A Study on Fuzzy Reliability Measures

K. Abdul Razak

Department of Mathematics
TRP Engineering College (SRM Group), Trichy, Tamil Nadu, India
arrazak76@gmail.com

K. Rajakumar

Department of Mathematics,
Trichy Engineering College, Trichy, Tamil Nadu, India.
skrjakumartec@gmail.com

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Abstract

This paper presents a new method for fuzzy system reliability analysis based on fuzzy semi-markov model with fuzzy transitions. The definition and the basic equation for interval transitions of a semi-markov model with fuzzy transitions are provided. The transitions for the above model with fuzzy states is provided for the non-homogeneous case. We apply this fuzzy technique to ATM transactions and this model gives the accessibility assessment that are based on uncertainties. The definitions and results for the fuzzy model are provided by means of the fuzzy probabilities and are modeled by trapezoidal fuzzy numbers.

Mathematics Subject Classification: 90B25, 90C40, 60K15, 60K50

Keywords: Fuzzy Reliability, Fuzzy probability, Trapezoidal fuzzy number, Semi-markov model with fuzzy transition

1. Introduction

The reliability engineering is one of the important engineering tasks in design and development of a technical system. It is well known that the conventional reliability analysis using the probabilities has been found to be inadequate to handle uncertainty of failure data and modeling. To overcome this problem, the
concept of fuzzy approach has been used in the evaluation of the reliability of a system Singer [20] presented a fuzzy set approach for fault tree and reliability analysis.

The Probabilistic method is used in the reliability analysis. A systematic theory of reliability is based on probability theory. For many systems due to uncertainties and imprecision of data, the estimation of precise values of probabilities is very difficult. For this reason the concept of fuzzy reliability has been introduced and formulated in Cai [13] and pointing out that there are various forms of fuzzy reliability theories namely PROBIST, POSBIST, PROFUST and application for fuzzy reliability were found in Zhang M.L. [1,2,3], Chen [4,5], Don-Lin Mon [6], Jiang Q [12] and Cai [13].

Semi-Markov model has found important applications in manpower systems. Semi Markov model has found important applications with transition found in Praba. B [17,18]. Now we extend the classical Semi-Markov model by assuming transitions as fuzzy transitions together with fuzzy states. Important theoretical results and applications for Semi-Markov models can be found in Losifescu-Manu [9], Howard [8], Meclean [15], Janssen [11], Usman Yusuf Abubakar [20], Janssen and Limnios [10] and in Limios and Oprisan [14].

The concept of non-homogeneous Markov systems was first introduced in Vassiliou [16]. The non homogeneous Semi-Markov system in discrete time was examined in Vassiliou and papadopoulou [21] and the asymptotic behavior of the same model was studied in papadopoulou and vissiliou [16].

In this paper we present a new method to find ATM transactions using fuzzy reliability of a non-homogeneous fuzzy probabilistic and Semi-Markov model consisting of set of states. Whose transitions are fuzzy transition between the states based on the assumption of fuzzy Profust reliability theory. The proposed method models explore the fuzzy system reliability through transition fuzzy probabilities and as a consequence of the fuzzy reliability it is represented as a trapezoidal fuzzy number.

This paper is constructed as follows Section 2 recalls the preliminaries needed for this paper and Section 3 gives the basic equation of a non homogeneous fuzzy probabilistic semi-Markov model with fuzzy transition. In Section 4 the basic equation for fuzzy reliability modeling using semi-Markov model with fuzzy transitions and fuzzy states is provided. Section 5 illustrates the above constructed model for ATM transaction in Section 6, conclusion is discussed.

2. Preliminaries

2.1 Fuzzy Number
A fuzzy number is a fuzzy set with the following conditions.

- Convex Fuzzy Set.
- Normalized Fuzzy Set.
It’s Membership Function is Piecewise Continuous. It is defined in the real number.

2.2 Trapezoidal Fuzzy Number

A fuzzy number $A(a_1, a_2, a_3, a_4)$ with the following membership function is called the trapezoidal fuzzy number.

$$
\mu_A(x) = \begin{cases} 
0 & \text{if } x < a_1 \\
\frac{x - a_1}{a_2 - a_1} & \text{if } a_1 \leq x \leq a_2 \\
1 & \text{if } a_2 \leq x \leq a_3 \\
\frac{a_4 - x}{a_4 - a_3} & \text{if } a_3 \leq x \leq a_4 \\
0 & \text{if } x > a_4
\end{cases}
$$

In this paper we have used the trapezoidal fuzzy number on $[0, 1]$.

