Motion of Curves Specified by Acceleration Field in $R^n$

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Abstract

Kinematic of moving generalized space curves in $R^n$ is formulated in terms of intrinsic geometries. The model for the dynamics is specified by acceleration fields. The acceleration is assumed to be local in the sense that it is a functional of the heir curvatures and their derivatives. By solving the nonlinear partial differential equations which governing the motion of the curves, we get on there curvatures and integrating the Seret-Frenet equations we display family of curves in plane and space.

Keywords: Motion of Curves; Integrability Condition; Seret-Frenet Equations

1 Introduction

A lot of physical processes can be modeled in terms of the motion of curves, including the dynamics of vortex filaments in fluid dynamics [2], the growth of dendritic crystals in a plane [3], and more generally, the planar motion of interfaces [4]. The subject of how space curves evolve in time is of great interest and has been investigated by many authors. Pioneering work is attributed to Hasimoto who showed in [2] the nonlinear Schrödinger equation describing the motion of an isolated non-stretching thin vortex filament. Lamb [5] used the Hasimoto transformation to connect other motions of curves to the mKdV and sine–Gordon equations. Nakayama, et al [6] obtained the sine–Gordon equation by considering a nonlocal motion. Also Nakayama
and Wadati [7] presented a general formulation of evolving curves in two dimensions and its connection to mKdV hierarchy. Nassar, et al [8, 9, 10, 22, 12, 13] have studied evolution of manifolds and obtained many interesting results. R. Mukherjee and R. Balakrishnan [1] applied their method to the sine-Gordon equation and obtained links to five new classes of space curves, in addition to the two which were found by Lamb [5]. For each class, they displayed the rich variety of moving curves associated with the one—soliton, the breather, the two—soliton and the soliton—antisoliton solutions.

The conventional geometrical model [6] is specified by velocity fields,

$$\frac{\partial \mathbf{r}}{\partial t} = W t + U n + V b$$

(1)

Here, t, n and b are the unit tangential, normal and binormal vectors along the curve, and W, V and V are the tangential, normal and binormal velocities.

In [14] the motion of curves were specified by the acceleration field

$$\frac{\partial^2 \mathbf{r}}{\partial^2 t} = E t + F n + G b$$

(2)

where the velocities \{E, F, G\} depend on the local values of \{κ, τ\}.

In this paper, we introduce a new class of geometrical models concerning the time evolution of the generalized space curve \( \mathbf{r} = \mathbf{r}(s) \). This class of models is specified by acceleration fields,

$$\frac{\partial^2 \mathbf{r}}{\partial^2 t} = C F,$$

(3)

where

$$C = (c_1 \ c_2 \ \ldots \ c_n), \quad F = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$$

(4)

and C(s, t) is the acceleration vector function and c_i(s, t) are the component of the acceleration vector along the Frenet vectors \( e_i \) which are the function of the intrinsic quantities \( κ_i(s) \) of the curve \( \mathbf{r} = \mathbf{r}(s) \) through the time evolution 3. By solving the partial differential equations which governing the motion of the curve, we get on there curvatures and then integrating Seret-Frenet equations we display some plane and space curves.

2 Partial differential equations governing the motion of curves in \( R^n \)

A generalized space curve \( \mathbf{r} = \mathbf{r}(s) \) in an N-dimensional Euclidean space \( R^n \) can be regarded as a Riemannian submanifold of dimension one in \( R^n \), with metric


\[ g = 1 = (r_s, r_s) \] induced by the Euclidean metric \((\cdot, \cdot)\), in this case, the generalized curve is called a unit speed curve or a curve parameterized by the arc length. A generalized curve \(C\) in \(R^n\) is defined as an image of a diffeomorphism \(r : J \subset R \rightarrow R^n\) which is given in the following parametrization

\[ r(s) = (x_i(s)), \quad i = 1, 2, ..., n \tag{5} \]

