Curvilinear Interpolation Using

an Exponential and a Linear Term

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Abstract

This paper illustrates methods for interpolating curvilinear data using an exponential or an exponential plus a linear term. The methods are based on the least-squares principal. They require a minimum of three or four data, respectively. An error in a recent three-point hyperbola paper is noted.

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1. Introduction

Recent papers in This Journal illustrate curvilinear interpolation by means of a hyperbola or a hyperbola plus a linear term [1,2]. The four-point hyperbola has an analog in terms of an exponential function. It is Eq. (1). Equation (2), a special case, is treated separately in the Appendix. In both equations the letter y represents an interpolated number. A, B, C, D are numerical parameters to be determined.
2. Method

The minimum number of data is four for Eq. (1) and three for Eq. (2). Both equations can accommodate more data. The routines execute much faster if the data are evenly-spaced. Both approaches are based on the least-squares principal and both are developed similarly. Equation (1) is a guide for developing Eq. (2).

Let four trial \((x,y)\) data be \((1,24), (2,36), (3,50), (4,68)\), respectively. The sum of squared residuals, denoted \(\Sigma\), is Eq. (3) \([1]\).

\[
\Sigma = (24 - A - B - CD)^2 + (36 - A - 2B - CD^2)^2 + (50 - A - 3B - CD^3)^2 + (68 - A - 4B - CD^4)^2 = 0
\]

The partial derivative of \(\Sigma\) with respect to \(A\) is denoted \(dA\), the partial derivative of \(\Sigma\) with respect to \(B\) is denoted \(dB\). They appear as Eqs. (4) and (5), respectively. Expressions for \(dC\) and \(dD\) are developed similarly.

\[
dA = -356 + 8A + 20B + 2CD + 2CD^2 + 2CD^3 + 2CD^4 = 0
\]

\[
dB = -1036 + 20A + 60B + 2CD + 4CD^2 + 6CD^3 + 8CD^4 = 0
\]

The equations \(\{dA, dB, dC, dD\}\) form a simultaneous set. This set can often be solved for \(\{A, B, C, D\}\). Substitute \(\{A, B, C, D\}\) into Eq. (1) to yield Eq. (6).

\[
y = 12 + (10)x + (1)2^x
\]

Let the generating function \(u(x)\) be \(\cosh(x)\). Five curvilinear \([x,y]\) points are: \([-0.7, u(-0.7)], [-0.4, u(-0.4)], [-0.1, u(-0.1)]\), \([0.2, u(0.2)], [0.5, u(0.5)]\). The interpolating equation is Eq. (7). Plotting Eq. (7) reveals its U-shape like \(\cosh(x)\) when centered at \(x = 0\). In 15-digit precision, the sum of the squares of deviations of Eq. (7) from \(\cosh(x)\), over the range \(x = -0.7 \ldots 0.5\), is about \((5)(10^{-7})\).

\[
y = -103.38976 + (10.36340)x + 104.38923(0.9055203)^x
\]
Curvilinear interpolation

This example is apt to be misleading. It gives the impression that Eq. (1) can generate an interpolating equation for all data containing an interior extremum. That impression is untrue. It is also untrue for the four-point hyperbola [2].

Let a generating function be \( y = x^3 + 5x \). Four arbitrarily spaced (x, y) data are (3.1, 45.291), (4.2, 95.088), (5.5, 193.875), (6.9, 363.009). They are reproduced and interpolated by Eq. (8). Its coefficients have been rounded.

\[
y = -234.790 - 64.946x + 255.630(1.2265341)^x \tag{8}
\]

Table 1 lists test functions applied to four equidistant x-data. It also lists sums of squares of deviations of three four-point equations used to interpolate the data. Equation (1) appears to be a satisfactory choice for these trial data.

Table 1. Approximate sums of squares of deviations of three four-point interpolating equations from generating functions over the range \( x = 1 \) .. 4. (12-digit precision)

<table>
<thead>
<tr>
<th>Function</th>
<th>Cubic polynomial</th>
<th>Equation (1) (text)</th>
<th>Four-point hyperbola [2]</th>
</tr>
</thead>
<tbody>
<tr>
<td>sinh(x/4)</td>
<td>(1)(10^{-8})</td>
<td>(7)(10^{-8})</td>
<td>(2)(10^{-7})</td>
</tr>
<tr>
<td>ln(x^{3/2}+1)</td>
<td>(6)(10^{-7})</td>
<td>(5)(10^{-7})</td>
<td>(1)(10^{-8})</td>
</tr>
<tr>
<td>exp(1.1x)+100/x</td>
<td>11.7</td>
<td>11.7</td>
<td>11.7</td>
</tr>
<tr>
<td>1024/2^x + 111x</td>
<td>4.0</td>
<td>(5)(10^{-28})</td>
<td>0.62</td>
</tr>
</tbody>
</table>

Appendix: Curvilinear interpolation by Eq. (2)

The development of Eq. (2) as an interpolating equation parallels the development of Eq. (1). Let three trial (x,y) data be (1,24), (2,36), (3,60), respectively. The sum of squared residuals, denoted \( \Sigma \), is Eq. (9).

\[
\Sigma = (24 - A - BC)^2 + (36 - A - BC^2)^2 + (60 - A - BC^3)^2 \tag{9}
\]

The partial derivatives of \( \Sigma \) with respect to A, B, and C are denoted dA, dB, and dC, respectively. They form a set of three simultaneous equations.

\[
dA = -240 + 6A + 2BC + 2BC^2 + 2BC^3 \tag{10}
\]
\[
\]

\[
\]
Equations (10), (11), (12) have the solution $A = 12$, $B = 6$, $C = 2$. The equation interpolating these three curvilinear $(x,y)$ data is therefore Eq. (13).

$$y = 12 + 6(2^x)$$  \hfill (13)

Equation (13) reproduces the three original curvilinear data. Equation (2) is illustrated with only three $(x,y)$ data but it can accept more curvilinear data. Equation (2) is useful primarily for data that are monotonic-increasing or monotonic-decreasing.

Not all curvilinear data can be satisfactorily represented by Eqs. (1) or (2). In cases of failure or unsatisfactory results, try enhanced precision, improved software, or use another method [1,2].

References


   There is an error in this citation. The text beneath Eq. (5) therein should read: When linear Eq. (3) is solved for $A$, the result is Eq. (6).


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