Two Analytical Methods Applied to Study
Thin Film Flow of an Eyring-Powell Fluid
on a Vertically Moving Belt

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Abstract

The purpose of this article is to investigate the thin film flow of an Eyring-
Powell fluid on a vertically moving belt. The nonlinear equation governing the
flow problem is modeled and then solved by applying the variational iteration
method and the Adomian decomposition procedure. The numerical results
obtained by these methods are then compared through graphs and very good
agreement is observed. This study highlights the significant features of the
proposed methods and their ability for solving nonlinear problems arising in
non-Newtonian fluid mechanics.

Keywords: Thin film flow, Eyring-Powell fluid, Moving belt, Nonlinear bound-
ary value problem, Variational iteration method, Adomian decomposition method

1 Introduction

Recently, many powerful mathematical methods have been proposed in the litera-
ture. Among them, the homotopy perturbation method (HPM) [1-3], the variational
iteration method (VIM) [4-6] and the Adomian decomposition method (ADM) [7-9]
are relatively new techniques for obtaining exact or analytical approximate solutions to linear or nonlinear problems. These are computational methods and provide immediate and visible symbolic terms of analytical solutions as well as a numerical approximate solution to both linear and nonlinear differential equations without any discretization, linearization or any other physically unrealistic assumption. The interested reader can see the references [10–13] for last developments in these methods.

The aim of the present study is to apply He’s VIM and the ADM to find approximate solutions of a nonlinear differential equation that arises in the thin film flow of an Eyring-Powell fluid on a vertically moving belt. The realistic numerical solutions are obtained in the form of rapidly convergent series with easily computable components. Numerical results derived from these methods are then shown graphically and a high degree of accuracy is observed. The conclusion is that the VIM and the ADM may be considered as alternative and powerful methods for solving nonlinear problems arising in non-Newtonian fluid mechanics.

2 Governing Equations

For an incompressible fluid, the balance of mass and momentum, neglecting the thermal effects, are given by

\[ \nabla \cdot \mathbf{V} = 0, \quad (1) \]

\[ \rho \frac{d\mathbf{V}}{dt} = -\nabla p + \nabla \cdot \mathbf{S} + \rho f, \quad (2) \]

where \( \mathbf{V} \) is the velocity vector, \( \rho \) is the constant density of the fluid, \( \mathbf{S} \) is the Cauchy stress tensor, \( f \) is (represents) the specific body force, \( p \) is the dynamic pressure and \( \frac{d}{dt} \) is the material time derivative. The Cauchy stress tensor for the Eyring-Powell fluid is related [16, 17] to the fluid motion through the relations

\[ \mathbf{S} = \mu \nabla \mathbf{V} + \frac{1}{\beta} \sinh^{-1} \left( \frac{1}{C} \nabla \mathbf{V} \right) \quad (3) \]

and

\[ \sinh^{-1} \left( \frac{1}{C} \nabla \mathbf{V} \right) \approx \frac{1}{C} \nabla \mathbf{V} - \frac{1}{6} \left( \frac{1}{C} \nabla \mathbf{V} \right)^3, \quad \left| \frac{1}{C} \nabla \mathbf{V} \right| \ll 1, \quad (4) \]

where \( \mu \) is the coefficient of shear viscosity and \( \beta, C \) are the material constants.
3 Formulation of the Problem and Flow Eq.

We consider a container filled with an incompressible non-Newtonian Eyring-Powell fluid. A wide belt passes through this container and is moving vertically upward with a constant speed $V_0$. This moving belt picks up a thin film fluid of uniform thickness $\delta$. Due to gravity, the fluid film tends to drain down the belt. The flow is steady, laminar and uniform and the pressure is assumed to be atmospheric pressure. We choose the $x$-axis normal to the belt and the $y$-axis is taken along the belt which is in upward direction.

