Inferences of Type II Extreme Value Distribution Based on Record Values

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Abstract

It is known that if the maximum or minimum of $n$ independent and identically distributed random variables when standardized converges as $n$ tends to infinity to a non-degenerate distribution, then it converges to one of the three extreme values distributions. Type II extreme value distribution is one of them. In this paper we will consider the record values of Type II extreme value distribution. Some distributional properties of the record values of this distribution will be given. Based on these properties some recurrence relations of the moments and a characterization of the Type II extreme value distribution will be presented.

1. Introduction

Suppose that $X_1$, $X_2$, ... is a sequence of independent and identically distributed random variables with cumulative distribution function $F(x)$. Let $Y_n = \max (\min)\{X_1, X_2, ..., X_n\}$ for $n \geq 1$. We say $X_j$ is an upper(lower) record value of $\{X_n, n \geq 1\}$, if $Y_j > (<) Y_{j-1}$, $j > 1$. By definition $X_1$ is an upper as well as a lower record value. The indices at which the upper record values occur are given by the record times $\{U(n)\}$, $n > 0$, where $U(n) = \min\{j|j > U(n-1), X_j > X_{U(n-1)}, n > 1\}$ and $U(1) = 1$. We will denote $L(n)$ as the indices where the lower record values occur. By our assumption $U(1) = L(1) = 1$. The distribution of $U(n)$ or $L(n)$ does not depend on $F$. 

A random variable $X$ with location parameter $\mu$ and scale parameter $\sigma$ is said to have type II extreme value distribution if the cumulative distribution function (cdf) of $X$ is given by

$$F(X) = 0 \quad \text{for } x < \mu$$

$$= e^{-\frac{(x-\mu)}{\sigma}} - \delta \quad , x \geq \mu, -\infty < \mu < \infty, \sigma > 0, \delta > 0,$$

and the corresponding probability density function (pdf) is

$$f(x) = \frac{\delta}{\sigma} \left(\frac{x-\mu}{\sigma}\right)^{-1-\delta} e^{-\frac{(x-\mu)}{\sigma}}$$

(1.1)

(1.2)


In this paper we will consider records of type II distribution.

2. Main Results

We will consider in this paper the lower record values. Many properties of the lower record value sequence can be expressed in terms of $H(x)$, where $H(x) = -\ln F(x)$ and $h(x) = -\frac{d}{dx} H(x)$. Here 'ln' is used for the natural logarithm. If we define $f(n)(x)$ as the pdf of $X_{L,n}$ for $n \geq 1$, then we have (see Ahsanullah(2004))

$$f(n)(x) = \frac{(H(x))^{n-1}}{\Gamma(n)} f(x)$$

(2.1)

For Type II extreme value distribution

$$f(n)(x) = \frac{\sigma}{\Gamma(n)} e^{-\frac{(x-\mu)}{\sigma}}$$

(2.2)
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where $-\infty < \mu < \infty, \sigma > 0, \delta > 0$.

It can be shown that

$$X_{L(n)} \overset{d}{=} \frac{(W_1 + W_2 + \ldots + W_n)^{-1}}{\delta}$$

(2.3)

where $W_1, W_2, \ldots, W_n$ are independent and identically distributed with cdf as $F(X) = 1 - e^{-\frac{X-\mu}{\sigma}}$ and $\overset{d}{=} \text{ denotes the equality in distribution.}$

Using (2.3), we obtain

$$E\left(\frac{X_{L(n)} - \mu}{\sigma}\right) = \frac{\Gamma(n - \frac{1}{\delta})}{\Gamma(n)}, \quad n > \frac{1}{\delta};$$

$$E\left(\frac{X_{L(n)} - \mu}{\sigma}\right)^2 = \frac{\Gamma(n - \frac{2}{\delta})}{\Gamma(n)}, \quad n > \frac{2}{\delta};$$

(2.3)

and

$$\text{Var} (X_{L(n)}) = \sigma^2 \left[ \frac{\Gamma(n - \frac{2}{\delta})}{\Gamma(n)} - \left(\frac{\Gamma(n - \frac{1}{\delta})}{\Gamma(n)}\right)^2 \right].$$

For $n > \frac{1}{\delta}, E\left(\frac{X - \mu}{\sigma}\right)^k = \frac{\phi(n - \frac{k}{\delta})}{\Gamma(n)}$.

