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Constructing a Train of Soliton Solutions for the Three-Wave-Interaction Equations

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Abstract

An exact solution of the Three-Wave-Interaction (TWI) equations is well known by a compact formula called the $N-$soliton solution, a deep look to this solution shows that we need to find the inverse of some matrix whose entries are long formula of functions depend on time t and one spatial dimension x, which becomes very complicated for large N.

An interested physical problem is to study the analytic form of the solution of the (TWI) for large N, in order to build what is called a train of soliton solutions, which will enable us to construct a pulse of a lot number of humps.

Here, we simplified the form of the $(N+1)-$ soliton solutions, and wrote it in terms of the $N-$ soliton solutions plus some extra terms, then we approximated these terms and made them very simple. This will enable us to build the train of soliton solutions one by one by simply add these terms each time. We also examined successfully the $(N+1)-$soliton solutions for small values of N analytically and graphically.

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1 Introduction

We start from the well-known one dimensional (TWI) equations system which arises in the study of dispersive systems and describes the evolution in time t and one spatial dimension of the coherent three-waves resonant-interaction (3WRI) [1]

$$\begin{aligned} \frac{\partial Q_1}{\partial t} + C_1 \frac{\partial Q_1}{\partial x} &= \gamma_1 Q_2^* Q_3^*, \\ \frac{\partial Q_2}{\partial t} + C_2 \frac{\partial Q_2}{\partial x} &= \gamma_2 Q_1^* Q_3^*, \\ \frac{\partial Q_3}{\partial t} + C_3 \frac{\partial Q_3}{\partial x} &= \gamma_3 Q_1^* Q_2^*. \end{aligned} \tag{1}$$

Where $Q_i(x,t)$ for $i = 1,2,3$ are the slowly varying complex wave packets , $|Q_i(x,t)|$ represent the amplitude of the ith wave at time t. C_i is the velocity of the ith wave, called the group velocity where we choose the order $C_1 < C_2 < C_3$ so that the totally symmetric system in (1) is in its canonical form [1], $\gamma_i^2 = 1$ are the signs of the wave energies whose relative signs are determined from the center frequency ω_1 when $\omega_1 + \omega_2 + \omega_3 = 0$.

The equations in (1) describe the interaction in a dispersive medium of three optics waves $Q_i(x,t)$ with associated electric fields $E_i(x,t) = Q_i(x,t)e^{I(\omega_i - k_i x)}, I^2 = -1, k_i$ are the waves numbers that satisfy the resonant condition $k_1 + k_2 + k_3 = 0$, where ω_1 denotes the highest-frequency pulse.

The system in (1) is a special case of a more general system where the wave packets are functions of the time t and three spatial dimensions (x,y,z,t), the nature of this general system was noted by Newell and Benney (1967) besides others [2]. Many special cases of (1)were widely investigated and interpreted physically, for example: If we let $C_3 = C_2, Q_3 = Q_2$ then the (TWI) system becomes Two-Wave-Interaction. This case was discussed in some details in [3].

Another special case of (1) which received much attention occurred when the complex wave packets $Q_i(x,t)$ were all depend on one variable either x or t , i.e the three waves propagate along one axis, in this case we get what is called uniformly wave train, in this case the system in (1) can be solved exactly in terms of the elliptic functions [6],[7]. An important physical application of the nonlinear interaction between the three waves is that the pulses transfer energy between them, and this transferring can be enforced to reach 100% by pre delaying the faster incoming pulses [4]. More physical applications for the (TWI) equations (1) are briefly mentioned in [1].

Two forms of the system (1) can be studied and solved separately. Each form is usually classified according to the signs of the energies of the three waves and hence of the $\gamma_i' s$ [1]:

A: The explosive instability form:$\gamma_1 = \gamma_2 = \gamma_3 = \pm 1$, i.e all $\gamma_i' s$ have the same sign. Only the envelope with the middle group velocity can have solitons. In this case the solution develops a singularity and blows up after some period of time.

B: The decay instability form: One of the $\gamma_i' s$ has a different sign. In this case the solution develops no singularity and become stable after a long period of time.

2 Method of solution

The (TWI) equations (1) are solved by using the Inverse Scattering Transformation (IST) method. Using this method in solving the (TWI)equations was first established independently in 1976 by Zakharov and Manakov [1], [9]. To use the (IST) method, we build an appropriate eigenvalue problem (3) called the (ZM) eigenvalue problem, then we do three main steps:

1) Forward Scattering Transformation: In this step we build the initial scattering matrix $S(\zeta, t = 0)$ from the asymptotic behavior of the solution of the (ZM) eigenvalue problem (3). Thus, the initial data of the problem is mapped into the initial scattering matrix via the (ZM) eigenvalue problem.

2) The Time Evolution of the initial scattering matrix: Her we compute the scattering matrix at any later time $S(\zeta, t > 0)$ which contains all information needed to start the next step. The roots of the diagonal elements of $S(\zeta, t > 0)$ are in general complex numbers and used to compute the eigenvalues of the (ZM) eigenvalue problem.