2.3 Fuzzy Reliability

The fuzzy reliability of the proposed system is defined to be the fuzzy probability that the system eventually completes its task successfully without failing from transition from one state to other state till its termination state is reached.

3. Fuzzy Probabilistic analysis of Non Homogeneous Semi-Markov Model

Possibility measure is a mathematical measures for dealing types of uncertainty and is an alternative to probability measures. In this paper, we consider a random experiment, which has certainty in its outcomes and have the uncertainty in the probability of events and these uncertainties in the probabilistic usage information are represented by fuzzifying the probability values into a Trapezoidal fuzzy number on $[0, 1]$ for the system to perform its function properly. In this section we briefly review the main definitions and result from the theory of non-homogeneous fuzzy probabilistic Markov renewal processes which are directly relevant for our purpose. Now we define the non-homogeneous fuzzy probabilistic Semi-Markov model with fuzzy transitions.

Let $(\Omega, G, P)$ be a probability space where $\Omega$ denotes the sample space, $E$ be a finite state space and $P$ be probability measure. On our probability space, we define two random variable,

$$
X_n : \Omega \rightarrow G, \quad I_n : \Omega \rightarrow N
$$

$X_n$ represents the state at the $n^{th}$ transitions and $I_n$ is the time of the $n^{th}$ transition.
The process \((X, I)\) is a non-homogeneous fuzzy probabilistic Markov Renewal Process if \(\forall i, j \in G\) and \(\forall t \in \mathbb{N}\), the following condition holds:

\[
\mathbb{P}[X_{n+1} = j, I_{n+1} \leq t / X_n = i, I_n = s, X_{n-1}, I_{n-1}, ..., X_0, I_0]
\]

\[
= \mathbb{P}[X_{n+1} = j, I_{n+1} \leq t / X_n = i, I_n = s] \text{ and for } j \neq i,
\]

is the associated non homogeneous fuzzy probabilistic Semi-Markov Kernel \(\mathbb{Q}\).

The fuzzy probabilistic Semi-Markov Kernel is written again as

\[
\mathbb{Q}_i(s, t) = \mathbb{P}[X_{n+1} = j, I_{n+1} \leq t / X_n = i, I_n = s]
\]

The second argument of \(\mathbb{Q}\) namely \(x\) represents the duration time where as \(s\) represents the starting time. The fuzzy transition matrix \(\mathbb{P}(s)\) of the non homogeneous extended fuzzy probabilistic to Markov chain \(X_n\) is obtained

\[
\mathbb{P}_i(s) = \lim_{x \to \infty} \mathbb{Q}_i(s, x) \forall i, j \in G.
\]

This fuzzy probabilistic function is obtained by

\[
\mathbb{F}_i(s, x) = \begin{cases} \mathbb{Q}_i(s, x) / \mathbb{P}_i(s) & \text{if } \mathbb{P}_i(s) \neq 0 \\ 1 & \text{if } \mathbb{P}_i(s) = 0 \end{cases}
\]

And for more feasibility, it is supposed free of the time ‘s’ namely \(\mathbb{F}_i(x)\).

We define \(\forall i, j \in G\) and \((s, t) \in \mathbb{N} \times \mathbb{N}\), the fuzzy probabilistic Semi-Markov’s interval transition fuzzy probabilistic as

\[
\mathbb{d}_{i,j}(s, t) = \mathbb{P}[Z(t) = j / Z(s) = i]
\]

Satisfying the following system of equations

\[
\mathbb{d}_{i,j}(s, t) = \mathbb{d}_{i,j} \left(1 - H_i(s, t)\right) + \sum_{k \in E} \sum_{\tau = 1} \mathbb{P}_k(s) \mathbb{F}_k(t) \mathbb{d}_{i,k}(\tau, t)
\]

where

\[
\mathbb{d}_{i,j} = \begin{cases} (0, 0, 0) & i \neq j \\ (1, 1, 1) & i = j \end{cases}
\]

In this context, we explain briefly in the next section the fuzzy reliability modeling using semi-Markov model through its transition fuzzy probabilities and waiting time fuzzy probabilities.

4. Fuzzy Reliability Modeling Using Semi-Markov Model

Fuzzy reliability is a concept in which fuzzy sets can capture subjective, uncertain and ambiguous information in a system. Now we present the fuzzy
reliability modeling using fuzzy probabilistic Semi-Markov model based on the
fuzzy profust reliability theory through the transition fuzzy probabilities.

