On the Analogues of Tangent Numbers and Polynomials Associated with $p$-Adic Integral on $\mathbb{Z}_p$

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper we introduce the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$. Some interesting results and relationships are obtained.

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1 Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $= \exp(x \log q)$ for $|x|_p \leq 1$. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$
the fermionic $p$-adic invariant integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{0 \leq x < p^N} g(x)(-1)^x, \quad (\text{see}[4]). \quad (1.1)$$

If we take $g_1(x) = g(x + 1)$ in (1.1), then we see that

$$I_{-1}(g_1) + I_{-1}(g) = 2g(0), \quad (\text{see [4-6]}). \quad (1.2)$$

From (1.1), we obtain

$$\int_{\mathbb{Z}_p} g(x + n) d\mu_{-1}(x) = (-1)^n \int_{\mathbb{Z}_p} g(x) d\mu_{-1}(x) + 2 \sum_{l=0}^{n-1} (-1)^{n-1-l} g(l). \quad (1.3)$$

Let us define the tangent numbers $T_n$ and polynomials $T_n(x)$ as follows:

$$\int_{\mathbb{Z}_p} e^{2xt} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!}, \quad (\text{see [6]}). \quad (1.4)$$

$$\int_{\mathbb{Z}_p} e^{(x+2y)t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \quad (1.5)$$

Numerous properties of tangent number are known. More studies and results in this subject we may see references [2], [3], [7]. About extensions for the tangent numbers can be found in [7]. Many mathematicians have studied in the area of the analogues of the Bernoulli numbers, Euler numbers, and Genocchi numbers (see [1-7]). Our aim in this paper is to define twisted tangent polynomials $T_{n,w}(x)$. We investigate some properties which are related to twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$. We also derive the existence of a specific interpolation function which interpolate twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$ at negative integers.

## 2 Twisted tangent numbers and polynomials

Our primary goal of this section is to define twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$. We also find generating functions of twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$ and investigate their properties. Let $T_p = \bigcup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N}$, where $C_{p^N} = \{w | w^{p^N} = 1\}$ is the cyclic group of order $p^N$. For $w \in T_p$, we denote by $\phi_w : \mathbb{Z}_p \to \mathbb{C}_p$ the locally constant function $x \mapsto w^x$.

For $w \in T_p$, if we take $g(x) = \phi_w(x)e^{2xt}$ in (1.2), then we easily see that

$$I_{-1}(\phi_w(x)e^{2xt}) = \int_{\mathbb{Z}_p} \phi_w(x)e^{2xt} d\mu_{-1}(x) = \frac{2}{we^{2t} + 1}.$$
Let us define the twisted tangent numbers $T_{n,w}$ and polynomials $T_{n,w}(x)$ as follows:

$$I_1(\phi_w(y)e^{2yt}) = \int_{\mathbb{Z}_p} \phi_w(y)e^{2yt}d\mu_1(y) = \sum_{n=0}^\infty T_{n,w}\frac{t^n}{n!},$$

$$I_1(\phi_w(y)e^{(2y+x)t}) = \int_{\mathbb{Z}_p} \phi_w(y)e^{(x+2y)t}d\mu_1(y) = \sum_{n=0}^\infty T_{n,w}(x)\frac{t^n}{n!}.\quad (2.1)$$

By (2.1) and (2.2), we obtain the following Witt's formula.

**Theorem 2.1** For $n \in \mathbb{Z}_+$, we have

$$\int_{\mathbb{Z}_p} \phi_w(x)(2x)^nd\mu_1(x) = T_{n,w}, \quad \int_{\mathbb{Z}_p} \phi_w(y)(x+2y)^nd\mu_1(y) = T_{n,w}(x).$$

Clearly, setting $w = 1$ in (2.1) and (2.2), we have $T_{n,w} = T_n$ and $T_{n,w}(x) = T_n(x)$ in terms of the tangent numbers $T_n$ and the tangent polynomials $T_n$.

By using $p$-adic integral on $\mathbb{Z}_p$, we obtain,

$$\int_{\mathbb{Z}_p} \phi_w(x)e^{2xt}d\mu_1(x) = 2 \sum_{m=0}^\infty (-1)^m w^me^{2mt}.\quad (2.3)$$

Thus twisted tangent numbers $T_{n,w}$ are defined by means of the generating function

$$F_w(t) = \sum_{n=0}^\infty T_{n,w}\frac{t^n}{n!} = 2 \sum_{m=0}^\infty (-1)^m w^me^{2mt}.\quad (2.4)$$

Using similar method as above, by using $p$-adic integral on $\mathbb{Z}_p$, we have

$$\sum_{n=0}^\infty T_{n,w}(x)\frac{t^n}{n!} = \left(\frac{2}{we^{2t}+1}\right)e^{xt}.\quad (2.5)$$

By using (2.2) and (2.5), we obtain

$$F_w(t, x) = \sum_{n=0}^\infty T_{n,w}(x)\frac{t^n}{n!} = 2 \sum_{m=0}^\infty (-1)^m w^me^{(2m+x)t}.\quad (2.6)$$

By Theorem 2.1, we easily obtain that

$$T_{n,w}(x) = \int_{\mathbb{Z}_p} \phi_w(y)(x+2y)^nd\mu_1(y) = \sum_{k=0}^n \binom{n}{k}x^{n-k}T_{k,w}$$

$$= (x + T_w)^n$$

$$= 2 \sum_{m=0}^\infty (-1)^m w^m(x + 2m)^n.\quad (2.7)$$
The following elementary properties of twisted tangent polynomials $T_{n,w}(x)$ are readily derived from (2.1) and (2.2). We, therefore, choose to omit the details involved. More studies and results in this subject we may see references [4]-[6].

