An Investigation on Some Classes of Super Strongly Perfect Graphs

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Abstract

A Graph G is Super Strongly Perfect Graph if every induced sub graph H of G possesses a minimal dominating set that meets all the maximal complete sub graphs of H. Bipartite graphs, Complete graphs etc., are some of the most important classes of Super Strongly Perfect graphs. Here, we summarize the results concerning Super Strongly Perfect graphs. We investigate some classes of Super Strongly Perfect graphs and we investigate the structure of Super Strongly Perfect Graphs.

Keywords: Super Strongly Perfect Graph, maximal complete sub graph, minimal dominating set

1. Introduction

In this paper, graphs are finite, undirected and simple, that is, they have no loops or multiple edges. Let G be a graph. A Complete graph is a simple undirected graph in which every pair of distinct vertices is connected by
a unique edge. The complete graph on \( n \) vertices is denoted by \( K_n \). A \textit{Path} in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence. A path may be infinite, but a finite path always has a first vertex, called its start vertex, and a last vertex, called its end vertex. Both of them are called terminal vertices of the path. A \textit{Cycle} is a path such that the start vertex and end vertex are the same and it is denoted by \( C_n \). The number of vertices in \( C_n \) equals the number of edges. The cycle graph with even number of vertices is called an even cycle and it is denoted by \( C_{2n} \). The cycle graph with odd number of vertices is called an odd cycle and it is denoted by \( C_{2n+1} \).

A \textit{Maximal Complete Subgraph} in \( G \) is a set \( X \subseteq V(G) \) of pairwise adjacent vertices. A subset \( D \) of \( V(G) \) is called a \textit{Dominating set} if every vertex in \( V \setminus D \) is adjacent to at least one vertex in \( D \). A subset \( S \) of \( V \) is said to be a \textit{Minimal Dominating Set} if \( S - \{u\} \) is not a dominating set for any \( u \in S \).

2. Overview of the Paper

Super Strongly Perfect graph is a new graph which was defined by U. S. R. Murty and its Characterization has been given as an open problem. Many classes of Super Strongly Perfect graphs like Bipartite graphs, Complete graphs etc., have been discussed [1,4]. In this paper we have discussed some other classes of Super Strongly Perfect graphs like Barbell graphs, Lollipop graphs, Tadpole graphs, Pan graphs and Banner graphs.

3. Super Strongly Perfect Graph

A Graph \( G = (V, E) \) is Super Strongly Perfect if every induced sub graph \( H \) of \( G \) possesses a minimal dominating set that meets all the maximal complete sub graphs of \( H \). Every Super Strongly Perfect graph with maximal complete sub graph \( K_2 \) is isomorphic to any bipartite graph. Any arbitrary graph which does not contain an odd cycle of length at least five is isomorphic to every Super Strongly Perfect graph with maximal complete sub graph \( K_n \) where \( n \geq 3 \) [4]. Every Complete graph is Super Strongly Perfect [1]. Every Cycle graph \( C_{2n} \) where \( n \) is the number of vertices, is Super Strongly Perfect. Every Cycle graph \( C_{2n+1} \) where \( n \) is the number of vertices, is Non-Super Strongly Perfect [3].

Example 1

![Figure 1: Super Strongly Perfect Graph](image-url)
Classes of super strongly perfect graphs

Here, \{1, 3, 5\} is a minimal dominating set which meets all maximal complete subgraphs of G.

**Example 2**

![Figure 2: Non-Super Strongly Perfect Graph](image)

Here, \{1, 3, 6, 8, 10\} is a minimal dominating set which does not meet all maximal complete subgraphs of G.

**3. 1. Theorem [3]**

Let \( G = (V, E) \) be a graph with number of vertices \( n \), where \( n \geq 5 \). If \( G \) contains an odd cycle as an induced subgraph, then \( G \) is Non-Super Strongly Perfect.

**3. 2. Theorem [4]**

Let \( G = (V, E) \) be a graph with number of vertices \( n \), where \( n \geq 5 \). Then \( G \) is Super Strongly Perfect if and only if it does not contain an odd cycle as an induced subgraph.

**4. Barbell Graph**

The \( n \)-**Barbell graph** is the simple graph obtained by connecting two copies of a complete graph \( K_n \) by a bridge.

**Example 3**

![Figure 3: 4 - Barbell Graph](image)

**4. 1. Theorem**

Every \( n \)-Barbell graph is Super Strongly Perfect.
Proof:
Let G be an n - Barbell graph.
Since every Complete graph is Super Strongly Perfect.
If we join two Complete graphs with an edge (i.e.,) $K_2$, again the graph is Super Strongly Perfect, with a minimal dominating set of 2 vertices from the joining edge of two Complete graphs. Hence G is Super Strongly Perfect.

5. Lollipop Graph

The $(m, n)$ - Lollipop graph is the graph obtained by joining a complete graph $K_m$ to a path graph $P_n$ with a bridge and it is denoted by $L_{m,n}$.