Consider a fuzzy probabilistic Semi-Markov model \{\{S_n, I_n\}, n \in \mathbb{N}\}
consisting of \('n'\) states together with transition time. Let \(U = \{s_1, s_2, \ldots, s_n\}\) denote
the universe of discourse.

On this universe we define, a fuzzy success state \(S: S = \{S, \mu_s(s_i); i = 1, 2, \ldots, n\}\)
and a fuzzy failure state \(F: F = \{S, \mu_f(s_i); i = 1, 2, \ldots, n\}\) where \(\mu_s(s_i)\) and \(\mu_f(s_i)\)
are Trapezoidal fuzzy number.

A fuzzy state is just a fuzzy set and fuzzy states are defined to represent the
system level of performance when fuzziness of interest is discarded, the fuzzy
success state and the fuzzy failure state become a conventional success and
failure state respectively. In the conventional reliability theory, one is interested
in the event of transition from system success state to system failure state.
Accordingly we are here interested in the event denoted by \(I_{SF}\) of transition from
the fuzzy success state to the fuzzy failure state. we define

\[
\bar{R}(t_0, t_0 + t) = \bar{P} \left[ I_{SF} \text{ does not occur in the time interval starting}
\right.
\]

from \(t_0\) to \(t_0 + t\).

\(\bar{R}(t_0, t_0 + t)\) is referred as the fuzzy interval reliability of the system in the
time interval starting from \(t_0\) to \(t_0 + t\).

To compute the fuzzy interval reliability we must express \(I_{SF}\). Since both \(S\) and \(F\) are fuzzy states, the transition between them are consequently fuzzy. We
view as a fuzzy event. Apparently \(I_{SF}\) may occur only when some state transition
occur among of \(n\) system states \(\{s_1, s_2, \ldots, s_n\}\), So \(I_{SF}\) can be defined on the universe
\(U_T = \{P_{ij}(t_0), i, j = 1, 2, \ldots, n\}\), where \(P_{ij}(t_0)\) represents the transition fuzzy
probability from \(s_i\) to \(s_j\) with membership function: \(\{\mu_{ij}(P_{ij}(t_0)), i, j = 1, 2, \ldots, n\}\).
(i.e.) \(\bar{T}_{SF} = \{P_{ij}(t_0), \mu_{ij}(P_{ij}(t_0)), i, j = 1, 2, \ldots, n\}\)

Let \(\bar{P}_{F/S}(S_i) = \frac{\mu_f(S_i)}{\mu_f(S_i) + \mu_s(S_i)}\)

Then \(\bar{P}_{F/S}(S_i)\) can be viewed as the grade of membership of \(S_i\) relative \(t_0\), \(S\), \(F\). It is reasonable to say that the fuzzy transition from \(S_i\) to \(S_j\) makes the fuzzy
transition from \(S\) to \(F\) occurs to some extent if and only if the relation
\(\bar{P}_{F/S}(S_j) > \bar{P}_{F/S}(S_i)\) holds: We therefore define
\[ \mu_{\text{ISR}} \left( \overline{P}_0(t_0) \right) = \begin{cases} \overline{P}_{\text{ISR}}(S_j) \cdot \overline{P}_{\text{ISR}}(S_i) & \text{when } \overline{P}_{\text{ISR}}(S_j) > \overline{P}_{\text{ISR}}(S_i) \\ 0 & \text{when } \overline{P}_{\text{ISR}}(S_j) \leq \overline{P}_{\text{ISR}}(S_i) \end{cases} \]

Hence, the fuzzy interval reliability can be expressed as

\[ \overline{R}(t_0, t_0 + t) = 1 - \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_{\text{ISR}} \left( \overline{P}_0(t_0) \right) \left( \overline{P}_0(t_0) \cdot \overline{P}_0(t_0 + t) \cdot \overline{S}_i \right) \]

\[ \overline{R}(t_0, t_0 + t) \] may be further expressed as

\[ \overline{R}(t_0, t_0 + t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \mu_{\text{ISR}} \left( \overline{P}_0(t_0) \right) \cdot \overline{P}_0(t_0 + t) \cdot \overline{S}_i \right) \]

where \( \mu_{\text{ISR}} \left( \overline{P}_0(t_0) \right) = 1 - \mu_{\text{ISR}} \left( \overline{P}_0(t_0) \right) \)

where \( t_0 = 0 \) we have \( \overline{R}(t_0, t_0 + t) = \overline{R}(t) \)

\( \overline{R}(t) \) is referred to as the fuzzy reliability of the system at time \( t \).

