*, \eta^*) (\psi_0(\eta^*) + \gamma'(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) \cdot (z_0(\eta^*) + \hat{\gamma}'(\eta^*)) \\
-(\psi_0(\eta^*) + \gamma'(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) \cdot (z_0(\eta^*) + \hat{\gamma}'(\eta^*))
\end{array} \right\}.$$

This means that the function $\tilde{u} := \psi'_0(\tau^*), \tilde{z} := z'(\tau^*)$ and $\tilde{\lambda} := \lambda'(\tau^*)$ satisfy the equations

$$\left\{ \begin{array}{l}
(A + i\beta) \tilde{u} + \tilde{\lambda} \psi_0 - \tau^* \mu G(\cdot, \eta^*) (\tilde{u}(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) \cdot \tilde{z}(\eta^*) \\
= \mu G(\cdot, \eta^*) (\psi_0(\eta^*) + \gamma'(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) \cdot (z_0(\eta^*) + \hat{\gamma}'(\eta^*))
\end{array} \right\}$$

$$\left\{ \begin{array}{l}
(A_0 + i\beta) \tilde{z} + \tilde{\lambda} \xi_0 - \tau^* \hat{G}(\cdot, \eta^*) (\bar{u}(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) \cdot \bar{z}(\eta^*) \\
= \hat{G}(\cdot, \eta^*) (\psi_0(\eta^*) + \gamma'(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) (z_0(\eta^*) + \hat{\gamma}'(\eta^*) )
\end{array} \right\}$$

$$\tilde{\lambda} + \tau^* (\tilde{u}(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) \cdot \tilde{z}(\eta^*)$$

$$= - (\psi_0(\eta^*) + \gamma'(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) (z_0(\eta^*) + \hat{\gamma}'(\eta^*) ) .$$

By letting $\phi := \psi_0 + \mu G(\cdot, \eta^*)$ and $\xi := z_0 + \hat{G}(\cdot, \eta^*)$, we have

$$\left\{ \begin{array}{l}
(A + i\beta) \tilde{u} + \tilde{\lambda} \phi = 0,
\end{array} \right\}$$

$$\left\{ \begin{array}{l}
(A_0 + i\beta) \tilde{z} + \tilde{\lambda} \xi = 0
\end{array} \right\}$$

$$\left\{ \begin{array}{l}
\tilde{\lambda} + \tau^* (\tilde{u}(\eta^*) + a' \hat{\gamma}(\eta^*) - s_0) \cdot \tilde{z}(\eta^*)
\end{array} \right\}.$$
Multiplying (29) by $r^{-1} \tilde{\phi}$ and (30) by $\mu a' (\hat{\gamma}(\eta^*) - s_0) r^{-1} \tilde{\xi}$ and integrating the resulting equation, we obtain

$$0 = \mu (\eta^*)^{-n} (\tilde{u}(\eta^*) + \mu a' (\hat{\gamma}(\eta^*) - s_0) \tilde{z}(\eta^*)) + 2i\beta \left( \int r^{-1} \tilde{u} \tilde{\phi} + \mu a' (\hat{\gamma}(\eta^*) - s_0) \int r^{-1} \tilde{z} \tilde{\xi} \right) + \tilde{\lambda} (|r^{(n-1)/2} \tilde{\phi}|^2 + \mu a' (\hat{\gamma}(\eta^*) - s_0) |r^{(n-1)/2} \tilde{\xi}|^2).$$

Applying (31) and (26), we have

$$0 = \mu (\eta^*)^{-n} \frac{i\beta}{(\tau^*)^2} + 2i\beta \left( \int r^{-1} \tilde{u} \tilde{\phi} + \mu a' (\hat{\gamma}(\eta^*) - s_0) \int r^{-1} \tilde{z} \tilde{\xi} \right).$$

Multiplying (29) by $r^{-1} \tilde{u}$ and (30) by $\mu a' (\hat{\gamma}(\eta^*) - s_0) r^{-1} \tilde{z}$ and integrating the resulting equation, we obtain

$$0 = ||A^{1/2} r^{(n-1)/2} \tilde{u}||^2 + \mu a' (\hat{\gamma}(\eta^*) - s_0) ||A^{1/2} r^{(n-1)/2} \tilde{z}||^2 + i\beta \left( ||r^{(n-1)/2} \tilde{u}||^2 + \mu a' (\hat{\gamma}(\eta^*) - s_0) ||r^{(n-1)/2} \tilde{z}||^2 \right) + \tilde{\lambda} \left( \int r^{-1} \tilde{\phi} \tilde{u} + \mu a' (\hat{\gamma}(\eta^*) - s_0) \int r^{-1} \tilde{\xi} \tilde{z} \right).$$

