Finite-Time Convergent Sliding Mode Controllers for Robot Manipulators

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Abstract

The problem of robust finite-time controller designs of robot manipulator is studied in this paper. Two finite-time sliding mode controllers are developed to deal with tracking control of robot manipulators. The proposed control laws are designed by using terminal sliding mode and fast terminal sliding mode control algorithms which are able to guarantee finite-time reachability of a given desired tracking motion of robot manipulators. By using the second method of Lyapunov, stability of the closed-loop system can be achieved in finite time. An example of a robot tracking system is presented and simulation results are provided to demonstrate and verify the usefulness of the developed controllers.

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1 Introduction

Tracking control of robot manipulators has been a popular research area during the last few decades. In practical situations, the system uncertainties are frequently encountered in robot operations because of unknown and changing payload. Thus, robust tracking controller designs of robot manipulator problem to suppress uncertainties have also attracted a great deal of attention. Various nonlinear robust control approaches have been proposed for solving the robust tracking control of robot manipulator problem including adaptive
control [1],[2], sliding mode control [3]-[5], fuzzy control [6],[7]. Chen [8] proposed a mixed $H_2/H_\infty$ control design for tracking of rigid robotic system under parameter perturbations and external disturbances. Sliding mode control (SMC) has been shown to be a potential approach when applied to a system with disturbances which satisfy the matched uncertainty condition [9]. Robust tracking controllers based on the SMC scheme have been proposed in [10]-[12]. These control laws can achieve global asymptotic stability and provide good tracking results. However, these controller were designed based on an asymptotic stability analysis which implies that the system trajectories converge to the equilibrium with infinite settling time. It is well known that finite-time stabilization of dynamical systems may provide a faster disturbance attenuation besides giving faster convergence to the desired position. The terminal sliding mode (TSM) method [13],[14] can be used to design sliding mode controller that will guarantee a finite-time convergence to the origin. In [15] and [16] the tracking control of robot manipulator has been studied and the fast terminal sliding mode (FTSM) method was used to design finite-time controllers.

In this research, the main contribution is that proposed controllers based on TSM and FTSM concepts are designed so that the tracking errors of a robot manipulator will converge to the origin in finite-time. Although applications of finite-time sliding mode control schemes to robot control systems are not recent, we believe that much research remains to be done in this area. Since these algorithms have rarely been studied for applications to robot system, we hope that this paper will contribute to the popularity of the area and will enhance future development.

This paper is organized as follows. In Section 2 basic concepts of the finite-time stability are given. Several definitions and lemmas which are employed in finite-time convergence analysis are provided. Section 3 presents the control design problem of a robot manipulator system. In Section 4, two robust finite-time controllers are developed using TSM and FTSM concepts. The stability analysis of both controllers is performed using the second method of Lyapunov. In Section 5 a numerical example of robot tracking system is presented to verify the usefulness of the proposed controllers. In Section 6 we present conclusions.

2 Basic concepts

Definition 2.1. [19] Consider a system

$$\dot{x} = f(x), f(0) = 0, x \in \mathbb{R}^n,$$  \hfill (1)

where $f : D \rightarrow \mathbb{R}^n$ is continuous on an open neighborhood $D$ of the origin, the equilibrium point $x=0$ of the system is (locally) finite-time stable if it is Lyapunov stable and finite-time convergent in a neighborhood $U \subseteq D$. Here, the finite-time convergence means: for any initial condition $x_0 \in U \setminus \{0\}$, there
is a settling time function $T(x_0) : U/\{0\} \rightarrow (0, \infty)$ such that every solution $x(t, x_0)$ of the system (1) is defined with $x(t, x_0) \in U/\{0\}$ for $t \in [0, T(x_0))$ and satisfies $\lim_{t \to T(x_0)} x(t, x_0) = 0$ and $x(t, x_0) = 0$, if $t \geq T(x_0)$.