Example 4

![Figure 4: $L_{6,3}$](image)

5.1. Theorem
Every Lollipop graph $L_{m,n}$ is Super Strongly Perfect.
Proof:
Let G be a Lollipop graph.
⇒ G is constructed by joining $K_n$ a complete graph on n vertices with a $P_m$ a path on m vertices by an edge (i.e.,) $K_2$.
Since every Complete graph and Path are Super Strongly Perfect.
If we join two Super Strongly Perfect graphs with an edge (i.e.,) $K_2$, again the graph is Super Strongly Perfect, with a minimal dominating set of 2 vertices from the joining edge of two Super Strongly Perfect graphs.
Hence G is Super Strongly Perfect.

6. Tadpole Graph

The $(m, n)$ - Tadpole graph, also called a dragon graph, is the graph obtained by joining a cycle graph $C_m$ to a path graph $P_n$ with a bridge and it is denoted by $T_{m,n}$. The $(m,1)$-tadpole graph is sometimes known as the m-pan graph. The particular cases of the $(3,1)$- and $(4,1)$-tadpole graphs are also known as the paw graph and banner graph, respectively.
Example 5

6.1. Theorem
Every Tadpole graph $T_{2m,n}$ is Super Strongly Perfect.

Proof:
Let $G$ be Tadpole graph.
$\Rightarrow G$ is constructed by joining $C_{2m}$ an cycle graph on $2m$ vertices with a $P_n$ a path on $n$ vertices by an edge (i.e.,) $K_2$.
Since every Even Cycle graph and Path are Super Strongly Perfect.
If we join two Super Strongly Perfect graphs with an edge (i.e.,) $K_2$, again the graph is Super Strongly Perfect, with a minimal dominating set of 2 vertices from the joining edge of two Super Strongly Perfect graphs.
Hence $G$ is Super Strongly Perfect.

6.2. Theorem
Every Tadpole graph $T_{2m+1,n}$ is Non - Super Strongly Perfect.

Proof:
Let $G$ be Tadpole graph.
$\Rightarrow G$ is constructed by joining $C_{2m+1}$ an cycle graph on $2m+1$ vertices with a $P_n$ a path on $n$ vertices by an edge (i.e.,) $K_2$.
Since every Odd Cycle graph is Non - Super Strongly Perfect, and every Path is Super Strongly Perfect.
If we join a Non - Super Strongly Perfect graph with a Super Strongly Perfect by an edge (i.e.,) $K_2$, again the graph is Non -Super Strongly Perfect.
Hence $G$ is Non - Super Strongly Perfect.

7. Pan Graph

The n-pan graph is the graph obtained by joining a cycle graph $C_n$ to a singleton graph $K_1$ with a bridge. The special case of the 3-pan graph is sometimes known as the paw graph [2]. Every n- Pan graph has n vertices and n edges.
7.1. Theorem
Every n-Pan graph, $n \geq 4$ where $n$ is even, is Super Strongly Perfect.

Proof:
Let $G$ be an n-Pan graph, $n \geq 4$ where $n$ is even.
$\Rightarrow G$ does not contain an odd cycle as an induced sub graph.
Now, by the theorem 3.2, $G$ is Super Strongly Perfect.
Hence every n-Pan graph, $n \geq 4$ where $n$ is even, is Super Strongly Perfect.

7.2. Theorem
Every n-Pan graph, where $n$ is odd, $n > 3$, is non-Super Strongly Perfect.

Proof:
Let $G$ be an n-Pan graph, where $n$ is odd, $n > 3$.
$\Rightarrow G$ contains an odd cycle as an induced sub graph.
Now, by the theorem 3.1, $G$ is non-Super Strongly Perfect.
Hence every n-Pan graph, where $n$ is odd, $n > 3$, is non-Super Strongly Perfect.

7.3. Remark
Every 3-Pan graph is Super Strongly Perfect.

7.4. Proposition
Every n-Pan graph, $n \geq 4$ where $n$ is even, which is Super Strongly Perfect contains a minimal dominating set of cardinality $\frac{n}{2}$.

7.5. Proposition
Every n-Pan graph, $n \geq 4$ where $n$ is even, which is Super Strongly Perfect has $\gamma = \frac{n}{2}$ and $\overline{\gamma} = 2$.

7.6. Proposition
Every n-Pan graph, $n \geq 4$ where $n$ is even, which is Super Strongly Perfect has $n$ maximal complete sub graphs $K_2$. 
8. Banner Graph

The 4-pan graph is the banner graph. The banner graph is the (4,1)-tadpole graph illustrated below. It could perhaps also be termed the “P graph”. It has 5 vertices and 5 edges.

**Example 7**

![Banner Graph](image)

**Figure 7:** Banner Graph.

8.1. Theorem
Every Banner graph is Super Strongly Perfect.

**Proof:**
Let G be Banner graph.
⇒ G does not contain an odd cycle as an induced sub graph.
Now, by the theorem 3.2, G is Super Strongly Perfect.
Hence every Banner graph is Super Strongly Perfect.

8.2. Proposition
Every Banner graph which is Super Strongly Perfect contains a minimal dominating set of cardinality 2.

8.3. Proposition
Every Banner graph which is Super Strongly Perfect has $\gamma = 2$ and $\overline{\gamma} = 2$.

8.4. Proposition
Every Banner graph which is Super Strongly Perfect has 5 maximal complete sub graphs $K_2$.

9. Conclusion
We have discussed some graphs classes like Barbell graphs, Lollipop graphs, Tadpole graphs, Pan graphs and Banner graphs to characterize the structure of Super Strongly Perfect Graphs.
References


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