On Dynamics of Randomly Alternating Prisoner's Dilemma Game (RAPDG)

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Abstract

In this article, our model consists of two players and two choices for each player. In this model, there is one option for one player in each round called leader. The two players have the same chance to be leader. In this model each two consecutive rounds represent one unit. We consider strategies realized by simple transition rules depending on the previous outcome. We consider homogeneous population of strategy $\Omega$, and ask for the most favorable adaptation. Any parameter $s$ in $\Omega$ changes according to the adaptive dynamics $\dot{s} = \frac{\partial F(\Omega, \Omega')}{\partial s}$ where $F$ is the payoff for $\Omega$-player against $\Omega'$-player and this evaluated at $\Omega' = \Omega$.

Keywords: Adaptive Dynamics, Alternating Prisoner's Dilemma Game, Transition matrix.

1 Introduction

Many examples of reciprocal altruism are modeled by an Alternating Prisoner's Dilemma Game (APDG). In this game we have two players I and II and two choices $C$ to cooperate and $D$ to defect. In each round of the game one of the two
players choose his option and the other player reply with his option in another round. For each round there is a single option for one of the two players. This player is called leader (or donor) and the other is called recipient. Consequently the leader control what the outcome is going to be.

Each player in this game has the same chance to be leader. That is, the two players in this game alternate the leader role. There are two types of this model, Strictly Alternating (SA) where the two players exchange the leader's role every round and the Randomly Alternating (RA) where the two players exchange the leader's role randomly. In this paper we shall be interested in (RAPDG).

It is argued in Boyd (1988), Nowak and Sigmund (1994), and Frean (1994) that mutual aid is often given alternatingly: a good turn to a partner in the hope of an eventual return. A vampire bat feeding a hungry fellow bat is obviously performing a cooperative move; this can be repaid, not simultaneously, but with a time lag. The same holds for a wolf that joins a fight to help its fellow, a bird emitting a warning call, or a monkey scratching another monkey's back. It needs no help right now; it may need it latter. A similar principle operates in economic exchanges within simple social groups, for instance in the bartering of goods and services in households and among neighbors.

2 The Transition Matrix of RAPDG

For the RAPDG, leader decides between the two choices $C$ and $D$, option $C$ yields $\alpha$ points to the donor and $\beta$ points to the recipient, while option $D$ yields $\gamma$ points to the donor and $\delta$ points to the recipient. In a single round, option $D$ is better than $C$ for the leader. We shall assume that the cost to the donor is less than the benefit that brings to the recipient. The loss is $\gamma - \alpha$ and the benefit is $\beta - \delta$, then we have

$$0 < \gamma - \alpha < \beta - \delta$$

(1)

Now we consider two consecutive rounds in which the players exchange the leader role in turn. If both play $C$, both earn $\alpha + \beta$, which we denote by $R$, if both play $D$, both earn $\gamma + \delta$, which we denote by $P$, but if one plays $C$ and the other $D$, then the cooperator earns $\alpha + \delta$ which we denote by $S$ and the defectors earns $\gamma + \beta$ which we denote by $T$. Thus from (1), we have

$$T > R > P > S$$

(2)

and

$$2R > T + S$$

(3)

The inequalities (2) and (3) are usual conditions for the payoff for the Simultaneous Prisoner's Dilemma Game (SPDG), where the two players choose their options at the same time. Adding, we obtain

$$T + S = P + R$$

(4)

Conditions (2), (3) and (4) describes the APD game, while condition (4) means in the SPDG that the cost from switching $D$ to $C$ is the same against a defector as against a cooperator ($P - S = T - R$). The four states of outcomes $\alpha, \beta, \gamma$ and $\delta$
represents the possible payoff obtained by one of the two players for one round. If we denote these outcomes by 1, 2, 3 and 4 respectively, then the possible strategies for each player can be represented by the quadruple \((u_1, u_2, u_3, u_4)\), where \(u_i\) denotes the probability to play \(C\) after outcome \(i\). These probabilities are independent of the random decision of who is going to be leader. If a player with strategy \(P = (p_1, p_2, p_3, p_4)\) and probability \(y\) to play \(C\) in the first round match against a player with strategy \(Q = (q_1, q_2, q_3, q_4)\) and probability \(y'\) to play \(C\) in the first round, then the transition matrix for the match from one round to the next is given by

\[
M = \frac{1}{2} \begin{pmatrix}
    p_1 & q_2 & 1 - p_1 & 1 - q_2 \\
    p_2 & q_1 & 1 - p_2 & 1 - q_1 \\
    p_3 & q_4 & 1 - p_3 & 1 - q_4 \\
    p_4 & q_3 & 1 - p_4 & 1 - q_3
\end{pmatrix}
\]

For instance the transition probability from state 1 to state 2 for the \(P\)-player equals the product of the probability that the \(Q\)-player is leader, which equals \(\frac{1}{2}\) and the probability \(q_2\) that the \(Q\)-player cooperates after receiving \(\beta\) in the previous round.