The appropriate boundary conditions for this problem are

$$v = V_0 \quad \text{at} \quad x = 0 \quad \text{(no slip condition)},$$

$$S_{xy} = 0 \quad \text{at} \quad x = \delta \quad \text{(free surface)},$$

where $S_{xy}$ is the shear stress component of the Eyring-Powell fluid.

We seek the velocity field and the extra stress tensor of the form

$$\mathbf{V} = [0, v(x), 0], \quad \mathbf{S} = \mathbf{S}(x). \quad (7)$$

The continuity equation (1) is satisfied identically, while the non-zero component of the momentum equation, in the absence of pressure gradient, takes the form

$$\frac{dS_{xy}}{dx} - \rho g = 0, \quad (8)$$

which on integration yields

$$S_{xy} = \rho gx + C, \quad (9)$$

where $C$ is a constant of integration. On invoking the boundary condition (6), we obtain

$$S_{xy} = \rho g (x - \delta). \quad (10)$$

Substituting (7) into (3) and (4) and on simplifying, the component $S_{xy}$ is given by

$$S_{xy} = \left( \mu + \frac{1}{\beta C} \right) \frac{dv}{dx} - \frac{1}{6\beta C^3} \left( \frac{dv}{dx} \right)^3. \quad (11)$$

Combining (11) and (10), the final form of the governing system is given by

$$\left( \mu + \frac{1}{\beta C} \right) \frac{dv}{dx} - \frac{1}{6\beta C^3} \left( \frac{dv}{dx} \right)^3 = \rho g (x - \delta), \quad (12)$$

$$v = 1 \quad \text{at} \quad x = 0 \quad \text{(no slip condition)}. \quad (13)$$
We introduce the non-dimensional variables as follows
\[ \eta = \frac{x}{\delta}, \quad f = \frac{v}{V_0}, \quad m = \frac{\rho g \delta^2}{\mu V_0}, \quad M = \frac{1}{\mu \beta C}, \quad K = \frac{MV_0^2}{6 \delta^2 C^2}. \] (14)

Thus, the dimensionless form of the system (12) and (13) is
\[ (1 + M) \frac{df}{d\eta} - K \left( \frac{df}{d\eta} \right)^3 - m(\eta - 1) = 0, \] (15)
\[ f(0) = 1, \] (16)
where \( M \) and \( K \) are the dimensionless parameters of the Eyring-Powell fluid. It is to be noted that (15) along with the boundary condition (16) is a highly nonlinear first order differential equation. It is a well-posed problem but difficult to find its closed form solution. Thus, we are interested to obtain approximate analytical solution of this problem using the VIM and the ADM.

4 Solution using the VIM

To apply the VIM (variational iteration method), we write equation (15) in the form
\[ (1 + M) L(f) + N(f) = g(\eta), \] (17)
where \( L(f) = \frac{df}{d\eta} \) is the linear term, \( N(f) = -K \left( \frac{df}{d\eta} \right)^3 \) is the nonlinear term and \( m(\eta - 1) \) is the source term. We can construct a correction functional according to the variational iteration method as
\[ f_{n+1} = f_n + \int_0^\eta \lambda(x) \left( (1 + M) L f_n(x) + N \tilde{f}_n(x) - g(x) \right) dx, \] (18)
where \( \lambda(x) \) is a general Lagrange multiplier which can be identified optimally via the variational iteration theory where the subscript \( n \) denotes the \( n \)th approximation and \( \tilde{f}_n(x) \) is considered as a restricted variation, that is, \( \delta \tilde{f}_n(x) = 0 \). After some calculations, we obtain the following stationary condition
\[ \lambda'(\eta) = 0, \quad 1 + (1 + M) \lambda(\eta) = 0. \] (19)

The Lagrange multiplier can therefore be identified as \( \lambda = -\frac{1}{1 + M} \). As a result, we obtain the following iteration formula
\[ f_{n+1}(\eta) = f_n(\eta) - \frac{1}{(1 + M)} \int_0^\eta \left( (1 + M) \frac{df_n}{dx} - K \left( \frac{df_n}{dx} \right)^3 - m(x - 1) \right) dx, \quad n \geq 0. \] (20)
Starting with the initial approximation \( f_0(\eta) = 1 \), the next iterates \( f_1(\eta), f_2(\eta), f_3(\eta), \ldots \), are given below respectively.