Using (2.3), we have for $m < n$.

$$E\left(\frac{X_{L(m)} - \mu}{\sigma}\right) \left(\frac{Y_{L(n)} - \mu}{\sigma}\right),$$

$$= \int_{0}^{\infty} \int_{0}^{\infty} x^{\frac{1}{\delta}} \left(1 - e^{-\frac{y}{\delta}}\right)^{m \sigma - 1} e^{-\frac{y}{\delta}} \frac{1}{\Gamma(m)} e^{-\frac{y}{\Gamma(n-m)}} e^{-y} dxdy.$$
Substituting \( u = \frac{x}{x+y} \) and \( v = x + y \), we obtain

\[
E\left(\frac{X_L(m) - \mu}{\sigma}\right)\left(\frac{Y_L(n) - \mu}{\sigma}\right)
\]

\[
= \int_0^\infty \frac{1}{\Gamma(m)} \frac{1}{\Gamma(n-m)} u^{m-1} \delta^{-1} (1-u)^{n-m-1} \delta^{-1} v^{n-2} \delta^{-1} e^{-vdv} du
\]

\[
= \frac{B(m-\frac{1}{\delta}, n-m)}{\Gamma(m)} \frac{\Gamma(n-\frac{2}{\delta})}{\Gamma(n)} = \frac{\Gamma(m-\frac{1}{\delta})}{\Gamma(m)} \frac{\Gamma(n-\frac{2}{\delta})}{\Gamma(n)}
\]

Thus

\[
\text{Cov}(X_L(m), X_L(n)) = \sigma^2 \left[ \frac{\Gamma(m-\frac{1}{\delta})}{\Gamma(m)} - \frac{\Gamma(n-\frac{2}{\delta})}{\Gamma(n)} \right]
\]

We can write

\[
\text{Cov}(X_L(m), X_L(n)) = \sigma^2 \alpha_m \beta_n
\]

where

\[
\alpha_n = \frac{\Gamma(m-\frac{1}{\delta})}{\Gamma(m)} \quad \text{and} \quad \beta_n = \frac{\Gamma(n-\frac{2}{\delta})}{\Gamma(n-\frac{1}{\delta})} - \frac{\Gamma(n-\frac{1}{\delta})}{\Gamma(n)}
\]

**Recurrence Relation Between Moments**

We will assume without loss of generality \( \mu = 0 \) and \( \sigma = 1 \).

To prove the next two theorems, we use the relation \( f(x) = \delta x^{-1-\delta} F(x) \).
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Theorem 2.1.
For $r \geq \delta$, $n = 1, 2, \ldots$

\[ E(X_{L(n+1)}^{r-\delta}) = E(X_{L(n)}^{r-\delta}) - \frac{r-\delta}{\delta} E(X_{L(n+1)}^{r}) \]

Proof.

\[ E(X_{L(n+1)}^{r}) = \int_{0}^{\infty} x^{r-1} (-\ln F(x))^{n} f(x) dx \]

\[ = \delta \int_{0}^{\infty} x^{r-1-\delta} (-\ln F(x))^{n} f(x) dx \]

\[ = \frac{\delta}{r-\delta} \int_{0}^{\infty} x^{r-\delta} (-\ln F(x))^{n-1} f(x) dx \]

\[ - \frac{\delta}{r-\delta} \int_{0}^{\infty} x^{r-\delta} (-\ln F(x))^{n} f(x) dx \]

\[ = \frac{\delta}{r-\delta} [E(X_{L(n)}^{r-\delta}) - E(X_{L(n+1)}^{r-\delta})] \]

Corollary 2.1.

\[ [E(X_{L(n)}^{m}) - E(X_{L(n+1)}^{m})] = \frac{m}{\delta} e(X_{L(n+1)}^{\delta+m}) \), \ m=1.2. \]

Theorem 2.2.