3 The Payoff of a Player in RAPD Game

If \(\rho\) denote the probability that the game proceed in the next round, then we have two cases to study, when \(\rho = 1\) (limiting case) and \(\rho < 1\).

1-For \(\rho = 1\)

The stationary distribution of transition matrix \(M\) is the eigenvector \(\Pi\) where

\[
\Pi = (\pi_1, \pi_2, \pi_3, \pi_4) \ ; \ \sum_{i=1}^{4} \pi_i = 1
\]

which correspond to the left eigenvalue of \(M\). Hence \(\Pi\) is given by

\[
\Pi M = \Pi
\]

This gives us the following four equations

\[
\begin{align*}
\pi_1(p_1 - 2) + \pi_2p_2 + \pi_3p_3 + \pi_4p_4 &= 0 \quad \text{(i)} \\
\pi_1q_2 + \pi_2(q_1 - 2) + \pi_3q_4 + \pi_4q_3 &= 0 \quad \text{(ii)} \\
\pi_1(1 - p_1) + \pi_2(1 - p_2) - \pi_3(1 + p_3) + \pi_4(1 - p_4) &= 0 \quad \text{(iii)} \\
\pi_1(1 - q_2) + \pi_2(1 - q_1) + \pi_3(1 - q_4) - \pi_4(1 + q_3) &= 0 \quad \text{(iv)}
\end{align*}
\]

under the condition \(\sum_{i=1}^{4} \pi_i = 1\)

From equations (i) to (iii) (or (ii) and (iv)) we deduce that

\[
\pi_1 - \pi_2 + \pi_3 - \pi_4 = 0 \quad \text{(vi)}
\]

Equations (v) and (vi) leads to

\[
\pi_3 = \frac{1}{2} - \pi_1
\]

(7)

Also for equations (i), (iii) and (v) we get

\[
\pi_4 = \frac{1}{2} - \pi_2
\]

(8)

Let \(\pi_1 = \pi\) and \(\pi_2 = \pi'\), then (7) and (8) leads to
\[ \Pi = (\pi, \pi', \frac{1}{2} - \pi, \frac{1}{2} - \pi') \]

Substituting with \( \Pi \) in (6), yield the following equations

\[ \pi(p_1 - 2) + \pi'p_2 + (\frac{1}{2} - \pi)p_3 + (\frac{1}{2} - \pi')p_4 = 0 \]
\[ \pi q_2 + \pi'(q_3 - 2) + (\frac{1}{2} - \pi)q_4 + (\frac{1}{2} - \pi')q_3 = 0 \]
\[ \pi(1 - p_1) + \pi'(1 - p_2) - \left(\frac{1}{2} - \pi\right)(1 + p_3) + \left(\frac{1}{2} - \pi'\right)(1 - p_4) = 0 \]
\[ \pi(1 - q_2) + \pi'(1 - q_1) + \left(\frac{1}{2} - \pi\right)(1 - q_4) - \left(\frac{1}{2} - \pi'\right)(1 + q_3) = 0 \]

This set of equations can be reduced to the following two equations

\[ (p_3 - p_1 + 2)\pi + (p_4 - p_2)\pi' - \frac{1}{2}(p_3 + p_4) = 0 \quad (vii) \]
\[ (q_2 - q_4)\pi + (q_1 - q_3 - 2)\pi' + \frac{1}{2}(q_3 + q_4) = 0 \quad (viii) \]

Solving equations (vii) and (viii) leads to

\[ \pi' = \frac{(q_3 + q_4)(2 + p_3 - p_1) - (p_3 + p_4)(q_4 - q_2)}{2[(2 + p_3 - p_1)(2 + q_3 - q_1) - (p_4 - p_2)(q_4 - q_2)]} \quad (9) \]

and

\[ \pi = \frac{(p_3 + p_4)(2 + q_3 - q_1) - (q_3 + q_4)(p_4 - p_2)}{2[(2 + p_3 - p_1)(2 + q_3 - q_1) - (p_4 - p_2)(q_4 - q_2)]} \quad (10) \]