3) Inverse Scattering Transformation (IST): In this step we use the elements of the scattering matrix $S(\zeta, t > 0)$ and the eigenvalues of the (ZM) eigenvalue problem to find the phases D_j defined in (10) . The jth soliton solution of the (TWI) system (1) is then recognized by a pair of the jth complex eigenvalue ζ_j and the jth phase D_j . If we obtain a real eigenvalue ζ , then we use the elements of the scattering matrix $S(\zeta, t > 0)$ to compute what is called the reflection coefficients [1]. The computed reflection coefficients are then used to set up the inverse scattering equations, called the Marchenko or the Gelfand Levitan integral equation, [1]. These inverse equations which are considered a linear extension of the inverse Fourier transformation, will help us to construct the radiation part of the soliton solution of the (TWI) at any later time t. The above steps are summarized in the following diagram [1].

$$Q_i(x,0) \underset{\text{problem}}{\overset{\text{Eigenvalue}}{\Longrightarrow}} S(\zeta, t=0) \underset{\text{Evolution}}{\overset{\text{Time}}{\Longrightarrow}} S(\zeta, t>0) \underset{\text{equations}}{\overset{\text{Inverse scattering}}{\Longrightarrow}} Q_i(x,t) \tag{2}$$

3 The Zakharov-Manakov eigenvalue problem [1]

We consider the following Zakharov Manakov (ZM) eigenvalue problem in the interval $|x| < \infty$.[1],[8].

$$\begin{aligned} \mathrm{I}\,\mathbb{V}_x &= (V + \zeta D)\mathbb{V} \\ \mathrm{I}\,\mathbb{V}_t &= Y\mathbb{V} \end{aligned} \tag{3}$$

Where $\mathbb{V} = (\vec{V}_1, \vec{V}_2, \vec{V}_3)$ is the matrix of three linearly independent eigenvectors, ζ is the corresponding eigenvalue, the matrices V, D are given by:

$$V=\begin{pmatrix} 0 & V_{12} & V_{13}\\ V_{21} & 0 & V_{23}\\ V_{31} & V_{32} & 0 \end{pmatrix}, D=\begin{pmatrix} C_1 & 0 & 0\\ 0 & C_2 & 0\\ 0 & 0 & C_3 \end{pmatrix} \tag{4}$$

$\{C_1, C_2, C_3\}$ are the group velocities, V_{ij} are called potential functions and given by:

$$\begin{array}{rcl} V_{23} & = & \frac{-IQ_1}{\sqrt{\beta_{21}\beta_{31}}}, \qquad V_{32}=-\gamma_2\gamma_3V_{23}^{*}\\ V_{31} & = & \frac{-IQ_2}{\sqrt{\beta_{21}\beta_{32}}}, \qquad V_{13}=\gamma_1\gamma_3V_{31}^{*}\\ V_{12} & = & \frac{-IQ_3}{\sqrt{\beta_{31}\beta_{32}}}, \qquad V_{21}=-\gamma_1\gamma_2V_{12}^{*}\\ \beta_{ij} & = & C_i-C_j \end{array} \tag{5}$$

Y is an unknown matrix which satisfies the following compatibility condition that comes from the cross partial differentiation $\mathbb{V}_{xt}=\mathbb{V}_{tx}$

$$IY_x=[V,Y]+\zeta[D,Y]+V_t \tag{6}$$

Where $[A,B]=AB-BA$. The (ZM) eigenvalue problem (3) can be written as [1]:

$$\begin{array}{rcl} \mathrm{L}\mathbb{V} & = & \zeta\mathbb{V}\\ \mathrm{I}\,\mathbb{V}_t & = & Y\mathbb{V} \end{array} \tag{7}$$

Where the operator L is given by:

$$L=D^{-1}(I(Id)\partial_x-V) \tag{8}$$

(Id) is the identity matrix of order 3. The entries of the matrix V are required to satisfy the restriction

$$V_{ij}\to 0 \qquad \text{sufficiently rapidly as } |x|\to\infty. \tag{9}$$

4 The N-soliton solution formula

In the direct scattering problem, we build the scattering matrix $S(x,t>0)$ whose elements are functions of the eigenvalues of the (ZM) eigenvalue problem (3) , and we determine the locations of the eigenvalues ζ_n . This gives us enough information to start the next step and use the (IST) method and construct the final state solution of the (TWI) equations (1) . Under the assumption that all the obtained eigenvalues are discrete and pure imaginary number $\left\{\zeta_j = I\eta_j\right\}_{j=1}^{N}$, a derived formula called N-soliton solution formula is applicable (where N is the number of the eigenvalues found in the direct scattering step) and gives us the final state solution $q_r(x)$, which is a curve representing the shape of the separated three pulses. This formula was introduced in [1] in a closed form given by:

$$q_r(x) = \sum_{j,k=1}^{N} D_j e^{-(\eta_j+\eta_k)x} \left[Id + M^2\right]_{jk}^{-1}, \quad \text{where} \quad |D_j| = 2\eta_j e^{2\eta_j x_j}, j = 1, 2, ..., N \tag{10}$$

r is the index for the corresponding wave, and x_j are the x -coordinates of the center of the jth soliton, which has an amplitude equal η_j. Id is the identity matrix of order N ,The elements of the matrix M are given by:

$$M_{ij}(x) = \frac{D_j e^{-(\eta_i+\eta_j)x}}{(\eta_i+\eta_j)}. \tag{11}$$

For complex eigenvalues $\{\lambda_j\}_{j=1}^{N}$ a similar formula to $q_r(x)$ in (10) was introduced in [5].

$$q_r(x) = -2\Gamma_r \sum_{j,k=1}^{N} D_j^* e^{-I(\lambda_j^*+\lambda_k^*)x} [Id + M \times M^*]_{jk}^{-1}, \tag{12}$$

where

$$\begin{array}{rcl} \Gamma_1 & = & \sqrt{\nu_{12}\nu_{13}}, \qquad \Gamma_2 = \sqrt{\nu_{12}\nu_{23}}, \\ \Gamma_3 & = & \sqrt{\nu_{13}\nu_{23}}, \qquad \nu_{ij} = |C_i^{-1} - C_j^{-1}| \end{array} \tag{13}$$

and the matrix M is given by:

$$M_{ij}(x) = \frac{D_j^* e^{I(\lambda_i-\lambda_j^*)x}}{\lambda_i - \lambda_j^*}. \tag{14}$$