**Definition 2.2.** [19] Consider a controlled system

$$\dot{x} = f(x) + g(x)u, x \in \mathbb{R}^n, u \in \mathbb{R}^m,$$

with $g(x) \neq 0$. It is finite-time stabilizable if there is a feedback law $u(x)$ such that $x=0$ is a (locally) finite-time stable equilibrium of the closed-loop system.

**Lemma 2.3.** [20] Consider the nonlinear system described in (1), suppose there is a $C^1$ (continuously differentiable) function $V(x)$ defined in a neighborhood $D \subset \mathbb{R}^n$ of the origin, and there are real numbers $\beta > 0$ and $0 < \gamma < 1$, such that $V(x) > 0$ on $D$ and

$$\dot{V}(x) + \beta V^\gamma(x) \leq 0$$

(along the trajectory) on $D$. Then the origin of the system is finite-time stable. Moreover, the settling time, depending on the initial state $x(0) = x_0$, is given by

$$T(x_0) \leq \frac{1}{\beta(1 - \gamma)} V^{1-\gamma}(x_0)$$

for $x_0$ in some open neighborhood of the origin. If $U = \mathbb{R}^n$ and $V(x)$ is also radially unbounded, the origin is globally finite-time stable.

**Definition 2.4.** [21] The TSM and FTSM can be described by the following first-order nonlinear differential equations

$$s = \dot{x} + \beta x^\gamma = 0, \quad s = \dot{x} + \alpha x + \beta x^\gamma = 0,$$

where

$$\alpha, \beta > 0, \quad 0 < \gamma < 1.$$ 

In [14], the expression (5) was reported as the TSM and FTSM

$$s = \dot{x} + \beta x^p = 0, \quad s = \dot{x} + \alpha x + \beta x^p = 0,$$

where $\alpha, \beta > 0, \quad p > q > 0$ are integers, $p$ is odd. The equation (5) should be the exact expression of TSM in spite that we have been suggesting only real solution for (6) is considered because this suggestion has been involved in (5).
Lemma 2.5. [20] The equilibrium point $x = 0$ of the continuous non-Lipschitz differential equations (5) is globally finite-time stable, i.e., for any given initial condition $x(0) = x_0$, the system state converges to $x = 0$ in finite-time

$$T(x_0) = \frac{1}{\beta(1-\gamma)} |x_0|^{1-\gamma}$$

$$T(x_0) = \frac{1}{\alpha(1-\gamma)} \ln \frac{\alpha |x_0|^{1-\gamma} + \beta}{\beta}$$

respectively and stay there forever. Lemma 2.5 can be easily proved with Definition 2.1 of finite-time stability. Furthermore, another extended Lyapunov function description of finite-time stability of Lemma 2.1 can be described with the form of fast TSM as

$$\dot{V}(x) + \alpha V(x) + \beta V^\gamma(x) \leq 0$$

and the settling time can be given by

$$T(x_0) \leq \frac{1}{\alpha(1-\gamma)} \ln \frac{\alpha V^{1-\gamma}(x_0) + \beta}{\beta}.$$  

It is obvious that the inequalities (8) and (9) include exponential stability plus finite-time stability. This provides faster finite-time stability when compared with the TSM.

3 Control Design Problem

We consider the general robot system model for a 2-DOF manipulator (ignore the friction) which is modeled as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = u(t) + d(t),$$

where $q, \dot{q}, \ddot{q} \in \mathbb{R}^n$ are the vectors of joint angular position, velocity and acceleration respectively. $M(q) \in \mathbb{R}^{n \times n}$ is a symmetric positive definite inertia matrix, $C(q, \dot{q})\dot{q} \in \mathbb{R}^n$ is the Coriolis and centrifugal vector, $G(q) \in \mathbb{R}^n$ is the gravitational vector, $d(t) \in \mathbb{R}^n$ denotes the vector of system disturbance. We assume that the elements of $M^{-1} \cdot d(t)$ are bounded, let $M^{-1} \cdot d(t) = L \in \mathbb{R}^n$ and $|L_i| < L_c$ where $L_i$ is the $i$th component of $L$ and $L_c$ is a positive scalar. $u(t) \in \mathbb{R}^n$ is the vector of applied joint torques, which is actually control inputs.