Now if \( F \) is the payoff for the \( P \)-player, then \( F \) is given by

\[ F = \Pi\left(\begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array}\right) = \left(\begin{array}{cccc} \pi & \pi' & \frac{1}{2} - \pi & \frac{1}{2} - \pi' \end{array}\right)\left(\begin{array}{c} \alpha \\ \beta \\ \gamma \\ \delta \end{array}\right) = \pi(\alpha - \gamma) + \pi'(\beta - \delta) + \frac{1}{2}(\gamma + \delta) \]

Using (9) and (10) in \( F \), we get

\[ F = \frac{1}{2}(\gamma + \delta) + \frac{(\alpha - \gamma)[\tau(2 + \nu') - \tau\mu] + (\beta - \delta)[\tau'(2 + \nu) - \tau\mu']}{2[2(2 + \nu)(2 + \nu') - \mu\nu']} \quad (11) \]

Where \( \nu = p_3 - p_1 \), \( \mu = p_4 - p_2 \) and \( \tau = p_3 + p_4 \) for \( P \)-player and similarly \( \nu', \mu' \), and \( \tau' \) for \( Q \)-player. In this case we note that the initial probability \( y \) and \( y' \) play no role.

2- For \( \rho < 1 \)

The total payoff \( F \) for the \( P \)-player against \( Q \)-player in this case is given by

\[ F = \frac{1}{2}[\gamma A + (1 - \gamma)\Gamma + \gamma' B + (1 - \gamma')\Delta] \]
\[ = \frac{1}{2}[\Gamma' + \Delta + \gamma(A - \Gamma) + \gamma'(B - \Delta)] \quad (12) \]

Where \( A, B, \Gamma \) and \( \Delta \) are the expected payoffs for the \( P \)-player, given that the first round of the game resulted in \( \alpha, \beta, \gamma \) and \( \delta \) respectively.

Then we have
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\[ A = \alpha + \frac{\rho}{2} [p_1 A + q_2 B + (1 - p_1) \Gamma + (1 - q_2) \Delta] \]

\[ B = \beta + \frac{\rho}{2} [p_2 A + q_1 B + (1 - p_2) \Gamma + (1 - q_1) \Delta] \]

\[ \Gamma = \gamma + \frac{\rho}{2} [p_3 A + q_4 B + (1 - p_3) \Gamma + (1 - q_4) \Delta] \]

\[ \Delta = \delta + \frac{\rho}{2} [p_4 A + q_3 B + (1 - p_4) \Gamma + (1 - q_3) \Delta] \]

Where \( \frac{\rho}{2} \) is the probability that P-player will be leader. We can rewrite equations (13) as following

\[
\begin{align*}
(1 - \frac{\rho}{2} p_1) A - \frac{\rho}{2} q_2 B - \frac{\rho}{2} (1 - p_1) \Gamma - \frac{\rho}{2} (1 - q_2) \Delta & = \alpha \\
-\frac{\rho}{2} p_2 A + (1 - \frac{\rho}{2} q_1) B - \frac{\rho}{2} (1 - p_2) \Gamma - \frac{\rho}{2} (1 - q_1) \Delta & = \beta \\
-\frac{\rho}{2} p_3 A - \frac{\rho}{2} q_4 B + (1 - \frac{\rho}{2} (1 - p_3)) \Gamma - \frac{\rho}{2} (1 - q_4) \Delta & = \gamma \\
-\frac{\rho}{2} p_4 A - \frac{\rho}{2} q_3 B - \frac{\rho}{2} (1 - p_4) \Gamma - (1 - \frac{\rho}{2} (1 - q_3)) \Delta & = \delta
\end{align*}
\]

Now if

\[ H = \begin{pmatrix}
1 - \frac{\rho}{2} p_1 & -\frac{\rho}{2} q_2 & -\frac{\rho}{2} (1 - p_1) & -\frac{\rho}{2} (1 - q_2) \\
-\frac{\rho}{2} p_2 & 1 - \frac{\rho}{2} q_1 & -\frac{\rho}{2} (1 - p_2) & -\frac{\rho}{2} (1 - q_1) \\
-\frac{\rho}{2} p_3 & -\frac{\rho}{2} q_4 & 1 - \frac{\rho}{2} (1 - p_3) & -\frac{\rho}{2} (1 - q_4) \\
-\frac{\rho}{2} p_4 & -\frac{\rho}{2} q_3 & -\frac{\rho}{2} (1 - p_4) & 1 - \frac{\rho}{2} (1 - q_3)
\end{pmatrix} = I - \rho M
\]

Where \( I \) is the 4-unit matrix has full rank, hence equations (13') in matrix form takes the form

\[ HK = X \]

where

\( K = \begin{pmatrix} A \\ B \\ \Gamma \\ \Delta \end{pmatrix} \), and \( X = \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \)

If \( \rho \to 0 \), then the two players have a very small chance to play with each other again, put \( \rho \to 0 \) in (13') we get \( H \to I, K \to X \), which means that the expected payoffs for the two players equal to the ones obtained in the first round of the game. If \( \rho \to 1 \), there is a high probability that the game will proceed in the next round.