\[
f_1(\eta) = 1 + \frac{m}{(1 + M)} \left( \frac{\eta^2}{2} - \eta \right), \tag{21}
\]

\[
f_2(\eta) = 1 + \frac{m}{(1 + M)} \left( \frac{\eta^2}{2} - \eta \right) + \frac{Km^3}{4(1 + M)^4} ((\eta - 1)^4 - 1), \tag{22}
\]

\[
f_3(\eta) = 1 + \frac{m}{(1 + M)} \left( \frac{\eta^2}{2} - \eta \right) + \frac{Km^3}{4(1 + M)^4} ((\eta - 1)^4 - 1) + \frac{K^2m^5}{2(1 + M)^7} ((\eta - 1)^6 - 1) + \frac{3K^3m^7}{8(1 + M)^{10}} ((\eta - 1)^8 - 1)
+ \frac{K^4m^9}{10(1 + M)^{13}} ((\eta - 1)^{10} - 1). \tag{23}
\]

Hence, the solution series in general gives

\[
f(\eta) = 1 + \frac{m}{(1 + M)} \left( \frac{\eta^2}{2} - \eta \right) + \frac{Km^3}{4(1 + M)^4} ((\eta - 1)^4 - 1) + \frac{K^2m^5}{2(1 + M)^7} ((\eta - 1)^6 - 1) + \frac{3K^3m^7}{8(1 + M)^{10}} ((\eta - 1)^8 - 1)
+ \frac{K^4m^9}{10(1 + M)^{13}} ((\eta - 1)^{10} - 1) + \ldots. \tag{24}
\]

Here it should be noted that for \( M = 0 \) and \( K = 0 \) we get the solution for the Newtonian fluid.

5 Solution using the ADM

To apply the ADM (Adomian decomposition methods) to our nonlinear equation (15), we first rewrite it in the following operator form [7-9]

\[
(1 + M) \ L f(\eta) = KNf(\eta) + m(\eta - 1), \tag{25}
\]

where \( L = \frac{d}{d\eta} \) is the linear invertible operator and \( Nf(\eta) = (\frac{df}{d\eta})^3 \) is the nonlinear term.

Applying the inverse operator \( L^{-1} \) on both sides of (25), we get

\[
(1 + M) \ L^{-1} f(\eta) = KL^{-1}Nf(\eta) + mL^{-1}(\eta - 1), \tag{26}
\]

so that

\[
(1 + M) \ (f(\eta) - f(0)) = m \left( \frac{\eta^2}{2} - \eta \right) + KL^{-1}Nf(\eta). \tag{27}
\]
And, by using the boundary condition (16), we obtain

$$f(\eta) = 1 + \frac{m}{(1 + M)} \left( \frac{\eta^2}{2} - \eta \right) + \frac{K}{(1 + M)} L^{-1} N f(\eta).$$  \hfill (28)

We decompose \( f(\eta) \) and the nonlinear term \( N f(\eta) = \left( \frac{df}{d\eta} \right)^3 \) respectively, as

$$f(\eta) = \sum_{n=0}^{\infty} f_n(\eta), \quad N f(\eta) = \sum_{n=0}^{\infty} A_n.$$  \hfill (29)

The first few terms of the Adomian polynomials \( A_n \) are given by

$$A_0 = \left( \frac{df_0}{d\eta} \right)^3,$$  \hfill (30)

$$A_1 = 3 \left( \frac{df_0}{d\eta} \right)^2 \frac{df_1}{d\eta},$$  \hfill (31)

$$A_2 = 3 \frac{df_0}{d\eta} \left( \frac{df_2}{d\eta} \right)^2 + 3 \left( \frac{df_0}{d\eta} \right)^2 \frac{df_2}{d\eta},$$  \hfill (32)

$$A_3 = \left( \frac{df_1}{d\eta} \right)^3 + 6 \left( \frac{df_0}{d\eta} \right) \left( \frac{df_1}{d\eta} \right) \left( \frac{df_2}{d\eta} \right) + 3 \left( \frac{df_0}{d\eta} \right)^2 \frac{df_3}{d\eta},$$  \hfill (33)

etc.