Assume that $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \dot{q} \\ \ddot{q} \end{bmatrix}$, from the equation (1), we can derive the state equations of robot manipulator. We know that the state equations of robot system, can be expressed as normalized form

$$\dot{z}_1 = z_2$$

$$\dot{z}_2 = f(z) + g(z)u + g(z)d,$$
where \( f(z) = M^{-1}(q)[-C(q, \dot{q})\dot{q} - G(q)] \) and \( g(z) = M^{-1}(q) \).

Let \( q_d \in \mathbb{R}^n \) be a given twice differentiable desired trajectory and the tracking error be defined as \( q_e = q - q_d \). The control objective is to find a feedback control law \( u(q, \dot{q}) \) such that the manipulator output \( q \) tracks the desired trajectory \( q_d \). In other word, the tracking error converges to zero in finite time.

4 Finite-time controller designs

4.1 Sliding surface design

We consider the following sliding surfaces of the TSM and FTSM [8] as

\[
S = \dot{q}_e + \beta \dot{q}_e^\alpha
\]

(12)

\[
S = \dot{q}_e + \alpha q_e + \beta \dot{q}_e^\alpha
\]

(13)

where \( S \in \mathbb{R}^n \), \( \alpha, \beta > 0 \) and \( a, b (a > b) \) are positive odd number. The vector \( \dot{q}_e^\alpha \) is defined as \( \dot{q}_e^\alpha = [\dot{q}_e^\alpha_1, \dot{q}_e^\alpha_2, \dot{q}_e^\alpha_3] \). Only in this condition, the nonlinear phase \( \beta \dot{q}_e^\alpha \) will be significant, which is also one of the requirements to ensure stability of the system.

4.2 TSM Control law

We propose the following controller

\[
u = C(q, \dot{q})\dot{q} - \beta M\frac{d}{dt}\dot{q}_e^\alpha + G(q) - kM \text{sign}(S)^r + M\ddot{q}_d,
\]

(14)

where \( \alpha, \beta, k \) are positive scalars and \( r \in (1, 2) \). The function \( \text{sign}(S)^r \) is defined as

\[
\text{sign}(S)^r = [\|S_1\|\text{sign}(S_1) \ |S_2|\text{sign}(S_2) \ |S_3|\text{sign}(S_3)]^T.
\]

**Theorem 4.1.** Under the controller (14) with the sliding surface (12), the trajectories of the closed-loop system (11) can be driven onto the sliding surface in a finite time.

To prove the theorem above we select a Lyapunov function to ensure the finite-time stability of closed-loop system.

**Proof.** Consider the following Lyapunov function

\[
V_1 = \frac{1}{2} S(t)^T S(t).
\]