and is called a regular representation of \(C\) in \(R^n\) if \(r_s \neq 0\) and \(s\) is the arc length parametrization. Along the curve there is existed an \(n\)-vectors \((e_1, e_2, ..., e_n)\) called Frenet n-frame and satisfy the following equation

\[
\frac{\partial F}{\partial s} = AF, \quad F(s) = (e_1(s), e_2(s), ..., e_n(s))^\top, \quad F(0) = I, \quad s \in J \tag{6}
\]

where \(A\) is a skew symmetric matrix in the form

\[
A = \begin{pmatrix}
0 & \kappa_1(s) & \cdots & 0 & 0 \\
-\kappa_1(s) & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \kappa_{n-1}(s) \\
0 & 0 & \cdots & -\kappa_{n-1}(s) & 0
\end{pmatrix} \tag{7}
\]

which \(A\) is called Frenet matrix or attitude matrix along the curve. In the sense of lie algebra the \(n\)-frame is defined as a linear mapping

\[ F : J \rightarrow \text{So}(n) \tag{8} \]

where \(\text{So}(n)\) is an orthogonal group of parameter \(s\), i.e., \(\text{So}(n)\) is 1-parametric group and is denoted by \(\text{So}(1, n)\) (one parametric lie group). The matrix \(A\) is the infinitesimal generator of the lie group \(\text{So}(1, n)\) defined on the generalized space curve \(r = r(s)\).

The elements \(\kappa_1(s), \kappa_2(s), ..., \kappa_{n-1}(s)\) are functions of the arc length parameter and are called the heir curvature functions or Euclidean curvatures of the generalized curve. The curvature \(\kappa_{n-1}(s)\) gives the speed of rotation of the osculating \((n-1)\)-plane (hyper plane) around the \((n-2)\)-plane. Thus the higher curvature are well defined through the formula 10.

**Remark 2.1.** The vector \(e_1\) is a tangent vector and \(e_1 \in T_p r(s), e_i \in T_p^\perp r(s)\).

**Remark 2.2.** The vector spaces defined along the curve \(r = r(s)\) which are generated by the tangent \(e_1\) and the set \((e_{i_1}, e_{i_2}, ..., e_{i_j})\) are called osculating spaces with dimension \(j+1\).

The general temporal evolution in which the Frenet frame \((e_1, e_2, ..., e_n)\) remains orthonormal adopts the form

\[
\frac{\partial F}{\partial t} = EF, \tag{9}
\]
where $E$ is a skew symmetric matrix in the form

$$
E = \begin{pmatrix}
0 & E_{12} & E_{13} & \ldots & E_{1n} \\
E_{21} & 0 & E_{23} & \ldots & E_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
-E_{n1} & -E_{n2} & \ldots & -E_{n(n-1)} & 0
\end{pmatrix}
$$

wish is called the evolution matrix along the curve. The elements $E_{ij}$ are functions of the arc length parameter $s$ and the time $t$. The differential equations (6) and (9) represent soliton geometry of dimension $(1 + 1)$ where $A = A(s, t)$ and $B = B(s, t)$.

In order that $F = (e_1, e_2, \ldots, e_n)\top$ be well-defined, it must satisfy a compatibility condition,

$$
\frac{\partial^2 F}{\partial s \partial t} = \frac{\partial^2 F}{\partial t \partial s},
$$

(11)

using (6) and (9) we have

$$
\frac{\partial A}{\partial t} - \frac{\partial E}{\partial s} + [A, E] = 0_{n \times m}
$$

(12)

where $[A, E] = AE - EA$ is called Lie bracket of $A$ and $E$. Equating both sides of the matrix form (12) we have $\frac{n(n-1)}{2}$ integrability conditions in the intrinsic invariants $\kappa_1(s), \kappa_2(s), \ldots, \kappa_{n-1}(s)$.

