Stability Analysis Solutions for the Fourth-Order Nonlinear Ablowitz-Kaup-Newell-Segur Water Wave Equation

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Abstract

Ablowitz-Kaup-Newell-Segur (AKNS) hierarchy appears naturally in the model besides a string equation. The Korteweg-de Vries hierarchy is contained in the AKNS hierarchy as a reduction, and therefore appears in the model before one takes the double scaling limit. The AKNS hierarchy is a complexified version of the Nonlinear Schrödinger (NLS) hierarchy and also contains the modified Korteweg-de Vries hierarchy. By using the extended auxiliary equation method, we obtained some new soliton like solutions for the two-dimensional fourth-order nonlinear (AKNS) equation with variable coefficient. These solutions include symmetrical,
non-symmetrical kink solutions, bell-shaped solitary wave solutions, solitary wave solutions and traveling wave solutions. The stability analysis for these solutions are discussed.

1 Introduction

During the early 1970s, motivated by the applications to nonlinear optics, Ablowitz, Kaup, Newell, and Segur derived the AKNS equations on the basis of the generalized Zakharov-Shabat spectral problem [1-4]. The AKNS equations are very important because they can be reduced to some well-known nonlinear evolution equations such as the KdV, the mKdV, the nonlinear Schrödinger, and the sine-Gordon equations, and others, which have many applications in physics and other nonlinear sciences. Various methods have been developed for obtaining explicit solutions of the AKNS equations, for instance, the inverse scattering transformation [1-4], the Bäcklund transformation [5], the Darboux transformation [6].

Some solutions of AKNS equation have been obtained by using Hirota bilinear method, tanh-coth method, Exp-function method, Darboux transformation method and multi-linear variable separation approach [7-11]. In addition, Lü et al. [12] have studied the (2 + 1)-dimensional AKNS equation with variable coefficients and have got some new explicit solutions by applying Lie symmetry method.

2 Analysis of the auxiliary equation method

The following is a given nonlinear differential equation with three variables \( x, y \) and \( t \)

\[
F(u, u_x, u_y, u_t, u_{xx}, u_{yy}, u_{tt}, \ldots) = 0
\]

where \( F \) is a polynomial function with respect to the indicated variables or some function which can be reduced to a polynomial function by using some transformations.

**Step 1:** Assume that Eq. (1) has the following formal solution:

\[
u(\xi) = \sum_{i=0}^{n} e^{it} \varphi^i(\xi),
\]

with the variable \( \varphi \) satisfying

\[
\frac{d\varphi}{d\xi} = c_0 + c_1 \varphi(\xi) + c_2 \varphi^2(\xi) + c_3 \varphi^3(\xi) + c_4 \varphi^4(\xi),
\]

\[
\xi(x, y, t) = x + y - \kappa t
\]
where \( c_i (i = 0, 1, 2, 3, 4) \) are constant.

**Step 2**: Balancing the highest order derivative term and the highest order nonlinear term of Eq.(1) with homogeneous balance method, the parameters \( n \) in (2) can be determined.

**Step 3**: Substituting (2), (3) and (4) into Eq.(1) and collecting coefficients of \( \varphi^k \varphi^{(l)} \), then setting coefficients equal zero, we will obtain a set of over-determined equations for \( \kappa \) and \( c_i \). By solving the system, we may determine these parameters.

**Step 4**: Substituting \( \kappa, c_i \) and \( \varphi(\xi) \) obtained in step 3 into Eq.(2) can derive the solutions of Eq.(1)

### 3 Traveling wave solutions for Ablowitz, Kaup, Newell, and Segur water equation (AKNS)

In this section, we apply the method introduced in section 2 to AKNS equation with variable coefficient

\[
4u_{xt} + u_{xxxt} + 8u_xu_{xy} + 4u_{xx}u_y - \alpha u_{xx} = 0 \tag{5}
\]

Making use of the variable \( \xi(x, t) = x + y - \kappa t \), we transform Eq.(5) into the following ordinary differential equation and integrated

\[
(4\kappa - \alpha)u' + u''' + 6u'^2 = 0 \tag{6}
\]

According to the above method in section 2, we consider homogeneous balance between \( u'^2 \) and \( u''' \) in Eq.(6), which give \( n = 2 \). We suppose the solution of Eq.(6) is of the form

\[
u = 1 + e^t \varphi(\xi) + e^{2t} \varphi^2(\xi) \tag{7}\]

Substituting (3), (4) and (7) into Eq.(6) and collecting coefficients of \( \varphi^k \varphi^{(l)} \), \( k = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 \), \( l = 0 \), and letting each coefficient equals zero, we obtain a set of over-determined equations:

with mathematic software, we obtain the parameters of \( \alpha, \kappa \) and \( c_i \) via symbolic computation.