5. Numerical Example

In the following, we illustrate the above defined fuzzy reliability model with a real time example. The problem is relating to transaction of different unit and different places. Let us consider the ATM transactions of various places of Tiruchirappalli, Tamil Nadu, India. The operational unit of ATM transactions: S₁ as State Bank of India (SBI), S₂ as Indian Bank (IB), S₃ as Industrial Credit and Investment Corporation of India (ICICI) Bank and S₄ as City Union Bank (CUB). The decisions are usually associated with four states namely Low, Moderate, Medium and High, which are the set of states and the associated connections are the transitions. Since there exists uncertainties in the probabilistic usage information between the state transitions, for each transition we associate fuzzy transition defined as transition fuzzy probabilities obtained as follows.

With the data extracted from the ATM transactions for the period of one day, we find N-th total number of transitions from state i to state j and s the number of success among them the ratio for transition probability \( \frac{s}{N} \).
Since these values (N and S) which are extracted from the path that exists in the specified period of time are not exact, we fuzzy these values using the formula
\[
p - (1 - \alpha) \sqrt{\frac{p(1-p)}{N}}, p + (1 - \alpha) \sqrt{\frac{p(1-p)}{N}}, \text{for } \alpha \in (0,1)
\]
to form the transition fuzzy probabilities as trapezoidal fuzzy number. We have modeled fuzzy probabilistic semi-Markov model with state space \( U = \{\text{SBI, IB, ICICI, CUB}\} \) and transition as transition fuzzy probabilities and depicted were shown in figure 1.

The fuzzy success state for the given state space are interpreted as follows.
\[
S = \{S_1/(0.2526,0.2551,0.2649,0.2674), S_2 / (0.1040,0.1065,0.1135,0.1160), S_3/(0.0745,0.0770,0.0830,0.0855), S_4  / 0.8941,0.8966,0.9034,0.9059)\}
\]

The transition fuzzy probabilities for the given state space are interpreted were given in table 1.

<table>
<thead>
<tr>
<th></th>
<th>SBI</th>
<th>IB</th>
<th>ICICI</th>
<th>CUB</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(0.2526,0.2551,0.2649,0.2674)</td>
<td>(0.1435,0.1460,0.1540,0.1565)</td>
<td>(0.1633,0.1658,0.1742,0.1767)</td>
<td>(0.5419,0.5444,0.5556,0.5581)</td>
</tr>
<tr>
<td></td>
<td>(0.1831,0.1846,0.1944,0.1969)</td>
<td>(0.1040,0.1065,0.1135,0.1160)</td>
<td>(0.1040,0.1065,0.1135,0.1160)</td>
<td>(0.3820,0.3845,0.3955,0.3980)</td>
</tr>
<tr>
<td></td>
<td>(0.1139,0.1164,0.1236,0.1261)</td>
<td>(0.0646,0.0671,0.0729,0.0754)</td>
<td>(0.0745,0.0770,0.0830,0.0855)</td>
<td>(0.2427,0.2452,0.2548,0.2573)</td>
</tr>
<tr>
<td></td>
<td>(0.4220,0.4245,0.4355,0.4380)</td>
<td>(0.2427,0.2452,0.2548,0.2523)</td>
<td>(0.2625,0.2650,0.2750,0.2775)</td>
<td>(0.8941,0.8966,0.9034,0.9059)</td>
</tr>
</tbody>
</table>

for \( i, j = \text{SBI, IB, ICICI, CUB} \)

Table 1. Transition Fuzzy Probabilities

Thus the fuzzy reliability of ATM transactions is given by
We observe that the fuzzy reliability is reasonable for the fuzzy success. Which were shown in figure 2.

![Fuzzy Reliability](image)

**Figure 2. Fuzzy Reliability**

### 6. Conclusion

In this paper, we have defined a non-homogeneous fuzzy semi-Markov model and presented a non-homogeneous fuzzy semi-Markov model approach to the dynamic evolution of ATM transactions defined by transition probabilities. Also we have extended a new method for finding fuzzy system reliability using fuzzy Profust reliability theory. We have applied the above method for ATM transactions and obtained the result as a Trapezoidal fuzzy number. The above constructed model could be used as a predictive device to study flow of ATM transactions. Such predictions could be useful to the entire bank for improve the usage of ATM transactions.

### References


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