On the other hand, differentiation of (9) by time yields

$$
\frac{\partial^2 F}{\partial t^2} = (\frac{\partial E}{\partial t} + E^2)F.
$$

(13)

Differentiation of (3) by $s$ gives

$$
\frac{\partial}{\partial s} \frac{\partial^2 r}{\partial t^2} = (\frac{\partial C}{\partial s} + CA)F,
$$

(14)

On the left hand side we use

$$
\frac{\partial}{\partial s} \frac{\partial^2 r}{\partial t^2} = \frac{\partial^2 r}{\partial t^2} \frac{\partial}{\partial s} = \frac{\partial^2 r}{\partial t^2} e_1.
$$

(15)

and from (3) we have

$$
\frac{\partial^2}{\partial t^2} e_1 = (\frac{\partial C}{\partial s} + CA)F,
$$

(16)

from (9) we have

$$
\frac{\partial e_1}{\partial t} = \sum_{i=1}^{n} E_{1i} e_1,
$$

(17)

$$
\frac{\partial^2 e_1}{\partial t^2} = \sum_{i=1}^{n} \frac{\partial E_{1i}}{\partial t} e_i + \sum_{i} E_{1i} \frac{\partial e_i}{\partial t}.
$$

(18)
Also from (9) we obtain
\[ \frac{\partial e_i}{\partial t} = \sum_{j \neq i} E_{ij} e_i, \quad \frac{\partial e_n}{\partial t} = \sum_{j \neq n} E_{nj} e_j, \] (19)
where \( E_{ji} = -E_{ij}, E_{ii} = 0, i, j = 1, 2, \ldots, n \).

Thus we obtain
\[ \frac{\partial^2 e_1}{\partial t^2} = \sum_{i \neq 1} \frac{\partial E_{1i}}{\partial t} e_i + \sum_{i \neq 1} E_{1i} \sum_{i \neq j} E_{ij} e_j, \] (20)
from (16) and (20) we have
\[ \sum_{i \neq 1} \frac{\partial E_{1i}}{\partial t} e_i + \sum_{i \neq 1} E_{1i} \sum_{i \neq j} E_{ij} e_j = (\frac{\partial C}{\partial s} + CA)F. \] (21)

Comparing the coefficients of the linearly independent vectors \( e_i \) in (21) leads to \( n \) scaler differential equations. These, along with (12), define completely the motion of the curve \( r = r(s, t) \) in \( R^n \).

**Remark 2.3.** The formula (3) can be obtained using the differentiation of the matrix \( F \) and using (3) where \( \frac{\partial^2 e_1}{\partial t^2} \) is the first element of the matrix \( \frac{\partial^2 F}{\partial t^2} = (\frac{\partial^2 e_1}{\partial t^2})^T \).

To summarize, the motion of a curve whose accelerations are specified by (3) is defined by \( \frac{n(n+1)}{2} \) equations : \( \frac{n(n-1)}{2} \) in (12) and \( n \) equations resulting from (21). This set of equations is the main result of this paper. For a given \( (c_1, c_2, \ldots, c_n) \) component function of the acceleration \( C \), we can obtain the elements \( E_{ij} \) of the evolution matrix \( E \). Thus the motion of the curve is determined from these equations by solving the system of \( (n - 1) \) nonlinear partial differential equations in the invariants \( \kappa_1(s), \kappa_2(s), \ldots, \kappa_{n-1}(s) \). Thus if we obtain \( \kappa_i(s, t) \) for a given \( C \) we can use the fundamental theorem of curves to reconstruct the curve \( r = r(s, t) \) up to orientation and translation.

### 3 Reduction the problem to lower dimensional cases

For application we try to solve the system (12) and (21). In other words if we given \( c_i \) and from (21) we have \( E_{ij} \) and then from (12) we have the solution of the nonlinear evolution equations for \( n = 2, n = 3 \) as in the following sections.
3.1 Evolution of plane curves

In order to reduce the time-evolution equations (12) and (21) to those that describe time-volution of curves in a two-dimensional Euclidean space, we set

$$\tau = E_{13} = c_3 = E_{23} = 0.$$  \hspace{1cm} (22)