**Case 1**: When \( c_4 = 0 \), Eq.(3) possesses a bell-shaped solitary wave solutions

\[
\varphi_1 = \frac{1}{2} e^{-t} \left( -1 \pm \frac{2\sqrt{\alpha - 4\kappa}e^t}{\sqrt{e^{2t}(1 + e^{\sqrt{\alpha - 4\kappa}}\sqrt{\alpha - 4\kappa})}} \right) \tag{8}\]
Figure (1a) shown that the bell-shaped solitary wave solution of Eq.(11) with $\kappa = 0.1$, $\alpha = 0.9$ and $t = 1$; in the interval [-10,10] and [-10,10], Figure (1b) shown that the bright solitary wave solutions of Eq.(12) with $\kappa = 0.01$, $\alpha = 0.1$ and $t = 10$; in the interval [0,10] and [-5,5]

$$\varphi_2 = \frac{1}{2} e^{-2t} \left( -e^t \pm \frac{2 \sqrt{\alpha - 4\kappa} (-1 + e^{\sqrt{\alpha - 4\kappa}}) \sqrt{\alpha - 4\kappa}}{(-1 + e^{\sqrt{\alpha - 4\kappa}})} \right)$$  \hspace{1cm} (9)

The parameters $c_i$ can be derived.

$$c_0 = \frac{1}{16} e^{-t} (-1 \mp 4\sqrt{\alpha - 4\kappa}), \quad c_1 = \frac{1}{8} (-3 \mp 4\sqrt{\alpha - 4\kappa}), \quad c_2 = -\frac{3}{4} e^t, \quad c_3 = -\frac{1}{2} e^{2t}$$  \hspace{1cm} (10)

So the solution of Eq.(6) are shown as follows.

$$u_1 = \frac{3}{4} + \frac{\sqrt{\alpha - 4\kappa}}{-1 + e^{\sqrt{\alpha - 4\kappa}(x+y-\kappa t)}}$$  \hspace{1cm} (11)

$$u_2 = \frac{3}{4} + \sqrt{\alpha - 4\kappa} + \frac{\sqrt{\alpha - 4\kappa}}{-1 + e^{\sqrt{\alpha - 4\kappa}(x+y-\kappa t)}}$$  \hspace{1cm} (12)

**Case 2:** When $c_0 = c_4 = 0$, Eq.(3) possesses solitary wave solution

$$\varphi_3 = \frac{2}{-4e^t + e^{\xi} \mp e^{\frac{\xi}{2}} \sqrt{-4e^t + e^{\xi}}}$$  \hspace{1cm} (13)

The parameters $c_i$ can be derived.

$$c_1 = -\frac{1}{4}, \quad c_2 = -\frac{3}{4} e^t, \quad c_3 = -\frac{1}{2} e^{2t}, \quad \kappa = \frac{1}{64}(-1 + 16\alpha),$$  \hspace{1cm} (14)
Figure (2) shown that the dark solitary wave solution of Eq.(15) with $\kappa = 0.7$ and $t = 1$; in the interval [-10, 10] and [-10, 10]

So the solution of Eq.(6) are shown as follows.

$$u_3 = 1 + \frac{1}{-4 + e^{(-4 + \frac{\kappa t}{4})}}$$

(15)

**Stability analysis:** The Hamiltonian for the solutions Eq.(15) can be rewritten

$$\mu = \frac{1}{2} \int \int u^2 dx$$

The Sufficient condition for discuss the stability of solution $\frac{\partial \mu}{\partial \kappa} > 0$

$$\frac{1}{8} \left(1 - 4e^{-4 + \frac{\kappa}{4}}\right) + \frac{2}{-1 + 4e^{1 + \frac{\kappa}{4}}} + \frac{1}{1 - 4e^{6 + \frac{\kappa}{4}}} +$$

$$14 \log(1 - \frac{1}{4}e^{-1 + \frac{\kappa}{4}}) - 7 \log(1 - \frac{1}{4}e^{-6 + \frac{\kappa}{4}}) - 7 \log(1 - \frac{1}{4}e^{4 + \frac{\kappa}{4}})) > 0$$

(16)

**Case 3:** When $c_1 = c_4 = 0$, Eq.(3) possesses traveling wave solution

$$\varphi_4 = \frac{1}{2} e^{-2t} \left( -e^t + \frac{\sqrt{3} \sqrt{e^{2t+\frac{3\kappa}{4}}(2 + e^{\frac{3\kappa}{4}})}}{2 + e^{\frac{3\kappa}{4}}} \right)$$

(17)