Equations in (13') can be written respectively as follows

\[ \Gamma + \Delta + p_1 (A - \Gamma) + q_2 (B - \Delta) + \frac{2}{\rho} (\alpha - A) = 0, \]

\[ \Gamma + \Delta + p_2 (A - \Gamma) + q_1 (B - \Delta) + \frac{2}{\rho} (\beta - B) = 0, \]
Solving equations from (15) to (17) in \((A - \Gamma), (B - \Delta)\) and \((\Gamma + \Delta)\), we get

\[
A - \Gamma = 2 \frac{(\alpha - \gamma)(2 + \rho v') - (\beta - \delta) \rho \mu'}{(2 + \rho v') (2 + \rho v) - \rho^2 \mu' \mu},
\]

\[
B - \Delta = 2 \frac{(\beta - \delta)(2 + \rho v) - (\alpha - \gamma) \rho \mu}{(2 + \rho v') (2 + \rho v) - \rho^2 \mu' \mu}
\]

and

\[
(\Gamma + \Delta) = \frac{1}{1 - \rho} \left[ (\gamma + \delta) + \frac{\rho \tau}{2} (A - \Gamma) + \frac{\rho \tau'}{2} (B - \Delta) \right]
\]

Equations (19), (20) and (21) give the payoff \(F\) (see (12)) for the P-player.

4 The Adaptive Dynamics For The APDG

We consider a homogeneous population of \(\Omega\)-players, and ask for the most favorable adaptation. If an individual was permitted a small deviation from strategy \(\Omega\), which direction would be most favorable. If \(s\) is any parameter belonging to \(\Omega\) where \(s\) can be \(y, p_1, p_2, \ldots, \text{or } p_4\), then \(s\) changes according to the adaptive dynamics, by

\[
\dot{s} = \frac{\partial F}{\partial s}
\]

Where the right hand side of (22) evaluated at \(\Omega = \Omega'\). The differentiation of \(F\) yields very cumbersome computation, so we shall use the implicit function theorem. Equations (14) can be written as

\[
f_i(p_1, p_2, p_3, p_4, q_1, q_2, q_3, q_4, A, B, \Gamma, \Delta, \alpha, \beta, \gamma, \delta) = 0, \quad i = 1, 2, 3, 4
\]

where

\[
f_1 = A \left( 1 - \frac{\rho}{2} p_1 \right) - \frac{\rho}{2} q_2 B - \frac{\rho}{2} (1 - p_1) \Gamma - \frac{\rho}{2} (1 - q_2) \Delta - \alpha,
\]

\[
f_2 = -A \frac{\rho}{2} p_2 + \left( 1 - \frac{\rho}{2} q_1 \right) B - \frac{\rho}{2} (1 - p_2) \Gamma - \frac{\rho}{2} (1 - q_1) \Delta - \beta,
\]

\[
f_3 = -A \frac{\rho}{2} p_3 - \frac{\rho}{2} q_4 B + \left( 1 - \frac{\rho}{2} (1 - p_3) \right) \Gamma - \frac{\rho}{2} (1 - q_4) \Delta - \gamma,
\]

\[
f_4 = -A \frac{\rho}{2} p_4 - \frac{\rho}{2} q_3 B - \frac{\rho}{2} (1 - p_4) \Gamma - \left( 1 - \frac{\rho}{2} (1 - q_3) \right) \Delta - \delta,
\]

and

\[
H = \frac{\partial (f_1, f_2, f_3, f_4)}{\partial (A, B, \Gamma, \Delta)}
\]