For $s > \delta$, and $m > (r + s + 1)/\delta$

\[ E(X_{L(m)}^{s})^{X_{L(m+1)}^{s+\delta}} = \frac{\delta}{s-\delta} [E(X_{L(m)}^{r+s+1}) - E(X_{L(m)}^{r} X_{L(m+1)}^{s-\delta})] \]

For $s > \delta$, $m > \frac{r}{\delta}$, $n > \frac{s}{\delta}$

\[ E(X_{L(m)}^{r})^{X_{L(n)}^{s-\delta}} = \frac{\delta}{s-\delta} [E(X_{L(m)}^{r} X_{L(n-1)}^{s-\delta}) - E(X_{L(m)}^{r} X_{L(n)}^{s-\delta})] \]
Proof.

\[ E(X^{r}_{L(m)} X^{s}_{L(n)}) = \frac{1}{\Gamma(m)} \frac{1}{\Gamma(n-m)} \int_{0}^{\infty} x^{r} (H(x))^{m-1} f(x) I(x) \, dx , \]

where

\[ I(x) = \int_{0}^{x} y^{s-1} (H(x) - H(y))^{n-m-1} f(y) \, dy \]

For \( n-m+1 \)

\[ I(x) = \delta \int_{0}^{x} y^{s-\delta-1} f(y) \, dy \text{ for } n = m + 1 \]

\[ = \delta \frac{x^{s-\delta}}{s-\delta} F(x) - \delta \frac{x^{s-\delta}}{s-\delta} \int_{0}^{x} f(y) \, dy \]

Thus

\[ E(X^{r}_{L(m)} X^{s}_{L(m+1)}) = \frac{\delta}{s-\delta} [E(X^{r+s-\delta}_{L(m)}) - E(X^{r}_{L(m)} X^{s-\delta}_{L(m+1)})] \]

For \( n \geq m+2 \),

\[ I(x) = \int_{0}^{x} y^{s-\delta-1} (H(x) - H(y))^{n-m-1} f(y) \, dy \]

\[ I(x) = \frac{\delta}{s-\delta} y^{s-\delta} (H(x) - H(y))^{n-m-1} f(y) \bigg|_{0}^{x} \]

\[ = \frac{\delta(n-m-1)}{s-\delta} \int_{0}^{x} y^{s-\delta} (H(x) - H(y))^{n-m-2} f(y) \, dy - \frac{\delta}{s-\delta} \int_{0}^{x} y^{s-\delta} (H(x) - H(y))^{n-m-1} f(y) \, dy \]
Thus

\[ E(X^r_{L(m)}X^s_{L(n)}) = \frac{\delta}{\delta - \delta} [E(X^r_{L(m)}X^{s-\delta}_{L(n-1)}) - E(X^r_{L(m)}X^{s-\delta}_{L(n)})] \]

A Characterization

**Theorem 2.2.**

Let \( \{X_i, i=1,2,\ldots\} \) be a sequence of independent and identically distributed random variables with absolutely continuous (with respect to Lebesgue measure) distribution function \( F(x) \) with \( F(0) = 0 \) and \( \delta > 0 \). Then the following two statements are equivalent:

(a) \( F(x) = \exp(-x^{-\delta}), \ x > 0, \ \delta > 0 \)

(b) \( \text{Var}(X^{s-\delta}_{L(n)} - X^{s-\delta}_{L(n-1)} \mid X^s_{L(n-1)} = x) = b, n \geq 2, \)

where \( b \) is a constant independent of \( x \).

**Proof.**

From (2.3), we have

\[ X^{s-\delta}_{L(n)} - X^{s-\delta}_{L(n-1)} \xrightarrow{d} W_n. \]

Thus \( X^{s-\delta}_{L(n)} - X^{s-\delta}_{L(n-1)} \) is independent of \( X^{-\delta}_{L(n-1)} \).

