Travelling Wave Solution of Two-Dimensional Nonlinear KdV-Burgers Equation

A. R. Seadawy $^{1,2}$

1 Mathematics Department, Faculty of Science
Taibah University, Al-Ula, Saudi Arabia
aly742001@yahoo.com

2 Mathematics Department, Faculty of Science
Beni-Suef University, Egypt

Abstract

In this study, we present two different methods a sech-tanh method and extended tanh-method to obtained the soliton solutions of the two-dimensional Korteweg-de Vries-Burgers (KdVB) equation with the initial conditions. These solutions include bright and dark solitary wave solutions, triangular solutions and complex line soliton wave solution. These solutions are stable and have applications in physics.

Keywords: Nonlinear KdV-Burgers equation, sech-tanh method, Extended tanh method, Traveling wave solutions

1 Introduction

It is well known that nonlinear phenomena are very important in a variety of scientific fields, especially in fluid mechanics, solid state physics, plasma physics, plasma waves and
chemical physics. Searching for exact and numerical solutions, especially, for traveling wave solutions, of nonlinear equations in mathematical physics plays an important role in soliton theory [1-2]. Many powerful methods to seek exact solutions to the nonlinear differential equations have been proposed. Among these are Backlund transformation [3-4], Darboux transformation [5], the inverse scattering method [6], Hirota’s bilinear method [7], the tanh method [8], the sine-cosine method [9-10], the homogeneous balance method [11-12], and the Riccati expansion method with constant coefficients [13]. Burgers equation has been found to describe various kinds of phenomena such as a mathematical model of turbulence [14] and the approximate theory of flow through a shock wave travelling in viscous fluid [15].

The Korteweg-de Vries-Burgers (KdVB) equation is one of the most famous nonlinear PDEs. It was derived in fluid mechanics to describe shallow water waves in a rectangular channel it also plays an important role in plasma physics. The Burgers equation is a nonlinear partial differential equation of second order, it is used in fluid dynamics and engineering as a simplified model for turbulence, boundary layer behavior, shock wave formation and mass transport. The KdV-Burgers equation arises in many different physical contexts as a model equation incorporating the effects of dispersion, dissipation and nonlinearity [16]. Some examples are provided by the propagations of waves on an elastic tube filled with a viscous fluid [17], the flow of liquids containing gas bubbles [18] and turbulence [19].

A number of theoretical issues concerning the KdV-Burgers equation have received considerable attention. In particular, the travelling wave solution to the KdV-Burgers equation has been studied extensively. Johnson [17], Demiray [20], Antar and Demiray [21] derived KdV-Burgers equation as the governing evolution equation for waves propagating in fluid-filled elastic or viscoelastic tubes in which the effects of dispersion, dissipation and nonlinearity are present.

Several studies in the literature, employing a large variety of methods, have been conducted to derive explicit solutions for KdV-Burgers equation. Grad and Hu [22] used a steady-state version to describe a weak shock profile in plasmas. They studied the same problem using a similar method to that used by Johnson [17] and a related problem was studied by Jeffrey [23]. A numerical investigation of the problem was carried out by Canosa and Gaxdag [24]. Bona and Schonbek [25] studied the existence and uniqueness of bounded travelling wave solution which tend to constant states at plus and minus infinity. More recently, Jeffrey and Xu [26] introduced a transformation which reduced the KdV-Burgers equation to a quadratic form involving a new dependent variable and its partial derivatives. They also obtained exact solutions of the KdV-Burgers equation by solving this one in terms of a series of exponentials. A comprehensive account of the
travelling wave solution to the KdV-Burgers equation can also be found in the review paper by Jeffrey and Kakutani [27]. For other theoretical issues and more details about these investigations concerning the KdV-Burgers equation, the reader is kindly referred to Jian-Jun [28].