Then (12) and (21) yields

$$\frac{\partial \kappa}{\partial t} = \frac{\partial E_{12}}{\partial s},$$  \hspace{1cm} (23a)

$$\frac{\partial E_{12}}{\partial t} = \frac{\partial c_2}{\partial s} + \kappa c_1,$$  \hspace{1cm} (23b)

$$\frac{\partial c_1}{\partial s} = \kappa c_2 - E_{12}^2.$$  \hspace{1cm} (23c)

From (23c) into (23a)

$$\frac{\partial \kappa}{\partial t} = \frac{\partial}{\partial s} \sqrt{\kappa c_2 - \frac{\partial c_1}{\partial s}}.$$  \hspace{1cm} (24)

Thus for a given $c_1, c_2$ we obtain the curvature by solving the nonlinear PDE and then integrating Seret-Frenet we get the evolving curve graphically.

The first equation in (23) motivates us to introduce a function, $\phi$, by

$$\kappa = \phi_s, \quad E_{12} = \phi_t,$$  \hspace{1cm} (25)

$$-\phi_t^2 = \frac{\partial c_1}{\partial s} - \phi_s c_2,$$  \hspace{1cm} (26)

$$\phi_u = \frac{\partial c_2}{\partial s} + \phi_s c_1.$$

Eliminating $\phi_s$

$$c_2 \phi_u - c_1 \phi_t^2 = c_1 \frac{\partial c_1}{\partial s} + c_2 \frac{\partial c_2}{\partial s}.$$  \hspace{1cm} (27)

For a given $(c_1, c_2)$, these two equations, determine the motion of curves in two-dimensional Euclidean space.

3.1.1 Model (3.1)

In this subsection, we consider some applications of the two-dimensional time-evolution equations (12) and (21). Choosing $c_1 = 0, c_2 = \phi$ we get heat equation

$$\phi_u = \phi_s.$$  \hspace{1cm} (28)
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the general solution is

$$\phi(u, t) = \exp[-\frac{A s}{\lambda^2}](D \cos t/\lambda + E \sin t/\lambda),$$

(29)

where $A, \lambda, E$ and $D$ are arbitrary real constant.

The curvature is given by

$$\kappa = \phi_s = -\frac{A}{\lambda^2} \exp[-\frac{A s}{\lambda^2}](D \cos t/\lambda + E \sin t/\lambda).$$

(30)

In [22] they have studied the motion of curve under heat flow.

3.1.2 Model (3.2)

Choosing $c_1 = 0, c_2 = \phi_s$ we get wave equation

$$\phi_{tt} - \phi_{ss} = 0,$$

(31)

the general solution is

$$\phi(s, t) = \alpha(s + t) + \beta(-s + t)$$

(32)

where $\alpha, \beta$ are arbitrary real functions. The curvature is given by

$$\kappa = \phi_s = \dot{\alpha}(s + t) - \dot{\beta}(-s + t).$$

(33)

In figure 2 we have used $\alpha = \text{sech}(s + t), \beta = \sin(-s + t)$.

The figures 1, 2, 4 and 6 represent snapshot of the evolving space curve obtained by solving the Frenet–Serret Eqs. (6) for a specified curvature and torsion using Mathematica [20]. Any moving space curve can be studied from two perspectives, namely, the shape of the curve and the evolution of the curve. At every fixed $t$, we clearly have a representation of the corresponding static space curve at that instant. In all the figures we have used the total curve length of 20 ($-10 \leq s \leq 10$). Figure 3 shows the behavior of the curvature functions for model 1, 2.