(15)
The derivative of \( V_1 \) can be written as
\[
\dot{V}_1 = S^T [\ddot{q}_e + \alpha \dot{q}_e + \beta \frac{d}{dt} \ddot{q}_e^w]. \tag{16}
\]
Substituting \( \ddot{q}_e = \ddot{q} - \dot{q}_d \) into (16), we obtain
\[
\dot{V}_1 = S^T [M^{-1}(-C(q, \dot{q})q_e - G(q) + u(t) + d(t)) - \ddot{q}_d + \beta \frac{d}{dt} \ddot{q}_e^w)]. \tag{17}
\]
Substituting \( u(t) \) into (17), one has
\[
\dot{V}_1 = S^T [M^{-1}(C(q, \dot{q})q_e - \beta M \frac{d}{dt} \ddot{q}_e^w) - G(q) - kM \text{sign}(S)^r + M \ddot{q}_d) + M^{-1}d - \ddot{q}_d + \beta \frac{d}{dt} \ddot{q}_e^w] \]
\[
= S^T [-k \text{sign}(S)^r + M^{-1}d] \]
\[
= -k \sum_{i=1}^{n} |S_i|^{r+1} + S^T L \]
\[
= -k \sum_{i=1}^{n} |S_i|^{r+1} - S_i L_i \]
\[
\leq - \sum_{i=1}^{n} (|S_i|^{r+1}k - |S_i| L_i) \]
\[
= - \sum_{i=1}^{n} |S_i|(k|S_i|^r - L_i). \tag{18}
\]
From (18), letting \( \Omega_i = \frac{L_i}{|S_i|^r} \), if the parameter \( k \) is chosen such that \( k > |\Omega_i| \). We can obtain
\[
\dot{V}_1 \leq -\lambda \sum_{i=1}^{n} |S_i| \]
\[
\text{where } \lambda = \min(k - \Omega_i, k + \Omega_i). \text{ Also, Using (15), (19) can be written as}
\]
\[
\dot{V} \leq -\lambda \sqrt{2V_1}
\]
To apply Lemma 2.3, (19) is rewritten as
\[
\dot{V}_1 + \tilde{\lambda} V_1^{\frac{1}{2}} \leq 0, \tag{20}
\]
where \( \tilde{\lambda} = \sqrt{2\lambda} \). With Lemma 2.3, the setting time, which is time interval to reach \( S = 0 \) is obtained as
\[
T(x_0) \leq \frac{2}{\lambda} V_1^{\frac{1}{2}}(x_0). \tag{21}
\]
Therefore, \( S = 0 \) is achieved in finite time by Lemma 2.3, and the proof is completed. \( \square \)
4.3 FTSM Control law

We propose the following controller
\[
    u = C(q, \dot{q})\ddot{q} - \alpha M\dot{q}_e - \beta M\frac{d}{dt}\dot{q}_e^2 + G(q) - M\rho S - M\bar{k}\text{sign}(S) + M\ddot{q}_d, \tag{22}
\]
where $\alpha$, $\beta$, $\bar{k}$ are positive scalars. The function $\text{sign}(s)$ is defined as
\[
    \text{sign}(S) = [\text{sign}(S_1) \; \text{sign}(S_2) \; \text{sign}(S_3)]^T.
\]

**Theorem 4.2.** Under the controller (22) with the sliding surface (13), the trajectories of the closed-loop system (11) can be driven onto the sliding surface in a finite time.

To prove the theorem above we select another Lyapunov function, the finite-time stability of the states can be guaranteed.

**Proof.** Consider the following Lyapunov function
\[
    V_2 = \frac{1}{2}S(t)^TS(t). \tag{23}
\]
The derivative of $V_2$ can be written as
\[
    \dot{V}_2 = S^T[\ddot{q}_e + \alpha\dot{q}_e + \beta\frac{d}{dt}\dot{q}_e^2]. \tag{24}
\]
Substituting $\ddot{q}_e = \ddot{q} - \ddot{q}_d$, into (24), we obtain
\[
    \dot{V}_2 = S^T[M^{-1}(-C(q, \dot{q}) - G(q) + u(t) + d(t)) - \ddot{q}_d + \alpha\dot{q}_e + \beta\frac{d}{dt}\dot{q}_e^2]. \tag{25}
\]
Substituting $u(t)$ into (25), one has
\[
    \dot{V}_2 = S^T[-M^{-1}C(q, \dot{q})\ddot{q} - M^{-1}G(q) + M^{-1}d + M^{-1}(C(q, \dot{q})\ddot{q} - \alpha M\dot{q}_e - \beta M\frac{d}{dt}\dot{q}_e^2 + G(q) - M\rho S - M\bar{k}\text{sign}(S) + M\ddot{q}_d) - \ddot{q}_d + \alpha\dot{q}_e + \beta\frac{d}{dt}\dot{q}_e^2]
\]
\[
    \dot{V}_2 = S^T[-\rho S - M\bar{k}\text{sign}(S) + M^{-1}d].
\]
\[
    \leq -\bar{k}\sum_{i=1}^{n}|S_i| - \rho\sum_{i=1}^{n}|S_i|^2 + \sum_{i=1}^{n}L_i|S_i| \tag{26}
\]
From(26), if $\bar{k}$ is chosen such that $\bar{k} > |L_i|$. We can obtain
\[
    \dot{V}_2 \leq -\bar{k}\sum_{i=1}^{n}|S_i| - \rho\sum_{i=1}^{n}|S_i|^2, \tag{27}
\]
where $\hat{k} = \min(\bar{k} - L_i, \bar{k} + L_i)$. Thus, using (23),(27) becomes