5 Another suggested form for $q_r(x)$ in (12)

Here, we introduce a new derivation of a similar formula to (12) under the assumption that all the obtained eigenvalues are discrete and equal arbitrary complex number. $\{\lambda_j = \mu_j + I\nu_j, \quad \mu_j \neq 0\}_{j=1}^{N}$. From this new form, we will see how to update (12) to be $(N+1)$-soliton solution. redefine D_j and $M_{ij}(x)$ by:

$$D_j = -2I\lambda_j e^{-2I\lambda_j\zeta_j}, j = 1, 2, 3, ..., N, \tag{15}$$

and

$$M_{ij}(x) = D_j \frac{e^{Ix(\lambda_j-(\lambda_i)^*)}}{(\lambda_j-(\lambda_i)^*)} = \frac{-2I\lambda_j}{(\lambda_j-(\lambda_i)^*)} e^{-2I\lambda_j\zeta_j} e^{Ix(\lambda_j-(\lambda_i)^*)}. \tag{16}$$

Then the general term of the matrix $(Id + (M.M^*))$ becomes:

$$\begin{aligned}[Id+(M.M^*)]_{jk} &= \delta_j^k+\sum_{i=1}^N M_{ji}M_{ik}^* = \delta_j^k+\sum_{i=1}^N D_iD_k^* \frac{e^{Ix(\lambda_i-\lambda_j^*)}}{(\lambda_i-\lambda_j^*)}\frac{e^{-Ix((\lambda_k)^*-\lambda_i)}}{(\lambda_k)^*-\lambda_i}\\ &= \delta_j^k+\sum_{i=1}^N \frac{4\lambda_i(\lambda_k)^*}{(\lambda_i-(\lambda_j)^*)((\lambda_k)^*-\lambda_i)}e^{-2I\lambda_i(\zeta_i-x)}e^{2I(\lambda_k)^*((\zeta_k)^*-x)}e^{-Ix(\lambda_j)^*}e^{Ix(\lambda_k)^*}\\ &= \delta_j^k+e^{-Ix(\lambda_j)^*}(\sum_{i=1}^N \frac{4\lambda_i(\lambda_k)^*}{(\lambda_i-(\lambda_j)^*)((\lambda_k)^*-\lambda_i)}e^{-2I\lambda_i(\zeta_i-x)}e^{2I(\lambda_k)^*((\zeta_k)^*-x)})e^{Ix(\lambda_k)^*}\\ &= e^{-Ix(\lambda_j)^*}(\delta_j^k+\sum_{i=1}^N \frac{4\lambda_i(\lambda_k)^*}{(\lambda_i-(\lambda_j)^*)((\lambda_k)^*-\lambda_i)}e^{-2I\lambda_i(\zeta_i-x)}e^{2I(\lambda_k)^*((\zeta_k)^*-x)})e^{Ix(\lambda_k)^*}\end{aligned} \tag{17}$$

So the general term of the matrix $(Id+(M.M^*))_{jk}^{-1}$ becomes:

$$\begin{gathered}[Id+(M.M^*)]_{jk}^{-1} = \\ e^{-Ix(\lambda_j)^*}(\delta_j^k+\sum_{i=1}^N \frac{4\lambda_i(\lambda_k)^*}{(\lambda_i-(\lambda_j)^*)((\lambda_k)^*-\lambda_i)}e^{-2I\lambda_i(\zeta_i-x)}e^{2I(\lambda_k)^*((\zeta_k)^*-x)})^{-1}e^{Ix(\lambda_k)^*}\end{gathered} \tag{18}$$

Substituting the definition of the nonlinear phases D_j in (15) and the general term of $(I+(M.M^*))_{jk}^{-1}$ in (18) in the shape function $q_r(x)$ in (12) , we get:

$$\begin{gathered}q_r(x) = -2\Gamma_r\sum_{j=1}^N\sum_{k=1}^N(2I(\lambda_j)^*e^{2I(\lambda_j)^*(\zeta_j)^*}e^{-Ix((\lambda_j)^*+(\lambda_k)^*)})(\\ e^{-Ix(\lambda_j)^*}(\delta_j^k+\sum_{i=1}^N \frac{4\lambda_i(\lambda_k)^*}{(\lambda_i-(\lambda_j)^*)((\lambda_k)^*-\lambda_i)}e^{-2I\lambda_i(\zeta_i-x)}e^{2I(\lambda_k)^*((\zeta_k)^*-x)})^{-1}e^{Ix(\lambda_k)^*})\\ = -2\Gamma_r\sum_{j=1}^N\sum_{k=1}^N(2I(\lambda_j)^*e^{2I(\lambda_j)^*(\zeta_j)^*}e^{-2Ix(\lambda_j)^*})(\delta_j^k+\\ \sum_{i=1}^N \frac{4\lambda_i(\lambda_k)^*}{(\lambda_i-(\lambda_j)^*)((\lambda_k)^*-\lambda_i)}e^{-2I\lambda_i(\zeta_i-x)}e^{2I(\lambda_k)^*((\zeta_k)^*-x)})^{-1}\end{gathered} \tag{19}$$