3.1.3 Model (3.3)

If we put $c_1 = c_2 = \phi$

$$\phi_{tt} - \phi_t^2 = 2\phi_u,$$

(34)

the general solution is given by the conjugate complex function

$$\phi_{1,2} = \frac{1}{4} \left(-2t^2 \pm 2i\sqrt{2}t, a_1(s) + a_2(s)^2\right)$$

(35)

where $a_1(s)$ and $a_2(s)^2$ are arbitrary functions in $s$ and $s^2$ respectively.
Figure 1: Plane curve corresponding to model (3.1) \( \kappa = -\exp(-s)(\cos t + \sin t), \tau = 0 \).
Figure 2: Plane curve corresponding to model (3.2) \( \kappa = -\text{sech}(-s + u)\tanh(-s + u) + \cos(s + u), \tau = 0. \)
3.2 Evolution of space curve in $R^3$

Putting $n = 3$ in the system of equations (12) and (21) we have the equation of evolution to the space curves as in the following

\[
\begin{align*}
\frac{\partial \kappa}{\partial t} - \frac{\partial E_{12}}{\partial s} + \tau E_{13} &= 0, \\
\frac{\partial \tau}{\partial t} - \frac{\partial E_{23}}{\partial s} - \kappa E_{13} &= 0, \\
\frac{\partial E_{13}}{\partial s} + \tau E_{12} - \kappa E_{23} &= 0,
\end{align*}
\]

(36)

\[
\begin{align*}
\frac{\partial E_{12}}{\partial t} - E_{13} E_{23} &= \frac{\partial c_2}{\partial s} + \kappa c_1 - \tau c_3, \\
\frac{\partial E_{13}}{\partial t} + E_{12} E_{23} &= \frac{\partial c_3}{\partial s} + \tau c_2, \\
E_{12}^2 + E_{13}^2 &= -\frac{\partial c_1}{\partial s} + \kappa c_2.
\end{align*}
\]

(37)

For given $(c_1, c_2, c_3)$ the equations (36) and (37) are determined the motion of the space curve in $R^3$.

Equations (36) and (37) can be written as

\[
\begin{align*}
\frac{\partial \kappa}{\partial t} &= \frac{\partial E_{12}}{\partial s} - \tau E_{13}, \\
\frac{\partial \tau}{\partial t} &= \frac{\partial}{\partial s} \left( \frac{\tau}{\kappa} E_{12} + \frac{1}{\kappa} \frac{\partial E_{13}}{\partial s} \right) + \kappa E_{13},
\end{align*}
\]

(38)

Figure 3: The behavior of the curvature functions for model 3.1 and 3.2.
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\[
E_{12}(\frac{\partial c_2}{\partial s} + \kappa c_1 - \tau c_3) + E_{13}(\frac{\partial c_3}{\partial s} + \tau c_2) = \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\partial c_1}{\partial s} \right),
\]
\[
E_{12}^2 + E_{13}^2 = \kappa c_2 - \frac{\partial c_1}{\partial s}.
\]

(39)

3.2.1 Model 3.4

If we put $c_1 = 0, c_2 = \kappa, c_3 = 0$ in (39) we have

\[
E_{12}\kappa_s + E_{13}\kappa\tau = \kappa\kappa_t
\]
\[
E_{12}^2 + E_{13}^2 = \kappa^2
\]

(40)

If we chose the curve s.t. $\kappa_t = \kappa_s$ then (40) has a solution in the form

\[
E_{12} = \kappa,
\]
\[
E_{13} = 0
\]

(41)

\[
E_{12} = \kappa - \frac{2\kappa^3\tau^2}{\kappa^2\tau^2 + \kappa_s^2}
\]
\[
E_{13} = \frac{2\kappa^2\tau\kappa_s}{\kappa^2\tau^2 + \kappa_s^2}
\]

(42)

By inserting the above values of (41) in (38) we get a coupled partial differential equations in $\kappa, \tau$ as follows

\[
\kappa_t = \kappa_s,
\]
\[
\tau_t = \tau_s
\]

(43)

the general solution is

\[
\kappa = f(s + t),
\]
\[
\tau = g(s + t)
\]

(44)

where $f$ and $g$ are arbitrary real functions. The figure 4 and 6 represent the evolving curve corresponding to $\kappa = f(s + t), \tau = g(s + t)$