**Theorem 2.2** For any positive integer $n$, we have

$$T_{n,w}(x) = (-1)^n w^{-1} T_{n,w-1}(2 - x).$$

**Theorem 2.3** For any positive integer $m (= \text{odd})$, we have

$$T_{n,w}(x) = m^n \sum_{a=0}^{m-1} (-1)^a w^a T_{n,w}^{m} \left( \frac{2a + x}{m} \right), \quad n \in \mathbb{Z}_+.$$  

By (1.3), (2.1), and (2.2), we easily see that

$$2^{m+1} \sum_{l=0}^{n-1} (-1)^{n-1-l} w^l l^m = w^n T_{m,w}(2n) + (-1)^{n-1} T_{m,w}.$$  

Hence, we have the following theorem.

**Theorem 2.4** Let $m \in \mathbb{Z}_+$. If $n \equiv 0 \pmod{2}$, then

$$w^n T_{m,w}(2n) - T_{m,w} = 2^{m+1} \sum_{l=0}^{n-1} (-1)^{l+1} w^l l^m.$$  

If $n \equiv 1 \pmod{2}$, then

$$w^n T_{m,w}(2n) + T_{m,w} = 2^{m+1} \sum_{l=0}^{n-1} (-1)^l w^l l^m.$$  

From (1.3), we note that

$$2 = w \int_{\mathbb{Z}_p} \phi_w(x) e^{(2x + 2)t} d \mu_{-1}(x) + \int_{\mathbb{Z}_p} \phi_w(x) e^{2xt} d \mu_{-1}(x)$$

$$= \sum_{n=0}^{\infty} \left( w \int_{\mathbb{Z}_p} \phi_w(x)(2x + 2)^n d \mu_{-2}(x) + \int_{\mathbb{Z}_p} \phi_w(x)(2x)^n d \mu_{-1}(x) \right) \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} (w T_{n,w}(2) + T_{n,w}) \frac{t^n}{n!}.$$  

Therefore, we have the following theorem.
Theorem 2.5 For \( n \in \mathbb{Z}_+ \), we have

\[
wT_n,w(2) + T_n,w = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0. \end{cases}
\]

By (2.7) and Theorem 2.5, we have the following corollary.

Corollary 2.6 For \( n \in \mathbb{Z}_+ \), we have

\[
w(T_w + 2)^n + T_n,w = \begin{cases} 2, & \text{if } n = 0, \\ 0, & \text{if } n \neq 0, \end{cases}
\]

with the usual convention of replacing \((T_w)^n\) by \(T_n,w\).

Theorem 2.7 For \( n \in \mathbb{Z}_+ \), we have

\[
T_{n,w}(x + y) = \sum_{k=0}^{n} \binom{n}{k} T_{k,w}(x) y^{n-k}.
\]

By Theorem 2.1, we easily get

\[
T_{n,w}(x) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} \int_{\mathbb{Z}_p} \phi_w(y) (2y)^l d\mu_{-1}(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} T_{l,w}.
\]

Therefore, we obtain the following theorem.

Theorem 2.8 For \( n \in \mathbb{Z}_+ \), we have

\[
T_{n,w}(x) = \sum_{l=0}^{n} \binom{n}{l} T_{l,w} x^{n-l}.
\]

3 Analogues of zeta function

In this section, by using twisted tangent numbers and polynomials, we give the definition for the twisted tangent zeta function and Hurwitz-type twisted tangent zeta functions. These functions interpolate the twisted tangent numbers and tangent polynomials, respectively. Let \( w \) be the \( p^N \)-th root of unity.

From (2.4), we note that

\[
\frac{d^k}{dt^k} F_w(t) \bigg|_{t=0} = 2 \sum_{m=0}^{\infty} (-1)^m w^m (2m)^k = T_{k,w}; (k \in \mathbb{N}).
\]

By using the above equation, we are now ready to define twisted tangent zeta functions.
**Definition 3.1** Let $s \in \mathbb{C}$ with $\text{Re}(s) > 1$.

$$\zeta_w(s) = 2 \sum_{n=1}^{\infty} \frac{(-1)^n w^n}{(2n)^s}. \quad (3.2)$$

Note that $\zeta_w(s)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_w(s)$ and $T_{k,w}$ is given by the following theorem.

**Theorem 3.2** For $k \in \mathbb{N}$, we have

$$\zeta_w(-k) = T_{k,w}. \quad (3.3)$$

Observe that $\zeta_w(s)$ function interpolates $T_{k,w}$ numbers at non-negative integers. By using (2.7), we note that

$$\left. \frac{d^k}{dt^k} F_w(t, x) \right|_{t=0} = 2 \sum_{m=0}^{\infty} (-1)^m w^m (x + 2m)^k$$

$$= T_{k,w}(x), \ (k \in \mathbb{N}),$$

and

$$\left. \left( \frac{d}{dt} \right)^k \left( \sum_{n=0}^{\infty} T_{n,w}(x) \frac{t^n}{n!} \right) \right|_{t=0} = T_{k,w}(x), \ \text{for} \ k \in \mathbb{N}. \quad (3.4)$$

By (3.2) and (3.4), we are now ready to define the Hurwitz-type twisted tangent zeta functions.

**Definition 3.3** Let $s \in \mathbb{C}$ with $\text{Re}(s) > 1$.

$$\zeta_w(s, x) = 2 \sum_{n=0}^{\infty} \frac{(-1)^n w^n}{(2n + x)^s}. \quad (3.2)$$

Note that $\zeta_w(s, x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_w(s, x)$ and $T_{k,w}(x)$ is given by the following theorem.

**Theorem 3.4** For $k \in \mathbb{N}$, we have

$$\zeta_w(-k, x) = T_{k,w}(x).$$
References


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