2 An analysis of the methods

Suppose there is a PDE of the form
\[ F(u, u_t, u_x, u_y, u_{xx}, u_{xy}, \ldots) = 0, \tag{1} \]
can be converted to an ODE:
\[ Q(u, u', u'', u''', \ldots) = 0, \tag{2} \]
by using a wave variable \( \xi = \alpha x + \beta y + ct \). Equation (2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

2.1 The extended tanh method

The tanh method developed by Malfliet in [29] introduces an independent variable:
\[ Y = \tanh(\mu \xi), \quad \xi = \alpha x + \beta y + ct, \]
is introduced that leads to the change of derivatives:
\[ \frac{d}{d\xi} = \mu(1 - Y^2) \frac{d}{dY}, \tag{3} \]
\[ \frac{d^2}{d\xi^2} = -2\mu^2 Y(1 - Y^2)(\frac{d}{dY})^2 + \mu^2(1 - Y^2)^2 \frac{d^2}{dY^2}. \]
The extended tanh method admits the use of the finite expansion:
\[ u(\mu \xi) = S(Y) = \sum_{k=0}^{m} a_k Y^k + \sum_{k=1}^{m} b_k Y^{-k}, \tag{4} \]
where \( m \) is a positive integer, in most cases, that will be determined. Expansion equation (4) reduces to the standard tanh method for \( b_k, 1 \leq k \leq m \). The parameter \( m \) is usually obtained by balancing the linear terms of highest order in the resulting equation with the highest order nonlinear terms. Substituting of equation (4) into the ODE of equation (2) results in an algebraic system of equations in powers of \( Y \) that will lead to the determination of the parameters \( a_k \), \( k = 0, \ldots, m \), \( \mu \) and \( c \).
2.2 The sech-tanh method

We suppose that \( u(x,y,t) = u(\xi) \) where \( \xi = \alpha x + \beta y + ct \), \( u(\xi) \) has the following formal travelling wave solution:

\[
u(\xi) = \sum_{i=1}^{n} \text{sech}^{i-1}(A_i \text{sech} \xi + B_i \text{tanh} \xi), \tag{5}\]

where \( A_0, A_1, ..., A_n \) and \( B_1, ..., B_n \) are constants to be determined.

**Step (1)** Equating the highest-order nonlinear term and highest-order linear partial derivative in equation (2) yields the value of \( n \).

**Step (2)** Setting the coefficients of \((\text{sech}^i \text{tanh}^j)\) for \( i = 0, 1 \) and \( j = 1, 2, ..., \) to zero, we have the following set of over determined equations in the unknowns \( A_0, A_i, B_i, \mu \) and \( c \) for \( i = 1, 2, ..., n \).

**Step (3)** Using Mathematica and Wu’s elimination methods, the algebraic equations in step (2) can be solved.

3 Application of the methods

3.1 Two-dimensional kdv-Burgers equation

In this section, we will employ the proposed methods to solve the two-dimensional kdv-Burgers equation:

\[
(u_t + uu_x + pu_{xxx} - qu_{xx})_x + ru_{yy} = 0. \tag{6}
\]

3.1.1 Using the extended tanh method

equivalently

\[
\alpha \left( c u' + \alpha uu' + \alpha^3 pu''' - q \alpha^2 u'' \right)' + r \beta^2 u'' = 0, \tag{7}
\]

or equivalently

\[
(\alpha c + r \beta^2) u' + \alpha^2 uu' + \alpha^4 pu''' - q \alpha^3 u'' = 0, \tag{8}
\]

obtained upon using the wave variable \( \xi = \alpha x + \beta y + ct \), integrating the resulting ODE equation (7) once and setting the constant of integration equal to zero. Balancing \( u''' \) with \( uu' \) in equation (8) gives \( m = 2 \). The extended tanh method equation (4) admits the use of the finite expansion:

\[
u(\xi) = a_0 + a_1 Y + a_2 Y^2 + \frac{b_1}{Y} + \frac{b_2}{Y^2}. \tag{9}\]
Substituting equation (9) into equation (8) and collecting the coefficient of $Y$, we obtain a system of algebraic equations for $a_0, a_1, a_2, b_1, b_2$ and $p$. Solving this system gives the following solution

$$I \quad a_0 = \frac{-5c\alpha \mp 6\mu \alpha^3 - 5r\beta^2}{5\alpha^2}, \quad a_1 = a_2 = 0, \quad p = \mp \frac{q}{10\mu\alpha}, \quad b_1 = -\frac{12}{5} \mu \alpha, \quad b_2 = \pm \frac{6}{5} \mu \alpha.$$ 