The parameters $c_i$ can be derived.

$$c_0 = \frac{e^{-t}}{8}, \quad c_2 = -\frac{3}{4} e^t, \quad c_3 = -\frac{1}{2} e^{2t}, \quad \alpha = \frac{1}{16} (9 + 64\kappa),$$

(18)

So the solution of Eq.(6) are shown as follows.

$$u_4 = \frac{3}{2} - \frac{3}{2(2 + e^{\frac{3(x+y-\kappa t)}{4}})}$$

(19)
Figure (3) shown that the traveling wave solution of Eq.(19) with $\kappa = 0.1$ and $t = 1$; in the interval $[-15,15]$ and $[-15,15]$

If we take

$$u(\xi) = \sum_{i=0}^{n} e^{(i+1)t} \varphi^i(\xi),$$  \hspace{1cm} (20)

**Case 4**: When $c_4 = 0$, Eq.(3) possesses solitary wave solutions

$$\varphi_5 = \frac{1}{2} e^{-2t} \left( -e^t \pm \frac{2 \sqrt{e^{2t+\sqrt{8-4\kappa}}}}{(e^{2t}+\sqrt{8-4\kappa})} \right)$$ \hspace{1cm} (21)

$$\varphi_6 = \frac{e^{-2t} \left( -e^{2t} + e^{t+\sqrt{8-4\kappa}} \pm 2 \sqrt{e^{2t}(-e^t + e^{\sqrt{8-4\kappa}})\sqrt{8-4\kappa}} \right)}{2(e^t - e^{\sqrt{8-4\kappa}})}$$ \hspace{1cm} (22)

The parameters $c_i$ can be derived.

$$c_0 = \frac{1}{16} (-1 \pm 4e^{-t}\sqrt{\alpha - 4\kappa}), \hspace{1cm} c_1 = \frac{1}{8} (-3e^t \pm 4\sqrt{\alpha - 4\kappa}), \hspace{1cm} c_2 = -\frac{3}{4} e^{2t}, \hspace{1cm} c_3 = -\frac{1}{2} e^{3t}$$ \hspace{1cm} (23)

So the solution of Eq.(6) are shown as follows.

$$u_5 = \frac{3}{4} e^t + \left( 1 + \frac{1}{-1 + e^{(t+\sqrt{8-4\kappa})}} \right) \sqrt{\alpha - 4\kappa}$$ \hspace{1cm} (24)

$$u_6 = \frac{1}{4} e^t \left( 3 + \frac{4\sqrt{\alpha - 4\kappa}}{-e^t + e^{\sqrt{8-4\kappa}(x+y-\kappa t)}} \right)$$ \hspace{1cm} (25)
shown that the bright and dark solitary wave solutions of Eq((24), (25)) with $\kappa = 0.1$, $\alpha = 0.9$ and $t = 2$; in the interval $[-2,2]$ and $[-2,2]$; the bright solitary wave (4a), the dark solitary wave (4b)

Figure (5) shown that the bell-shaped solitary wave solution of Eq.(28) with $\kappa = 0.6$ and $t = 2$; in the interval $[-10,10]$ and $[-10,10]$

**Case 5**: When $c_0 = c_4 = 0$, Eq.(3) possesses bell-shaped solitary wave solution

$$\varphi_7 = \frac{2}{-4e^t + e^{t\xi} \pm \sqrt{e^{t\xi} - 4e^{(t+\frac{1}{4}\xi)}}}$$ (26)

The parameters $c_i$ can be derived.

$$c_1 = -\frac{1}{4}e^t, \quad c_2 = -\frac{3}{4}e^{2t}, \quad c_3 = -\frac{1}{2}e^{3t}, \quad \alpha = \frac{1}{16}(e^{2t} + 64\kappa)$$ (27)

So the solution of Eq.(6) are shown as follows.

$$u_7 = e^t + \frac{e^{2t}}{-4e^t + e^{t(x+y-\kappa)}}$$ (28)

**Stability analysis**: The Hamiltonian for the solutions Eq.(28) can be rewritten
Figure (6) shown that the traveling wave solution of Eq.(32) with $\kappa = 0.6$ and $t = 2$; in the interval $[-10,10]$ and $[-10,10]$

$$\mu = \frac{1}{2} \int \int u^2 dx$$

The Sufficient condition for discuss the stability of solution $\frac{\partial \mu}{\partial \kappa} > 0$

$$\frac{1}{4} e^{2t} \left( \frac{1}{1 - 4 e^{2(2 + \frac{1}{2} e^{2(-10 + \kappa)})}} + \frac{2}{-1 + 4 e^{(2 + \frac{1}{2} e^{2})(10 + \kappa)}} + \frac{1}{1 - 4 e^{(2 + \frac{1}{2} e^{2})(10 + \kappa)}} \right) +$$

$$14 \log(1 - \frac{1}{4} e^{(-2 - \frac{3}{2} e^2)}) - 7 \log(1 - \frac{1}{4} e^{(-2 + e^2(5 - \frac{5}{2}))}) - 7 \log(1 - \frac{1}{4} e^{(-2 - \frac{3}{2} e^2(10 + \kappa))}) > 0 \quad (29)$$