Hence, by the implicit function theorem, we have
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\[
\frac{\partial (A, B, \Gamma, \Delta)}{\partial (p_1, p_2, p_3, p_4)} = -\left(\frac{\partial (f_1, f_2, f_3, f_4)}{\partial (A, B, \Gamma, \Delta)}\right)^{-1} \frac{\partial (f_1, f_2, f_3, f_4)}{\partial (p_1, p_2, p_3, p_4)}
\]

where

\[
\frac{\partial (f_1, f_2, f_3, f_4)}{\partial (p_1, p_2, p_3, p_4)} = \begin{pmatrix}
-\frac{\rho}{2} A + \frac{\rho}{2} \Gamma & 0 & 0 & 0 \\
0 & -\frac{\rho}{2} A + \frac{\rho}{2} \Gamma & 0 & 0 \\
0 & 0 & -\frac{\rho}{2} A + \frac{\rho}{2} \Gamma & 0 \\
0 & 0 & 0 & -\frac{\rho}{2} A + \frac{\rho}{2} \Gamma
\end{pmatrix}
\]

\[
= -\frac{\rho}{2} (A - \Gamma) \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Then, we have

\[
\frac{\partial (A, B, \Gamma, \Delta)}{\partial (p_1, p_2, p_3, p_4)} = -H^{-1} \left(-\frac{\rho}{2} (A - \Gamma) I\right) = \frac{\rho}{2} (A - \Gamma) Z,
\]

where \( Z = H^{-1} \) and since \( H = I - \rho M \), then we have

\[
Z = H^{-1} = (I - \rho M)^{-1} = I + \rho M + (\rho M)^2 + \ldots
\]

Such that

\[
l + \rho M + (\rho M)^2 = \begin{pmatrix}
z_{11} & z_{12} & z_{13} & z_{14} \\
z_{21} & z_{22} & z_{23} & z_{24} \\
z_{31} & z_{32} & z_{33} & z_{34} \\
z_{41} & z_{42} & z_{43} & z_{44}
\end{pmatrix} = Z, \text{ where}
\]

\[
z_{11} = \frac{1}{2} \rho p_1 + \frac{1}{4} \rho^2 p_2 q_2 + \frac{1}{4} \rho^2 p_1^2 + \frac{1}{4} \rho^2 p_3(1 - p_1) + \frac{1}{4} \rho^2 p_4(1 - q_2) + 1
\]

\[
z_{12} = \frac{1}{2} \rho q_2 + \frac{1}{4} \rho^2 p_1 q_2 + \frac{1}{4} \rho^2 p_3 q_2 + \frac{1}{4} \rho^2 q_4(1 - p_1) + \frac{1}{4} \rho^2 q_3(1 - q_2),
\]

\[
z_{14} = \frac{1}{2} \rho (1 - q_3) + \frac{1}{4} \rho^2 q_3(1 - q_4) + \frac{1}{4} \rho^2 p_4(1 - q_2) + \frac{1}{4} \rho^2 (1 - q_3)^2 + \frac{1}{4} \rho^2 (1 - q_4) + 1
\]

We note that

\[
det H = \frac{(1 - \rho)}{4} [2 + \rho(v + \mu)][2 + \rho(v - \mu)]
\]

\[
z_{11} = z_{22}, \quad z_{12} = z_{21}, \quad z_{13} = z_{24}, \quad z_{14} = z_{23}, \text{ and } z_{31} = z_{42}, \quad z_{32} = z_{41},
\]

\[
z_{33} = z_{44}, \quad z_{34} = z_{43}.
\]

Using (24), we get

\[
z_{11} + z_{12} = \frac{\rho}{4 \det H} (2 + \rho(v - \mu)) \left( \frac{2}{\rho} - 2 + p_3 + p_4 \right), \tag{26}
\]

\[
z_{13} + z_{14} = \frac{\rho}{4 \det H} (2 + \rho(v - \mu))(2 - p_1 - p_2), \tag{27}
\]

\[
z_{31} + z_{32} = \frac{\rho}{4 \det H} (2 + \rho(v - \mu))(p_3 + p_4), \tag{28}
\]
\[ z_{33} + z_{34} = \frac{\rho}{4 \det H} (2 + \rho(v - \mu)) \left(\frac{2}{\rho} - p_1 - p_2\right), \] (29)

and

\[ (z_{11} + z_{12}) - (z_{31} + z_{32}) = \frac{\rho}{4 \det H} (2 + \rho(v - \mu)) \left(\frac{2}{\rho} - 2\right), \] (30)

\[ (z_{13} + z_{14}) - (z_{33} + z_{34}) = \frac{\rho}{4 \det H} (2 + \rho(v - \mu)) \left(\frac{2}{\rho} - \frac{2}{\rho}\right) \] (31)