Now, If we define

$$\begin{aligned}B_{ij} &= \frac{2\lambda_j}{(\lambda_j-(\lambda_i)^*)}e^{-2I\lambda_j(\zeta_j-x)} = \Lambda_{ij}e^{-2I\lambda_j(\zeta_j-x)}\\ T_{jk} &= -4I(\lambda_j)^*e^{2I(\lambda_j)^*((\zeta_j)^*-x)}\end{aligned} \tag{20}$$

Then the above form of the shape function $q_r(x)$ becomes:

$$q_r(x) = \Gamma_r\sum_{j=1}^N\sum_{k=1}^N T_{jk}(\delta_j^k+\sum_{i=1}^n B_{ji}B_{ik}^*)^{-1} = \Gamma_r(T:(Id+BB^*)^{-1}) \tag{21}$$

Where the product : is defined as $X:Y = X_{ij}Y_{ij}$

6 Updated formula for $q_r(x)$

Let us assume that we obtained N-soliton solutions for the system in (1). The question now is: If we have N-eigenvalues, and we already know the shape of $q_r(x)$ is it possible to know the shape of $q_r(x)$ in case we have $(N+1)$-soliton solutions?. Here, we answer these questions, and derive an updated formula for $q_r(x)$ given in

(21). The matrix B defined in (20) has the form:

$(B_{N+1})_{(N+1)\times(N+1)} = \left(\begin{array}{c|c} (B_N)_{N\times N} & (b)_{N\times 1} \\ \hline (c)_{1\times N} & (d)_{1\times 1} \end{array}\right)$. Let $(M_N)_{N\times N} = (Id + B_N \times B_N^*)_{N\times N}$. Define $X_{N\times N}, \vec{y}_{N\times 1}, \vec{z}_{1\times N}$, and $w_{1\times 1}$ as:

$$\begin{pmatrix} X & \vec{y} \\ \vec{z} & w \end{pmatrix} = \left(\begin{array}{c|c} Id + B_N B_N^* + bc^* & B_N b^* + bd^* \\ \hline cB_N^* + dc^* & 1 + cb^* + dd^* \end{array}\right) = M_{N+1} \tag{22}$$

We wish to find $(M_{N+1})^{-1}$. Define S_w as:

$$S_w = \left(X - \tfrac{1}{w}\vec{y}\otimes\vec{z}\right) \tag{23}$$

where the direct (or dyadic) product $\otimes$ is defined as: $(\vec{y}\otimes\vec{z})_{ij} = y_i z_j$. To see how S_w^{-1} looks like, we prove the following claim:

Claim:

$$S_w^{-1} = (X - \tfrac{1}{w}\vec{y}\otimes\vec{z})^{-1} = (X^{-1} + \tfrac{(X^{-1}\vec{y})\times(\vec{z}X^{-1})}{w - \vec{z}X^{-1}\vec{y}}) \tag{24}$$

Proof : Let $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{Z}$, $\mathbb{W}$ be arbitrary block matrices, and Id be the identity matrix. Then, the matrix $\left(\begin{array}{c|c} \mathbb{X} & \mathbb{Y} \\ \hline \mathbb{Z} & \mathbb{W} \end{array}\right)$ can be diagonalized in two ways by noticing that:

$$\begin{gathered}
\left(\begin{array}{c|c} \mathbb{X} & \mathbb{Y} \\ \hline \mathbb{Z} & \mathbb{W} \end{array}\right) = \left(\begin{array}{c|c} Id & 0 \\ \hline \mathbb{Z}\mathbb{X}^{-1} & Id \end{array}\right)\left(\begin{array}{c|c} \mathbb{X} & 0 \\ \hline 0 & S_{\mathbb{X}} \end{array}\right)\left(\begin{array}{c|c} Id & \mathbb{X}^{-1}\mathbb{Y} \\ \hline 0 & Id \end{array}\right), \\
\left(\begin{array}{c|c} \mathbb{X} & \mathbb{Y} \\ \hline \mathbb{Z} & \mathbb{W} \end{array}\right) = \left(\begin{array}{c|c} Id & \mathbb{Y}\mathbb{W}^{-1} \\ \hline 0 & Id \end{array}\right)\left(\begin{array}{c|c} S_{\mathbb{W}} & 0 \\ \hline 0 & \mathbb{W} \end{array}\right)\left(\begin{array}{c|c} Id & 0 \\ \hline \mathbb{W}^{-1}\mathbb{Z} & Id \end{array}\right),
\end{gathered} \tag{25}$$

where $S_{\mathbb{X}} = \mathbb{W} - \mathbb{Z}\mathbb{X}^{-1}\mathbb{Y}$, and $S_{\mathbb{W}} = \mathbb{X} - \mathbb{Y}\mathbb{W}^{-1}\mathbb{Z}$. From (25) , $\left(\begin{array}{c|c} \mathbb{X} & \mathbb{Y} \\ \hline \mathbb{Z} & \mathbb{W} \end{array}\right)$ can be inverted as:

$$\begin{aligned}
\left(\begin{array}{c|c} \mathbb{X} & \mathbb{Y} \\ \hline \mathbb{Z} & \mathbb{W} \end{array}\right)^{-1} &= \left(\begin{array}{c|c} Id & -\mathbb{X}^{-1}\mathbb{Y} \\ \hline 0 & Id \end{array}\right)\left(\begin{array}{c|c} \mathbb{X}^{-1} & 0 \\ \hline 0 & S_{\mathbb{X}}{}^{-1} \end{array}\right)\left(\begin{array}{c|c} Id & 0 \\ \hline -\mathbb{Z}\mathbb{X}^{-1} & Id \end{array}\right) \\
&= \left(\begin{array}{c|c} \mathbb{X}^{-1}\left(Id + \mathbb{Y}S_{\mathbb{X}}{}^{-1}\mathbb{Z}\mathbb{X}^{-1}\right) & -\mathbb{X}^{-1}\mathbb{Y}S_{\mathbb{X}}{}^{-1} \\ \hline -S_{\mathbb{X}}{}^{-1}\mathbb{Z}\mathbb{X}^{-1} & S_{\mathbb{X}}{}^{-1} \end{array}\right), \\
\left(\begin{array}{c|c} \mathbb{X} & \mathbb{Y} \\ \hline \mathbb{Z} & \mathbb{W} \end{array}\right)^{-1} &= \left(\begin{array}{c|c} Id & 0 \\ \hline -\mathbb{W}^{-1}\mathbb{Z} & Id \end{array}\right)\left(\begin{array}{c|c} S_{\mathbb{W}}{}^{-1} & 0 \\ \hline 0 & \mathbb{W}^{-1} \end{array}\right)\left(\begin{array}{c|c} Id & -\mathbb{Y}\mathbb{W}^{-1} \\ \hline 0 & Id \end{array}\right) \\
&= \left(\begin{array}{c|c} S_{\mathbb{W}}{}^{-1} & -S_{\mathbb{W}}^{-1}\mathbb{Y}\mathbb{W}^{-1} \\ \hline -\mathbb{W}^{-1}\mathbb{Z}S_{\mathbb{W}}{}^{-1} & \mathbb{W}^{-1}\left(Id + \mathbb{Z}S_{\mathbb{W}}{}^{-1}\mathbb{Y}\mathbb{W}^{-1}\right) \end{array}\right)
\end{aligned} \tag{26}$$

From the $(1,1)$ element in(26)we have:

$$S_{\mathbb{W}}{}^{-1} = (\mathbb{X} - \mathbb{Y}\mathbb{W}^{-1}\mathbb{Z}\ )^{-1} = \mathbb{X}^{-1}(Id + \mathbb{Y}S_{\mathbb{X}}{}^{-1}\mathbb{Z}\mathbb{X}^{-1}\ ) \tag{27}$$

Our claim in (24)is now proven by (27). Moreover, From (22) and the second part in (26) we have:

$$\begin{aligned} (M_{N+1})^{-1} &= \begin{pmatrix} X & \vec{y} \\ \vec{z} & w \end{pmatrix}^{-1} = \begin{pmatrix} Id & 0 \\ -\frac{1}{w}\vec{z} & Id \end{pmatrix} \begin{pmatrix} S_w^{-1} & 0 \\ 0 & \frac{1}{w} \end{pmatrix} \begin{pmatrix} Id & -\frac{1}{w}\vec{y} \\ 0 & Id \end{pmatrix} \\ &= \begin{pmatrix} S_w^{-1} & -\frac{1}{w}S_w^{-1}\vec{y} \\ -\frac{1}{w}\vec{z}S_w^{-1} & \frac{1}{w} + \frac{1}{w^2}\vec{z}S_w^{-1}\vec{y} \end{pmatrix} \end{aligned} \tag{28}$$

Notice that the matrix X^{-1} can be simplified by using the famous Sherman-Morrison formula which is: $(A + cd^T\ )^{-1} = A^{-1} - \frac{A^{-1}c \otimes d^T A^{-1}}{1+d^T A^{-1}c}$ as follows:

$$\begin{gathered} X^{-1} = (Id + B_N B_N^* + bc^*)^{-1} = (M_N + bc^*)^{-1} = M_N^{-1} - \frac{M_N^{-1} b \otimes c^* M_N^{-1}}{1+c^* M_N^{-1} b} = M_N^{-1} - P, \\ \text{where} \quad P = \frac{M_N^{-1} b \otimes c^* M_N^{-1}}{1+c^* M_N^{-1} b} \end{gathered} \tag{29}$$

So from (27) we have:

$$S_w{}^{-1} = X^{-1} + \frac{\left(X^{-1}\vec{y}\right)\left(\vec{z}X^{-1}\right)}{w - \vec{z}X^{-1}\vec{y}} = M_N^{-1} - P + \frac{\left(M_N^{-1} - P\right)\vec{y} \times \vec{z}\left(M_N^{-1} - P\right)}{w - \vec{z}\left(M_N^{-1} - P\right)\vec{y}} \tag{30}$$

From (28) and (30) we have:

$$\begin{aligned} (M_{N+1})^{-1} &= \left(Id + B_{N+1}B_{N+1}^*\right)^{-1} = \begin{pmatrix} S_w^{-1} & -\frac{1}{w}S_w^{-1}\vec{y} \\ -\frac{1}{w}\vec{z}S_w^{-1} & \frac{1}{w} + \frac{1}{w^2}\vec{z}S_w^{-1}\vec{y} \end{pmatrix} \\ &= \begin{pmatrix} M_N^{-1} & \vec{0} \\ \vec{0} & 0 \end{pmatrix} + \begin{pmatrix} -P + \frac{\left(M_N^{-1} - P\right)\vec{y}.\vec{z}\left(M_N^{-1} - P\right)}{w - \vec{z}\left(M_N^{-1} - P\right)\vec{y}} & -\frac{1}{w}S_w^{-1}\vec{y} \\ -\frac{1}{w}\vec{z}S_w^{-1} & \frac{1}{w} + \frac{1}{w^2}vzS_w^{-1}\vec{y} \end{pmatrix} \end{aligned} \tag{31}$$