In this case, the generalized soliton solution can be written as

$$u_1(x, y, t) = \left(\frac{-5c\alpha \mp 6\mu \alpha^3 - 5r\beta^2}{5\alpha^2}\right) - \frac{12}{5} \mu \alpha \coth[\mu(\alpha x + \beta y + ct)] \pm \frac{6}{5} \mu \alpha \coth^2[\mu(\alpha x + \beta y + ct)].$$

Figures (a1-a2) shown that the travelling wave solutions with $(c = -0.5, p = 0.25, q = 0.5, r = 2, \alpha = 0.1, \beta = 0.1, t = 0.1, \mu = 2)$ in the interval $[-10, 10]$ and $[-10, 10]$.

$$II \quad a_0 = \frac{-5c\alpha \mp 6\mu \alpha^3 - 5r\beta^2}{5\alpha^2}, \quad a_1 = b_1 = -\frac{12}{5} \mu \alpha, \quad a_2 = b_2 = \pm \frac{3}{5} \mu \alpha, \quad p = \mp \frac{q}{20\mu\alpha}.$$ 

In this case, the generalized soliton solution can be written as

$$u_2(x, y, t) = \left(\frac{-5c\alpha \mp 6\mu \alpha^3 - 5r\beta^2}{5\alpha^2}\right) - \frac{12}{5} \mu \alpha (\tanh[\mu(\alpha x + \beta y + ct)] + \coth[\mu(\alpha x + \beta y + ct)])$$

$$\pm \frac{3}{5} \mu \alpha (\tanh^2[\mu(\alpha x + \beta y + ct)] + \coth^2[\mu(\alpha x + \beta y + ct)]).$$

(11)
Figure (a3-a4) Travelling waves solutions of equation (11) is plotted: the non-symmetrical bright and dark solitary waves.

Figures (a5) Travelling waves solutions of equation (12) is plotted: periodic solitary waves.

Figures (a3-a4) shown that the travelling wave solutions with \((c = -0.5, \ p = -0.125, \ q = 0.5, \ r = 2, \ \alpha = 0.1, \ \beta = 0.1, \ t = 0.1, \ \mu = 2)\); in the interval \([-10, 10]\) and \([-10, 10]\).

\[ III \] \(a_0 = \frac{-5c\alpha \mp 6\mu q\alpha^3 - 5r\beta^2}{5\alpha^2}, \ b_1 = b_2 = 0, \ p = \pm \frac{q}{10\mu\alpha}, \ a_1 = -\frac{12}{5}\mu q\alpha, \ a_2 = \pm \frac{6}{5}\mu q\alpha.\)

In this case, the generalized soliton solution can be written as

\[ u_3(x, y, t) = \left(\frac{-5c\alpha \mp 6\mu q\alpha^3 - 5r\beta^2}{5\alpha^2}\right) - \frac{12}{5}\mu q\alpha [\mu(\alpha x + \beta y + ct)] \pm \frac{6}{5}\mu q\alpha^2 [\mu(\alpha x + \beta y + ct)]. \quad (12) \]

Figures (a5) shown that the travelling wave solutions with \((c = -0.5, \ p = 0.25, \ q = 0.5, \ r = 2, \ \alpha = 0.1, \ \beta = 0.1, \ t = 0.1, \ \mu = 2)\); in the interval \([-100, 100]\) and \([-100, 100]\).
Figures (a6) Travelling waves solutions of equation (14) is plotted: periodic solitary waves.