We identify the zeroth component \( f_0(\eta) \) by

$$f_0(\eta) = 1 + \frac{m}{(1 + M)} \left( \frac{\eta^2}{2} - \eta \right)$$  \hfill (34)

and the remaining components \( f_{n+1}(\eta) \) by the recurrence relation

$$f_{n+1}(\eta) = \frac{K}{(1 + M)} L^{-1} (A_n), \quad n \geq 0.$$  \hfill (35)

Adopting the same algorithms as those used in [7-9], we obtain the following components:

$$f_1(\eta) = \frac{K m^3}{4 (1 + M)^4} ((\eta - 1)^4 - 1),$$  \hfill (36)

$$f_2(\eta) = \frac{K^2 m^5}{2 (1 + M)^6} ((\eta - 1)^6 - 1),$$  \hfill (37)

$$f_3(\eta) = \frac{3K^3 m^7}{2 (1 + M)^8} ((\eta - 1)^8 - 1),$$  \hfill (38)

$$f_4(\eta) = \frac{11K^4 m^9}{2 (1 + M)^{10}} ((\eta - 1)^{10} - 1),$$  \hfill (39)
and so on. In this manner, the rest of the terms in the decomposition series can be calculated.

Summing up, we write the solution in the decomposition series form

\[ f(\eta) = f_0 + f_1 + f_2 + f_3 + f_4 + \cdots. \]

This, after inserting the values of \( f_0, f_1, f_2, f_3 \) and \( f_4 \) from (34) and (36)–(39), becomes

\[
\begin{align*}
\ f(\eta) = 1 + & \frac{m}{(1+M)} \left( \frac{\eta^2}{2} - \eta \right) + \frac{Km^3}{4(1+M)^4} (\eta - 1)^4 - 1 \\
& + \frac{K^2m^5}{2(1+M)^7} ((\eta - 1)^6 - 1) + \frac{3K^3m^7}{2(1+M)^{10}} ((\eta - 1)^8 - 1) \\
& + \frac{11K^4m^9}{2(1+M)^{13}} ((\eta - 1)^{10} - 1) + \cdots.
\end{align*}
\]

Notice for \( M = 0 \) and \( K = 0 \), we get the solution for the Newtonian fluid.

6 Discussion and conclusion

Figs. 1–3 show a graphical comparison between three iteration solution of the VIM and the four terms of the ADM solution for different values of \( M, K, m \) and no visible difference was observed. This comparison indicates that the numerical results of these methods are in excellent accordance. It is also evident that the errors between these solutions can be reduced further and high accuracy can be achieved by evaluating more components of \( f(\eta) \). It is remarkable to observe that third iteration of the VIM is almost equivalent to the four terms of the ADM solution.

The quantitative effect of the material parameter \( M \) for \( K = 1.0, m = 0.5 \) on the velocity profile (24) is observed physically in Fig. 4. It is shown that as we decrease the parameter \( M \), the velocity profile tends to the Newtonian case when \( M = 0 \). Consequently, the present successful implementation of these methods verifies the wide range capabilities of these methods for handling other nonlinear problems arising in non-Newtonian fluid dynamics.
Figure 1: Comparison of the VIM solution and the ADM solution for $M = 0.3, K = 0.8, m = 0.6$.

Figure 2: Comparison of the VIM solution and ADM solution for $M = 1.0, K = 0.8, m = 1.0$.

Figure 3: Comparison of the VIM solution and the ADM solution for $M = 0.1, K = 0.5, m = 0.1$.

Figure 4: Effect of $M$ on velocity profile when $K = 1.0, m = 0.5$.

References


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