But, $q_r(x)$, given in (21), has the form:

$$\begin{aligned} q_r(x) &= \Gamma_r \textstyle\sum_{j=1}^N \sum_{k=1}^N (T_N\ )_{jk} (Id + B_N.B_N^*\ )_{j,k}^{-1} \\ &= \Gamma_r (T_N : (Id + B_N B_N^*\ )^{-1}) = \Gamma_r (T_N : (M_N\ )^{-1}) \end{aligned} \tag{32}$$

Let qq_r be the shape of the separated pulses in case we have $(N+1)$ eigenvalues, then by (21), (31) and (32) we have:

$$
\begin{aligned}
qq_r(x) &= \Gamma_r(T_{N+1} : (M_{N+1})^{-1}) = \Gamma_r\{\left(\begin{array}{c|c} T_N & \alpha_N \\ \hline \beta_N & \gamma_N \end{array}\right) : \\
&\left(\left(\begin{array}{cc} M_N^{-1} & \vec{0} \\ \vec{0} & 0 \end{array}\right) + \left(\begin{array}{cc} -P + \frac{(M_N^{-1}-P)\vec{y}.\vec{z}(M_N^{-1}-P)}{w-\vec{z}(M_N^{-1}-P)\vec{y}} & -\frac{1}{w}S_w^{-1}\vec{y} \\ -\frac{1}{w}\vec{z}S_w^{-1} & \frac{1}{w}+\frac{1}{w^2}\vec{z}S_w^{-1}\vec{y} \end{array}\right)\right)\} \\
&= \Gamma_r(\left(\begin{array}{c|c} T_N & \alpha_N \\ \hline \beta_N & \gamma_N \end{array}\right) : \left(\begin{array}{c|c} M_N^{-1} & \vec{0} \\ \vec{0} & 0 \end{array}\right) + \left(\begin{array}{c|c} T_N & \alpha_N \\ \hline \beta_N & \gamma_N \end{array}\right) : \\
&\left(\begin{array}{cc} -P + \frac{(M_N^{-1}-P)\vec{y}.\vec{z}(M_N^{-1}-P)}{w-\vec{z}(M_N^{-1}-P)\vec{y}} & -\frac{1}{w}S_w^{-1}\vec{y} \\ -\frac{1}{w}\vec{z}S_w^{-1} & \frac{1}{w}+\frac{1}{w^2}\vec{z}S_w^{-1}\vec{y} \end{array}\right)) \\
&= \Gamma_r((T_N : M_N^{-1}) + (T_N : (-P + \frac{(M_N^{-1}-P)\vec{y}.\vec{z}(M_N^{-1}-P)}{w-\vec{z}(M_N^{-1}-P)\vec{y}})) + (\alpha_N : -\frac{1}{w}S_w^{-1}\vec{y}) \\
&\quad +(\beta_N : -\frac{1}{w}\vec{z}S_w^{-1}) + (\gamma_N : (\frac{1}{w} + \frac{1}{w^2}\vec{z}S_w^{-1}\vec{y}))) \\
&= q_r(x) + \Gamma_r((T_N : (-P + \frac{(M_N^{-1}-P)\vec{y}.\vec{z}(M_N^{-1}-P)}{w-\vec{z}(M_N^{-1}-P)\vec{y}})) + (\alpha_N : -\frac{1}{w}S_w^{-1}\vec{y}) \\
&\quad +(\beta_N : -\frac{1}{w}\vec{z}S_w^{-1}) + (\gamma_N : (\frac{1}{w} + \frac{1}{w^2}\vec{z}S_w^{-1}\vec{y})))
\end{aligned} \tag{33}
$$

So, we get the following updated formula for $q_r(x)$:

$$
\begin{aligned}
qq_r(x) &= q_r(x) + \Gamma_r((T_N : (-P + \frac{(M_N^{-1}-P)\vec{y}.\vec{z}(M_N^{-1}-P)}{w-\vec{z}(M_N^{-1}-P)\vec{y}})) \\
&+(\alpha_N : -\frac{1}{w}S_w^{-1}\vec{y}) + (\beta_N : -\frac{1}{w}\vec{z}S_w^{-1}) + (\gamma_N : (\frac{1}{w} + \frac{1}{w^2}\vec{z}S_w^{-1}\vec{y})))
\end{aligned} \tag{34}
$$

The updated formula in (34) shows that the $(N+1)$ soliton formula is just the previous formula for the N-soliton solution plus the last terms. These last terms are the analytic form of the soliton numbered $N+1$.

7 Approximation of the updated formula in (34)

Here, we will approximate the last 4 terms in (34) to look more simple. From the definition of the matrices B and T in(20) we have :

$$
\begin{aligned}
(B_{N+1})_{(N+1\times N+1)} &= \left(\begin{array}{c|c} (B_N)_{N\times N} & (b)_{N\times 1} \\ \hline (c)_{1\times N} & (d)_{1\times 1} \end{array}\right) \\
(T_{N+1})_{(N+1\times N+1)} &= \left(\begin{array}{c|c} (T_N)_{(N\times N)} & (\alpha_N)_{N\times 1} \\ \hline (\beta_N)_{1\times N} & (\gamma_N)_{1\times 1} \end{array}\right)
\end{aligned} \tag{35}
$$