### 3.1.2 Using sech-tanh method

\[
    u(\xi) = A_0 + A_1 \text{sech}[\xi] + B_1 \text{tanh}[\xi] + A_2 \text{sech}^2[\xi] + B_2 \text{tanh}[\xi] \text{sech}[\xi].
\]  

Substituting from (13) into (7), setting the coefficients of \((\text{sech}^i \text{tanh}^j)\) for \(i = 0, 1\) and \(j = 1, 2, 3, 4\) to zero, we have the following set of over determined equations in the unknowns \(A_0, A_1, A_2, B_1, B_2\) and \(P\). Solve the set of equations of coefficients of \((\text{sech}^i \text{tanh}^j)\), by using Mathematica and Wu's elimination method, we obtain the following solutions:

- **I)** \(A_0 = -\frac{c\alpha + r\beta^2}{\alpha^2}, B_2 = A_1 = 0, B_1 = -\frac{12}{5} q\alpha, A_2 = \pm \frac{6}{5} q\alpha, p = \pm \frac{q}{10\alpha}.\)

So that, the generalized soliton solution can be written as

\[
    u_1(x, y, t) = -\left(\frac{c\alpha + r\beta^2}{\alpha^2}\right) - \frac{12}{5} q\alpha \text{tanh}[\alpha x + \beta y + ct] \pm \frac{6}{5} q\alpha \text{sech}^2[\alpha x + \beta y + ct].
\]  

Figure (a6) shown that the travelling wave solutions with \((c = 0.5, p = 0.5, q = 0.5, r = 0.5, \alpha = -0.1, \beta = 0.05, t = 0.1)\); in the interval \([-100, 100]\) and \([-100, 100]\).

- **II)** \(A_0 = -\frac{c\alpha + r\beta^2}{\alpha^2}, B_1 = A_2 = -\frac{6}{5} q\alpha, A_1 = \pm \frac{6}{5} q\alpha i, B_2 = \mp \frac{6}{5} q\alpha i, p = -\frac{1}{5} q\alpha.\)

So that, the soliton solution can be written as

\[
    u_2(x, y, t) = -\left(\frac{c\alpha + r\beta^2}{\alpha^2}\right) + \frac{6}{5} q\alpha \left(\pm \text{sech}[\alpha x + \beta y + ct] \mp \text{tanh}[\alpha x + \beta y + ct] \text{sech}[\alpha x + \beta y + ct]\right)
\]

\[
    -\frac{6}{5} q\alpha \left(\text{sech}^2[\alpha x + \beta y + ct] + \text{tanh}[\alpha x + \beta y + ct]\right) .
\]  

(15)
Figures (a7) Travelling waves solutions of equation (17) is plotted: periodic solitary waves.

\( III) \ A_0 = -\frac{c\alpha + r\beta^2}{\alpha^2}, \ B_2 = A_1 = \pm \frac{6}{5}q\alpha i, \ A_2 = -B_1 = \frac{6}{5}q\alpha, \ p = \frac{1}{5}q\alpha. \)

So that, the soliton solution can be written as

\[
\begin{align*}
\ u_3(x, y, t) &= -\frac{c\alpha + r\beta^2}{\alpha^2} \pm \frac{6}{5}q\alpha i \left( \text{sech}[\alpha x + \beta y + ct] + \tanh[\alpha x + \beta y + ct] \text{sech}[\alpha x + \beta y + ct] \right) \\
&\quad + \frac{6}{5}q\alpha \left( \text{sech}^2[\alpha x + \beta y + ct] - \tanh[\alpha x + \beta y + ct] \right). \tag{16}
\end{align*}
\]

\( IV) \ A_0 = -\frac{c\alpha + r\beta^2}{\alpha^2}, \ B_2 = A_1 = 0, \ B_1 = -q\alpha \pm \sqrt{q^2\alpha^2 + 96p^2\alpha^4}, \ A_2 = 12p\alpha^2. \)

So that, the generalized soliton solution can be written as

\[
\begin{align*}
\ u_4(x, y, t) &= -\left( \frac{c\alpha + r\beta^2}{\alpha^2} \right) + \left( -q\alpha \pm \sqrt{q^2\alpha^2 + 96p^2\alpha^4} \right) \tanh[\alpha x + \beta y + ct] + \\
&\quad 12p\alpha^2 \text{sech}^2[\alpha x + \beta y + ct]. \tag{17}
\end{align*}
\]

Figure (a7) shown that the travelling wave solutions with \((c = -0.5, \ p = 1, \ q = 0.5, \ r = 1, \ \alpha = 0.1, \ \beta = -0.1, \ t = 2); \) in the interval \([-100, 100]\) and \([-100, 100]\).

References

Travelling wave solution


Received: April 21, 2013