Applying (32)

$$0 = ||A^{1/2} r^{(n-1)/2} \tilde{u}||^2 + \mu a' (\hat{\gamma}(\eta^*) - s_0) ||A^{1/2} r^{(n-1)/2} \tilde{z}||^2 + i\beta \left( ||r^{(n-1)/2} \tilde{u}||^2 + \mu a' (\hat{\gamma}(\eta^*) - s_0) ||r^{(n-1)/2} \tilde{z}||^2 \right) - \frac{\tilde{\lambda} \mu (\eta^*)^{-n} 1}{2(\tau^*)^2},$$

we see that the real part of the resulting equation is

$$\frac{\mu}{2(\tau^*)^2} (\eta^*)^{-n} \Re \tilde{\lambda} = ||A^{1/2} r^{(n-1)/2} \tilde{u}||^2 + \mu a' (\hat{\gamma}(\eta^*) - s_0) \cdot ||A^{1/2} r^{(n-1)/2} \tilde{z}||^2.$$ 

Since $a$ is an increasing function, $\Re \tilde{\lambda}(\tau^*) > 0$ for $\beta > 0$ and thus by the Hopf-bifurcation theorem in [4], there exists a family of periodic solutions which bifurcates from the stationary solution as $\tau$ passes $\tau^*$.

The next theorem shows a critical Hopf point $\tau^*$ exists uniquely.

**Theorem 4.4.** Under the same condition as in Definition 3.2, there exists a unique, purely imaginary eigenvalue $\lambda = i\beta$ of (15) with $\beta > 0$ for a unique critical point $\tau^* > 0$ in order for $(0, 0, \eta^*, \tau^*)$ to be a Hopf point.

**Proof:** We only need to show that the function $(u, z, \beta, \tau) \mapsto E(u, z, i\beta, \tau)$ has a unique zero with $\beta > 0$ and $\tau > 0$. This means solving the system (15) with $\lambda = i\beta$, $\eta_0 = 1$, $\psi_0 = \phi - \mu G(\cdot, \eta^*)$ and $z_0 = \xi - \hat{G}(\cdot, \eta^*)$

$$\begin{align*}
(A + i\beta)\phi &= \mu \delta_{\eta^*}, \\
(A_0 + i\beta)\xi &= \hat{\delta}_{\eta^*}, \\
-\frac{i\beta}{\tau^*} &= \phi(\eta^*) - \mu G(\eta^*, \eta^*) + \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0) (\xi(\eta^*) - \hat{G}(\eta^*, \eta^*) + \hat{\gamma}'(\eta^*)).
\end{align*}$$
The third equation becomes
\[
\frac{-i\beta}{\tau^*} = \mu G_\beta(\eta^*, \eta^*) - \mu G(\eta^*, \eta^*) + \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0)(\hat{G}_\beta(\eta^*, \eta^*) - \hat{G}(\eta^*, \eta^*) + \hat{\gamma}'(\eta^*)) ,
\]
where $G_\beta$ and $\hat{G}_\beta$ are a Green’s function of $A + i\beta$ and $A_0 + i\beta$, respectively. The real and imaginary parts of this above equation are given by
\[
\begin{cases}
-\frac{\beta}{\tau^*} = \mu \text{Im}G_\beta(\eta^*, \eta^*) + a'(\hat{\gamma}(\eta^*) - s_0) \text{Im}\hat{G}_\beta(\eta^*, \eta^*) \\
0 = \mu \text{Re}G_\beta(\eta^*, \eta^*) - \mu G(\eta^*, \eta^*) + \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0)(\text{Re}\hat{G}_\beta(\eta^*, \eta^*) - \hat{G}(\eta^*, \eta^*) + \hat{\gamma}'(\eta^*)).
\end{cases}
\]
Since $\text{Im}G_\beta(\eta^*, \eta^*) < 0$ and $\text{Im}\hat{G}_\beta(\eta^*, \eta^*) < 0$ in Lemma 12 ([4]), there is a unique $\tau$ in the first equation if it does guarantee the existence of $\beta$. Now, we let
\[
T(\beta) := \mu \text{Re}G_\beta(\eta^*, \eta^*) - \mu G(\eta^*, \eta^*) + \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0)(\text{Re}\hat{G}_\beta(\eta^*, \eta^*) - \hat{G}(\eta^*, \eta^*) + \hat{\gamma}'(\eta^*)) \]
than $T(0) = \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0)\hat{\gamma}'(\eta^*) > 0$ and $T(\infty) = -\mu G(\eta^*, \eta^*) + \gamma'(\eta^*) + a'(\hat{\gamma}(\eta^*) - s_0)(-\hat{G}(\eta^*, \eta^*) + \hat{\gamma}'(\eta^*)) < 0$. Since $\text{Re}G_\beta(\eta^*, \eta^*)$ and $\text{Re}\hat{G}_\beta(\eta^*, \eta^*)$ are decreasing functions of $\beta$ in Lemma 12 ([4]), $T'(\beta) < 0$ for all $\beta > 0$.

There is a unique pure imaginary eigenvalue $\beta > 0$ and the critical point $\tau^*$ of (9) and thus, there exists a family of periodic solutions which bifurcates from the stationary solution as $\tau$ passes $\tau^*$ under the condition of Theorem 4.4. Thus we also found the relationship between $\mu$ and $a(w(\eta))$ for which Hopf bifurcation occurs for the problem (9).

References


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