If we assume that $\lambda_j = \mu_j + I\nu_j$ and $\zeta_j = \phi_j + I\psi_j$ where μ_j , ϕ_j, ψ_j are arbitrary real numbers and $\nu_j > 0, j = 1,2,3,....,N$, then we can approximate $qq_r(x)$ for large x ,were $\phi_1 < \phi_2 < ... < \phi_N < x < \phi_{N+1}$ as follows: from (20) we have

$$\begin{aligned} B_{ij} &= \tfrac{2\lambda_j}{\left(\lambda_j-(\lambda_i)^*\right)} e^{-2I\lambda_j\left(\zeta_j-x\right)} = \tfrac{2\left(\mu_j+I\nu_j\right)}{\left(\left(\mu_j+I\nu_j\right)-\left(\mu_i-I\nu_i\right)\right)} e^{-2I\left(\mu_j+I\nu_j\right)\left(\phi_j+I\psi_j-x\right)} \\ T_{jk} &= -4I\left(\lambda_j\right)^* e^{2I\left(\lambda_j\right)^*\left(\left(\zeta_j\right)^*-x\right)} = -4I\left(\mu_j-I\nu_j\right) e^{2I\left(\mu_j-I\nu_j\right)\left(\phi_j-I\psi_j-x\right)}. \end{aligned} \tag{36}$$

So from (36) we have:

$$\begin{gathered} |B_{ij}| = 2\sqrt{\tfrac{\left(\mu_j^2+\nu_j^2\right)}{\left(\mu_i-\mu_j\right)^2+\left(\nu_i+\nu_j\right)^2}} e^{2\psi_j\mu_j} e^{2\left(-x+\phi_j\right)\nu_j} = \chi_{ij} e^{2\psi_j\mu_j} e^{2\left(-x+\phi_j\right)\nu_j}, \\ \text{where} \quad \chi_{ij} = 2\sqrt{\tfrac{\left(\mu_j^2+\nu_j^2\right)}{\left(\mu_i-\mu_j\right)^2+\left(\nu_i+\nu_j\right)^2}} \quad \text{and} \\ |T_{jk}| = 4\sqrt{\left(\mu_j^2+\nu_j^2\right)} e^{2\mu_j\psi_j} e^{2\nu_j\left(-x+\phi_j\right)} \end{gathered} \tag{37}$$

Since $\nu_j > 0$ and $\phi_1 < \phi_2 < \ldots < \phi_N < x < \phi_{N+1}$ we have:

$$\begin{cases} e^{2\left(-x+\phi_j\right)\nu_j} < 1, & j < N+1 \\ e^{2\left(-x+\phi_j\right)\nu_j} > 1, & j = N+1 \end{cases} \tag{38}$$

If we neglect all terms which are < 1 ,then we can approximate $|B_{N+1}|$ and $|T_{N+1}|$ as follows:

$$\begin{aligned} |B_{N+1}| &= \left(\begin{array}{c|c} |B_N| & |b| \\ \hline |c| & |d| \end{array}\right) \approx \left(\begin{array}{c|c} 0 & |b| \\ \hline 0 & |d| \end{array}\right) \\ |T_{N+1}| &= \left(\begin{array}{c|c} |T_N| & |\alpha| \\ \hline |\beta| & |\gamma| \end{array}\right) \approx \left(\begin{array}{c|c} 0 & 0 \\ \hline |\beta| & |\gamma| \end{array}\right) \end{aligned} \tag{39}$$

So from (22) , (39) we have:

$$\begin{gathered} M_{N+1} = Id + B_{N+1}B_{N+1}^* = \left(\begin{array}{c|c} Id + B_N B_N^* + bc^* & B_N b^* + bd^* \\ \hline cB_N^* + dc^* & 1 + cb^* + dd^* \end{array}\right) \\ \approx \left(\begin{array}{c|c} Id & bd^* \\ \hline 0 & 1 + dd^* \end{array}\right) = \begin{pmatrix} X & \vec{y} \\ \vec{z} & w \end{pmatrix} \end{gathered} \tag{40}$$

and from (29) , (30) we have:

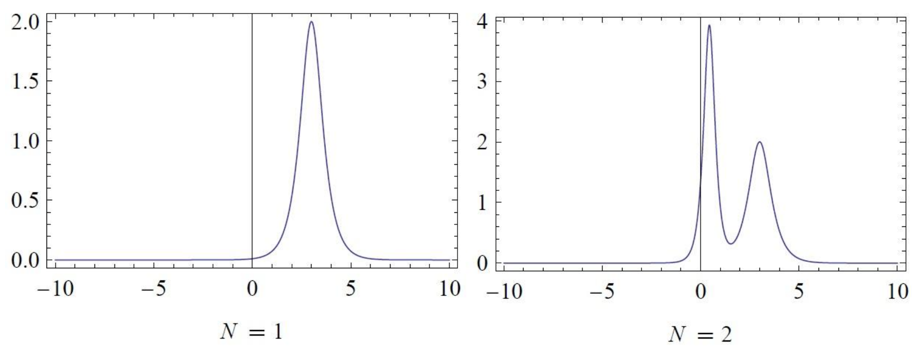
$$\begin{aligned} P &= \tfrac{M_N^{-1} b \otimes c^* M_N^{-1}}{1 + c^* M_N^{-1} b} \approx 0, \\ S_W^{-1} &= \left(X - YW^{-1}Z\right)^{-1} \approx X^{-1} = Id \end{aligned} \tag{41}$$

So the updated formula in (34) is approximated by:

$$qq_r(x) \approx q_r(x) + \Gamma_r\left(\gamma_N : \tfrac{1}{1+dd^*}\right) \tag{42}$$