**Case 6**: When $c_1 = c_4 = 0$, Eq.(3) possesses traveling wave solution

$$\varphi_8 = \frac{1}{2} e^{-2t} \left( -e^t \mp \frac{\sqrt{3} \sqrt{e^{(2t + \frac{3\kappa t}{4})}(2 + e^{\frac{9\kappa t}{4}})}}{2 + e^{-\frac{3\kappa t}{4}}} \right) \quad (30)$$

The parameters $c_i$ can be derived.

$$c_0 = \frac{1}{8}, \quad c_2 = -\frac{3}{4} e^{2t}, \quad c_3 = \frac{1}{2} e^{3t}, \quad \alpha = \frac{1}{16}(9 e^{2t} + 64\kappa) \quad (31)$$

So the solution of Eq.(6) are shown as follows.

$$u_8 = \frac{3}{2} e^t \left( 1 - \frac{1}{2 + e^{\frac{3\kappa t}{4}(x + y - \kappa t)}} \right) \quad (32)$$

**Stability analysis**: The Hamiltonian for the solutions Eq.(32) can be rewritten

$$\mu = \frac{1}{2} \int \int u^2 dx$$

The Sufficient condition for discuss the stability of solution $\frac{\partial \mu}{\partial \kappa} > 0$
Figure (7) shown that the bell-shaped solitary wave solution of Eq.(37) with $\kappa = 0.7$ and $t = 1$; in the interval $[-10,10]$ and $[-10,10]$

$$\frac{3}{4}e^{2} \left( \frac{1}{1 + 2e^{\frac{3}{2}e^{2}(-10+\kappa)}} \right) - \frac{2}{1 + 2e^{\frac{3}{2}e^{2}\kappa}} + \frac{1}{1 + 2e^{\frac{3}{2}e^{2}(10+\kappa)}} +$$

$$6\log\left(1 + \frac{1}{2}e^{-\frac{3}{2}e^{2}\kappa}\right) - 3\log\left(1 + \frac{1}{2}e^{-\frac{3}{2}e^{2}(-10+\kappa)}\right) - 3\log\left(1 + \frac{1}{2}e^{-\frac{3}{2}e^{2}(10+\kappa)}\right) > 0 \quad (33)$$

If we take

$$u(\xi) = \sum_{i=0}^{n} t^{i} \varphi^{i}(\xi), \quad (34)$$

**Case 7:** When $c_{0} = c_{4} = 0$, Eq.(3) possesses a bell-shaped solitary wave solutions

$$\varphi_{9} = \frac{2}{-4t + e^{\frac{3}{2}e^{2}\kappa} \pm \sqrt{e^{\frac{3}{2}e^{2}\kappa} - 4te^{\frac{3}{2}e^{2}\kappa}}} \quad (35)$$

The parameters $c_{i}$ can be derived.

$$c_{1} = -\frac{1}{4}, \quad c_{2} = -\frac{3}{4}t, \quad c_{3} = -\frac{1}{2}t^{2}, \quad \kappa = \frac{1}{64}(-1 + 16\alpha), \quad (36)$$

So the solution of Eq.(6) are shown as follows.

$$u_{9} = 1 + \frac{t}{-4t + e^{\frac{(x+y-t)}{4}}} \quad (37)$$

**Stability analysis:** The Hamiltonian for the solutions Eq.(15) can be rewritten

$$\mu = \frac{1}{2} \int \int u^{2}dx$$

The Sufficient condition for discuss the stability of solution $\frac{\partial \mu}{\partial \kappa} > 0$
\[
\frac{1}{8} \left( 1 - 4e^{(-5+\frac{\kappa}{4})} \right) + \frac{2}{1 - 4e^{(\frac{\kappa}{4})}} + \frac{1}{1 - 4e^{(5+\frac{\kappa}{4})}} + 14 \log \left( 1 - \frac{1}{4} e^{-\kappa} \right) - 7 \log \left( 1 - \frac{1}{4} e^{(-5-\kappa)} \right) - 7 \log \left( 1 - \frac{1}{4} e^{(5-\kappa)} \right) > 0 \quad (38)
\]

References


Received: April 21, 2013