Now from (12) and (22), we get the following

\[ \dot{y} = \frac{1}{2} \frac{\partial F}{\partial y} = \frac{1}{2} \frac{\partial}{\partial y} \left[ \Gamma + \Delta + y(A - \Gamma) + y'(B - \Delta) \right] \]

\[ \dot{y} = \frac{1}{2} (A - \Gamma), \] (32)

\[ \dot{p}_1 = \frac{\partial F}{\partial p_1} = \frac{1}{2} \left[ y \left( \frac{\partial A}{\partial p_1} + \frac{\partial B}{\partial p_1} \right) + (1 - y) \left( \frac{\partial \Gamma}{\partial p_1} + \frac{\partial \Delta}{\partial p_1} \right) \right] \]

\[ = \frac{\rho}{4} \left[ y(z_{11} + z_{12}) + (1 - y)(z_{31} + z_{32}) \right] (A - \Gamma), \] (33)

and

\[ \dot{p}_3 = \frac{\partial F}{\partial p_3} = \frac{1}{2} \left[ y \left( \frac{\partial A}{\partial p_3} + \frac{\partial B}{\partial p_3} \right) + (1 - y) \left( \frac{\partial \Gamma}{\partial p_3} + \frac{\partial \Delta}{\partial p_3} \right) \right] \]

\[ = \frac{\rho}{4} \left[ y(z_{13} + z_{14}) + (1 - y)(z_{33} + z_{34}) \right] (A - \Gamma) \] (34)

where, \( \dot{p}_1 = \dot{p}_2 \) and \( \dot{p}_3 = \dot{p}_4 \).

We note that the sign of \( \dot{p}_1 \) and \( \dot{p}_3 \) is the same as the sign of \( (A - \Gamma) \) which has been computed in (19). Using the previous expressions together with the equations (26) – (29), we get the following result.

**Theorem.** The adaptive dynamics for the RAPD game is given by

(1) For \( \rho < 1 \)

\[ \dot{y} = \frac{1}{2} (A - \Gamma) \]

\[ \dot{p}_1 = \dot{p}_2 = \frac{\rho^2}{4(1 - \rho)} (2 + \rho(v + \mu))^{-1} \left[ y \left( \frac{2}{\rho} - 2\right) + \tau \right] (A - \Gamma) \] (35)

and

\[ \dot{p}_3 = \dot{p}_4 = \frac{\rho^2}{4(1 - \rho)} (2 + \rho(v + \mu))^{-1} \left[ y \left( \frac{2}{\rho} - \frac{2}{\rho} \right) + \left( \frac{2}{\rho} - p_1 - p_2 \right) \right] (A - \Gamma) \] (36)

where,

\[ A - \Gamma = 2 \frac{(\alpha - \gamma)(2 + \rho v) - (\beta - \delta) \rho \mu}{(2 + \rho(v - \mu))(2 + \rho(v + \mu))} \] (37)

(2) For \( \rho = 1 \) (limiting case)
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\[
\dot{p}_1 = \dot{p}_2 = \frac{\tau[(\alpha - \gamma)(2 + v) - (\beta - \delta)\mu]}{2(2 + v + \mu)^2(2 + v - \mu)},
\]
and
\[
\dot{p}_3 = \dot{p}_4 = \frac{(2 + v + \mu - \tau)[(\alpha - \gamma)(2 + v) - (\beta - \delta)\mu]}{2(2 + v + \mu)^2(2 + v - \mu)}
\]

For every case we note that

1. \(\dot{p}_1 = \dot{p}_2\) and \(\dot{p}_3 = \dot{p}_4\) and this means that the optimal adaptation depends only on whether there was a C or a D in the previous round, and not on who actually implemented it.

2. \(\dot{y}\) and all \(\dot{p}_i\) has positive sign and this gives the zone of cooperation, which is defined by

\[
(\alpha - \gamma)(2 + \rho v) > (\beta - \delta)\rho \mu
\]

and it is independent of \(\gamma\).

If \(\frac{\beta - \delta}{\gamma - \alpha}\), which is just \(\frac{T - P}{T - R}\), and hence greater than one i.e. \(\eta > 1\) this deduced from

\[
0 < \frac{\gamma - \alpha < \beta - \delta}{\gamma - \alpha}
\]

dividing every side of the previous inequality by \(\gamma - \alpha\) then we have

\[
0 < 1 < \eta = \frac{\beta - \delta}{\gamma - \alpha}
\]

We get from (40), that \((2 + \rho v) < \frac{(\beta - \delta)}{(\alpha - \gamma)}\rho \mu\). Then

\[
(2 + \rho v) < -\frac{(\beta - \delta)}{(\gamma - \alpha)}\rho \mu = -\eta \rho \mu
\]

and \(2 < -\rho v - \eta \rho \mu\).