Hence (a) \( \Rightarrow \) (b).

Let \( Y = X^{-\delta} \), then \( \text{Var}(X^{s-\delta}_{L(n)}) - X^{s-\delta}_{L(n-1)} \mid X^s_{L(n-1)} = x)) = b, n \geq 2, \)

is equivalent to \( \text{Var}(Y_U(n)) - Y_{L(n-1)} \mid Y_{L(n)} = y)) = b, n \geq 2, \)

where \( b \) is independent of \( y \) and \( Y_U(n) \) and \( Y_{L(n-1)} \) are the upper record values from the sequence \( Y_i, i=1,2,\ldots \).

Thus since \( Y = X^{-\delta} \), then \( Y_{U(k)} \xrightarrow{d} X^{-\delta}_{L(k)} \) for all \( k = 1,2,\ldots \)

\[ b = E(Z_n \mid Y_U(n) = y) - (E(Z_n \mid Y_U(n) = y))^2 \]
where \( Z_n = Y_{U(n)} - Y_{U(n-1)}, n \geq 2 \).

\[
= \int_0^\infty z^2 (F^*(y))^{-1} dF^*(z+y) = 2 \int_0^\infty z (F^*(y))^{-1} F^*(z+y)\,dz
\]

and

\[
E(Z_n \mid Y_{U(n)} = y) = \int_0^\infty z (F^*(y))^{-1} dF^*(z+y) = \int_0^\infty (F^*(y))^{-1} F^*(z+y)\,dz,
\]

where \( F^*(y) = 1 - F^*(y) \) and \( F^*(y) \) is the cdf of \( Y_i \).

Substituting \( G(y) = \int_0^\infty z F^*(z+y)\,dz \) and denoting \( G^{(r)}(y) \) as the \( r \) th derivative of \( G^*(x) \), we have on simplification

\[
G^{(1)}(y) = \int_0^\infty F^*(z+y)\,dz, \quad G^{(2)}(y) = F^*(y) \quad \text{and} \quad G^{(3)}(y) = -f^*(y).
\]

Writing (2.4) and (2.5) in terms of \( G(y) \) and \( G^{(r)}(y) \), we have, \( 2G(y) \{G^{(r)}(y)\}^{-1} - \{G^{(1)}(y) (G^{(2)}(y))^{-1}\}^2 = b. \) for all \( y \geq 0 \). \( (2.6) \)

Differentiating (2.6) with respect to \( y \) and simplifying, we obtain

\[
2G^{(3)}(y) \{G^{(2)}(y)^{-3} - \{G^{(1)}(y)^2 - G(y) G^{(2)}(y)\} = 0
\]

Since \( G^{(3)}(y) \neq 0 \) for all \( y > 0 \), we must have

\[
\{G^{(1)}(y)^2 - G(y) G^{(2)}(y)\} = 0,
\]

i.e.

\[
\frac{d}{dx} \{ G(y) (G^{(1)}(y))^{-1} \} = 0, \quad \text{for all} \quad y \geq 0.
\]

The solution of (2.9) is

\[
G(y) = a e^{-cy}, \quad y \geq 0
\]

where \( a \) and \( c \) are arbitrary constants.

Hence

\[
F^*(y) = G^{(2)}(y) = ac^2e^{-cy}, \quad y > 0.
\]

Since \( F^*(x) \) is a distribution function of \( Y_i \) with \( F(0) = 0 \) and \( \text{Var}(Y_i) = 1 \), it follows that

\[
F^*(y) = e^{-y}.
\]

Now for all \( x \geq 0 \),
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\[ F(x) = P(X \leq x) = P(X^{-\delta} \leq x^{-\delta}) \]

\[ = P(X^{-\delta} \geq x^{-\delta}) = P(Y \geq x^{-\delta}) \]

\[ = e^{-x^{-\delta}} \]

This completes the proof.

References


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