$$\dot{V}_2 + \sqrt{2k}V_2^{\frac{1}{2}} + 2\rho V_2 \leq 0. \quad (28)$$

By Lemma 2.5, the time interval to reach $S = 0$ is obtained as

$$T(x_0) \leq \frac{2}{\sqrt{2k}} ln\frac{\sqrt{2k}V_2^{\frac{1}{2}}(x_0)}{\sqrt{2\rho}} \quad (29)$$

Therefore, $S = 0$ is achieved in finite time by Lemma 2.5, and the proof is completed.

\section{Simulation and Result}

Two-joint manipulator parameters are given as follow [22]:

$$M(q) = \begin{bmatrix} 0.1 + 0.01 \cos(q_2) & 0.01 \sin(q_2) \\ 0.01 \sin(q_2) & 0.1 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} 0.1g \cos(q_1 + q_2) \\ 0.1g \cos(q_1 + q_2) \end{bmatrix}, \quad g = 9.8$$

$$C(q, \dot{q}) = \begin{bmatrix} -0.005 \sin(q_2) \dot{q}_2 & 0.005 \cos(q_2) \dot{q}_2 \\ 0.005 \cos(q_2) \dot{q}_2 & 0 \end{bmatrix}.$$ 

The desired reference signals are given by $q_d = \begin{bmatrix} \sin(t) \\ 1 \end{bmatrix}$. The disturbance vector is given by $d = \begin{bmatrix} 2 \sin(2\pi t) \\ 1.5 \cos(2\pi t) \end{bmatrix}$. We choose the controller parameters $k = 10, a = 7, b = 5, \alpha = 2, \beta = 1.8, \rho = 5.0, r = 1.3$. The initial values of the system are selected as $q_1(0) = 1.0, q_2(0) = 0.5, \dot{q}_1(0) = 0.0, \dot{q}_2(0) = 0.0$.

The simulation results are shown in Fig.1-Fig.4. Fig.1 shows tracking response of joint 1 obtained by the control law (14). Tracking responses of joint 1 and 2 are driven to their desired positions. Evidently the effect of external disturbances on both responses is totally removed. The applied control signals $u_1, u_2$ of joint 1 and joint 2 are shown in Fig.3. From Fig.4, it can been seen that the sliding surface $s = 0$ is achieved in about 0.5 seconds.

As shown in Figs 6 and 7, under the controller (22), tracking responses of joint 1 and joint 2 are forced to the desired position. The control signals are smoother (Fig.8.) and the sliding manifold $s = 0$ is quicker attained.
Figure 1: Response of joint 1 - controller (14).

Figure 2: Response of joint 2 - controller (14).
6 Conclusion

In this research, finite-time sliding mode controllers have been designed for tracking control of robot manipulators in the presence of external disturbances. Based on TSM and FTSM concepts, the presented controllers have been proposed to force the state variables of the closed-loop system to converge to the desired state. By using the second method of Lyapunov, stability of the closed-loop system can be achieved in finite-time. An example of a robot tracking
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Figure 5: Response of joint 1 - controller (22).

Figure 6: Response of joint 2 - controller (22).

system is presented and simulation results are provided to demonstrate and verify the usefulness of the developed controllers.
Figure 7: Input torques-controller(22).

Figure 8: Sliding surface - controller (22).

References


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