Hence, after substituting with \(v\) and \(\mu\) and dividing by \(\rho\) we get

\[
\frac{2}{\rho} < p_1 - p_3 + \eta(p_2 - p_4)
\]

(41)

Condition (41) implies to the following:

The zone of cooperation is non-empty if and only if it contains the Tit For Tat strategy, which is given by \((p_1, p_2, p_3, p_4) = (1, 1, 0, 0)\), i.e. if and only if the condition \(\eta > \frac{2 - \rho}{\rho}\) holds. This condition agree with \(\eta > 1\) since \(\eta > \frac{2}{\rho} - 1\), \(\rho < 1\) implies to \(\frac{2}{\rho} > 2\).

In the limiting case \(\rho = 1\), we see that condition \(\eta > \frac{2 - \rho}{\rho}\) is always satisfied.

Cooperation is easier to achieve the larger of the zone, i.e. the temptation \(T - R\) to defect unilaterally is smaller compared with the gain \(R - P\) obtained by mutual cooperation.
In particular, if \( \eta > \frac{2}{\rho} \), then from (41) the zone of cooperation contains the strategy given by \((p_1, p_2, p_3, p_4) = (1, 1, 1, 0)\), which is always ready to cooperate except if it has been played for a sucker, i.e., if it has experienced a defect in the last round. In this case, it defects if it is leader in the next round, but it defects only once. We note that in the limiting case \( (\rho = 1) \), condition \( \eta > \frac{2}{\rho} \) simply means that the coast to the donor \( \gamma - \alpha \) is twice as large as the benefit to the recipient \( \beta - \delta \).

It is easy to find the consensus strategy with highest payoff which immune to defection. If all members of the population adopt this strategy, then the exploiters with a lower propensity to cooperate cannot invade, and the overall payoff for the population is maximal (subject to this non-invadability condition). This payoff is given by

\[
F(\Omega, \Omega) = \frac{1}{2} [\Gamma + \Delta + y(A - \Gamma + B - \Delta)]
\]

Then from (19), (20) and (21) when \( \Omega' = \Omega \) we get

\[
(A - \Gamma) + (B - \Delta) = 2 \frac{(\alpha - \gamma + \beta - \delta)}{(2 + \rho(\nu + \mu))}
\]

then we have

\[
F(\Omega, \Omega) = \frac{\gamma + \delta}{2(1 - \rho)} + \frac{(\alpha - \gamma + \beta - \delta)}{(2 + \rho(\nu + \mu))} \left[ 2y + \frac{\rho \tau}{1 - \rho} \right]
\]

since

\[
\eta = \frac{\beta - \delta}{\gamma - \alpha}, \text{ then } \eta - 1 = \frac{\beta - \delta}{\gamma - \alpha} - 1 = \frac{\beta - \delta - \gamma + \alpha}{\gamma - \alpha}
\]

then we have

\[
(\eta - 1)(\gamma - \alpha) = (\beta - \delta - \gamma + \alpha)
\]

then

\[
F(\Omega, \Omega) = \frac{\gamma + \delta}{2(1 - \rho)} + \frac{(\eta - 1)(\gamma - \alpha)}{(2 + \rho(\nu + \mu))} \left[ 2y + \frac{\rho \tau}{1 - \rho} \right]
\]

4.1 Special cases

I. Let \( \nu = 0 \), \( \mu = 0 \) and \( \tau = 0 \), then \((p_1, p_2, p_3, p_4) = (q_1, q_2, q_3, q_4) = (0, 0, 0, 0)\) (which we call \textit{AllD} strategy)

The adaptive dynamics for the RAPD game is given by

1. For \( \rho < 1 \)

\[
\dot{y} = \frac{1}{2} (\alpha - \gamma)
\]

\[
p_1 = p_2 = \frac{\rho}{4} y (\alpha - \gamma)
\]

\[
p_3 = p_4 = \rho \left[ \frac{1}{(1 - \rho)} - y \right] (\alpha - \gamma)
\]

The total payoff for the population in which all its members adopt the strategy with highest payoff which immune to defection is given by
Dynamics of randomly alternating prisoner's dilemma game

\[ F(\Omega, \Omega) = \frac{\gamma + \delta}{2(1 - \rho)} + \frac{\eta(\eta - 1)(\gamma - \alpha)}{2} \]

(2) For \( \rho = 1 \) (limiting case)

\[ \dot{p}_1 = \dot{p}_2 = 0 \]

\[ \dot{p}_3 = \dot{p}_4 = \frac{\alpha - \gamma}{4} \]

ii. Let \( \nu = 0 \), \( \mu = 0 \) and \( \tau = 2 \), then \( (p_1, p_2, p_3, p_4) = (q_1, q_2, q_3, q_4) = (1, 1, 1, 1) \)

(which we call AllC strategy)