From (42) in (38) we have

\[
\frac{\partial \kappa}{\partial t} = \frac{\partial}{\partial s} \left( \kappa - \frac{2\kappa^3\tau^2}{\kappa^2\tau^2 + \kappa_s^2} \right) - \tau \left( \frac{2\kappa^2\tau\kappa_s}{\kappa^2\tau^2 + \kappa_s^2} \right)
\]
\[
\frac{\partial \tau}{\partial t} = \frac{\partial}{\partial s} \left( \frac{\tau}{\kappa} \left( \kappa - \frac{2\kappa^3\tau^2}{\kappa^2\tau^2 + \kappa_s^2} \right) + \frac{1}{\kappa} \frac{\partial}{\partial s} \left( \frac{2\kappa^2\tau\kappa_s}{\kappa^2\tau^2 + \kappa_s^2} \right) + \kappa \left( \frac{2\kappa^2\tau\kappa_s}{\kappa^2\tau^2 + \kappa_s^2} \right) \right)
\]

(45)
Figure 4: The evolving space curve crossposting to model (3.4) $\kappa = \tanh(s + t)$, $\tau = \sin(s + t)$.

Figure 5: The behavior of the curvature and torsion for model (3.4).
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Figure 6: The evolving space curve crossposting to model (3.4) $\kappa = \text{sech}(s+t), \tau = \sin(s+t)$.

(a) $t=0$

(b) $t=1$

(c) $t=2$

(d) $t=3$

Figure 7: The behavior of the curvature and torsion for model (3.4).

(a) The graph of the function $\tau = \sin(s+t)$

(b) The graph of the function $\kappa = \text{sech}(s+t) \tanh(s+t)$
We used Mathematica package software for solving the system (45) which apply the \( \text{tanh} \)– and \( \text{sech} \)– methods [18].

\[
\begin{align*}
\kappa_{1,2} &= b_4 (-1 + \tanh[sb_1 + tb_2 + b_3]), \\
\tau_{1,2} &= \pm ib_1 (1 + \tanh[sb_1 + tb_2 + b_3]), \\
\kappa_{3,4} &= b_4 (1 + \tanh[sb_1 + tb_2 + b_3]), \\
\tau_{3,4} &= \pm ib_1 (-1 + \tanh[sb_1 + tb_2 + b_3]).
\end{align*}
\]

(46)

where \( b_1, b_2, b_3, b_4 \) are arbitrary real constants.

Here we can not reconstruct the curve from its curvatures because the integration of Seret-Frenet is very difficult so, we try to describe the curve from its curvatures through the figure 8 .

![Figure 8](image)

(a) The graph of the function \( \kappa_{1,2} \)  
(b) The graph of the function \( \kappa_{3,4} \)

Figure 8: The curvature function of the model 3.4

3.2.2 Model 3.5

If we put \( c_1 = 0, c_2 = \text{constant} = a, c_3 = 0 \) in (39) we have

\[
\begin{align*}
E_{13} \tau a &= \kappa \kappa_t \\
E_{12}^2 + E_{13}^2 &= \kappa a
\end{align*}
\]

(47)

by solving in \( E_{12}, E_{13} \) we have

\[
\begin{align*}
E_{12} &= -\sqrt{a \kappa - \frac{\kappa^2 \kappa_t^2}{a^2 \tau^2}}, \\
E_{13} &= \frac{\kappa \kappa_t}{a \tau},
\end{align*}
\]

(48)
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$$E_{12} = \sqrt{\kappa^2 - \frac{\kappa^2_k \kappa_{13}^2}{a^2 \tau^2}},$$
$$E_{13} = \frac{\kappa k_{13}}{a \tau}.$$  \hspace{1cm} (49)

By inserting the above values of $E_{12}, E_{13}$ in (38) we get a coupled nonlinear partial differential equations in $\kappa, \tau$. It's known that if we have $\kappa, \tau$ we can reconstruct the curve up to its position in space.

Conclusions

The problem of motion of curves specified by acceleration field in $R^n$ is investigated. In lower dimensional cases such as plane and space curve the evolution equations for curvature and torsion are derived. Curves corresponding to solutions of evolution equations are numerically constructed.

References


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