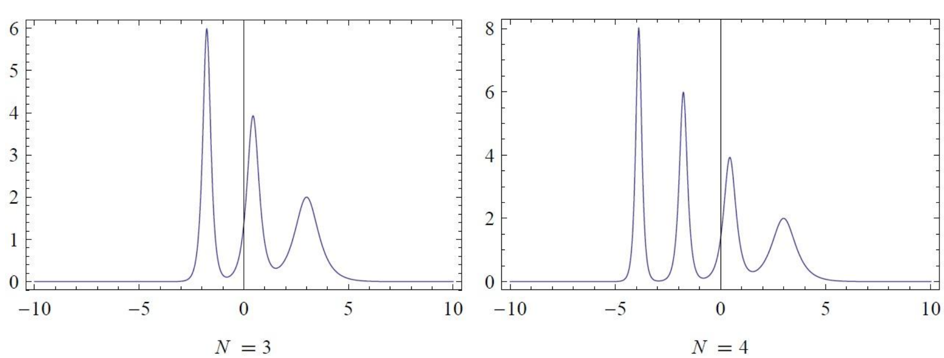
8 Graphs of $q_r(x)$ and $qq_r(x)$

By using a simple *Mathematica* code we plot the graph of 1,2,3,4-soliton solutions using (12) , then we do the same by using the updated formula in (34), the result shows that they both give the exact shape.



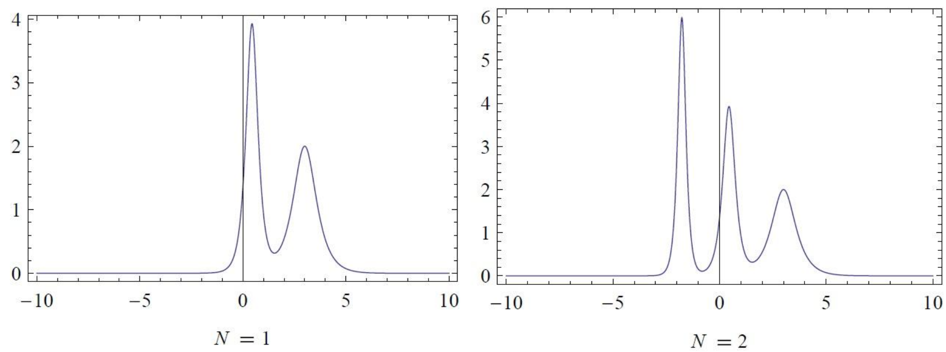
Figure(1)

Figure(1) shows the graph of $q_r(x)$ in (12) incase we have $N = 1$ and $N = 2$



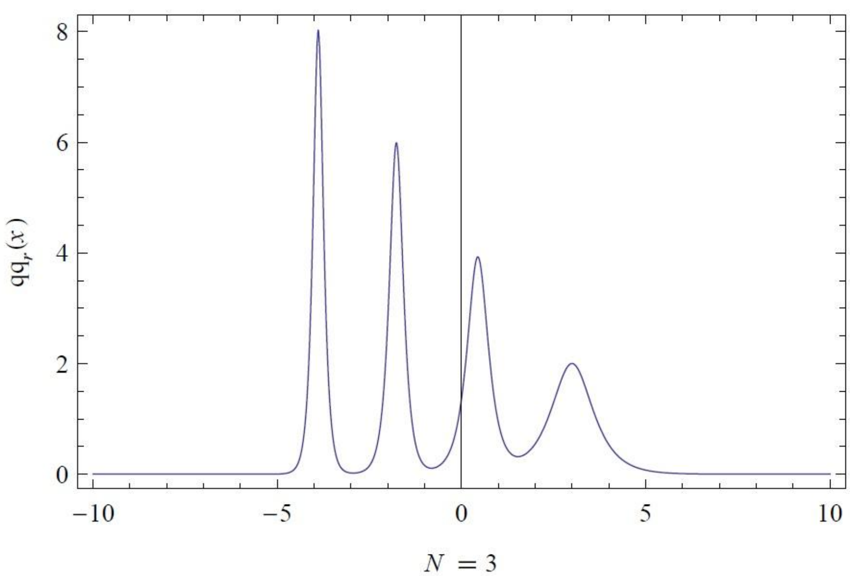
Figure(2)

Figure(2) shows the graph of $q_r(x)$ in (12) incase we have $N = 3$ and $N = 4$



Figure(3)

Figure(3) shows the graph of $qq_r(x)$ in (42) incase we have $N=1$ and $N=2$, Notice that in the first shape the left peak represents the shape of $q_r(x)$ in (12) in case $N=1$ while the right peak represents the extra term in $qq_r(x)$ in (42) Also in the second shape the left 2 peaks represent the shape of $q_r(x)$ in (12) in case $N=2$ while the right peak represents the extra term in $qq_r(x)$ in (42).



Figure(4)

Figure(4) shows the graph of $qq_r(x)$ in (42) incase we have $N=3$. Notice that the left 3 peaks represent the shape of $q_r(x)$ in (12) in case $N=3$, while the right peak represents the extra term in $qq_r(x)$ in (42). We are strongly expecting that the graphs of $q_r(x)$ and $qq_r(x)$ for larger values of N will give us same results, but because the analytic form of $q_r(x)$ in (12) becomes very complicated , we couldn't plot them by Mathematica.

9 Conclusion

The $(N+1)$-soliton solution for the system of Three-Wave-Interaction (TWI) equations is simplified as a sum of the N-soliton solution plus extra terms, these terms were approximated to look very simple. Doing this will help us to build a solution for the (TWI) equations for large N, and enable us to construct a train of soliton solutions, which is an interested physical problem. Since the N-soliton solution formula looks very complicated for large values of N, we chose small values of N and checked successfully our $(N+1)$-soliton solution formula analytically and graphically.

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