The adaptive dynamics for the RAPD game is given by

(1) For \( \rho < 1 \)

\[ \dot{y} = \frac{1}{2} (\alpha - \gamma) \]

\[ \dot{p}_1 = \dot{p}_2 = \frac{\rho^2}{4(1 - \rho)} \left[ (\frac{\gamma}{\rho} - 1) + 1 \right] (\alpha - \gamma) \]

\[ \dot{p}_3 = \dot{p}_4 = \frac{\rho}{4} (1 - \gamma)(\alpha - \gamma) \]

The overall payoff for the population in which all its members adopt the strategy with highest payoff which immune to defection is given by

\[ F(\Omega, \Omega) = \frac{\gamma + \delta}{2(1 - \rho)} + \frac{(\eta - 1)(\gamma - \alpha)}{2} \left[ y + \frac{\rho}{1 - \rho} \right] \]

(2) For \( \rho = 1 \) (limiting case)

\[ \dot{p}_1 = \dot{p}_2 = \frac{\alpha - \gamma}{4} \]

\[ \dot{p}_3 = \dot{p}_4 = 0 \]

iii. Let \( \nu = 1 \), \( \mu = 1 \) and \( \tau = 2 \), then \( (p_1, p_2, p_3, p_4) = (q_1, q_2, q_3, q_4) = (0, 0, 1, 1) \)

(which we call ATFT strategy)

The adaptive dynamics for the RAPD game is given by

(1) For \( \rho < 1 \)

\[ \dot{y} = \frac{\rho(\alpha + \delta - \beta - \gamma) + 2(\alpha - \gamma)}{4(1 + \rho)} \]

\[ \dot{p}_1 = \dot{p}_2 = \frac{\rho^2}{4(1 + \rho)} \left[ y(\frac{1}{\rho} - 1) + 1 \right] [\rho(\alpha + \delta - \beta - \gamma) + 2(\alpha - \gamma)] \]

\[ \dot{p}_3 = \dot{p}_4 = \frac{\rho(y(\rho - 1) + 1) [\rho(\alpha + \delta - \beta - \gamma) + 2(\alpha - \gamma)]}{8(1 - \rho^2)(1 + \rho)} \]

The overall payoff for the population in which all its members adopt the strategy with highest payoff which immune to defection is given by

\[ F(\Omega, \Omega) = \frac{\gamma + \delta}{2(1 - \rho)} + \frac{(\eta - 1)(\gamma - \alpha)}{2(1 + \rho)} \left[ y + \frac{\rho}{1 - \rho} \right] \]

(2) For \( \rho = 1 \) (limiting case)
\[ \dot{p}_1 = \dot{p}_2 = \dot{p}_3 = \dot{p}_4 = \frac{3(\alpha - \gamma) - (\beta - \delta)}{32} \]

iv. Let \( \alpha = 2, \beta = 1, \gamma = 3, \delta = -2 \) \((R = 3, S = 0, T = 4, P = 1)\)

The adaptive dynamics for the RAPD game is given by

1. For \( \rho < 1 \)
\[
\dot{y} = \frac{1}{2} (A - \Gamma)
\]
\[
\dot{p}_1 = \dot{p}_2 = \frac{\rho^2}{4(1 - \rho)} (2 + \rho(v + \mu))^{-1} \left[ y \left( \frac{2}{\rho} - 2 \right) + \tau \right] (A - \Gamma)
\]
\[
\dot{p}_3 = \dot{p}_4 = \frac{\rho^2}{4(1 - \rho)} (2 + \rho(v + \mu))^{-1} \left[ y \left( 2 - \frac{2}{\rho} \right) + \left( \frac{2}{\rho} - p_1 - p_2 \right) \right] (A - \Gamma)
\]

Where
\[
A - \Gamma = -\frac{2(2 + \rho(v + 3\mu))}{(2 + \rho(v - \mu))(2 + \rho(v + \mu))}
\]

The overall payoff for the population in which all its members adopt the strategy with highest payoff which immune to defection is given by
\[
F(\Omega, \Omega) = \frac{2(1 + 2y(1 - \rho) + \rho(2\tau + v + \mu))}{2(1 - \rho)(2 + \rho(v + \mu))}
\]

2. For \( \rho = 1 \) (limiting case)
\[
\dot{p}_1 = \dot{p}_2 = -\frac{\tau(2 + v + 3\mu)}{2(2 + v + \mu)^2(2 + v - \mu)}
\]
\[
\dot{p}_3 = \dot{p}_4 = -\frac{(2 + v + \mu - \tau)(2 + v + 3\mu)}{2(2 + v + \mu)^2(2 + v - \mu)}
\]

References


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