A Fixed Point Theorem for Weakly \( \alpha \)-Contractive Mappings with Application

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Abstract

In this paper, we introduce concept of weakly \( \alpha \)-contractive mappings, and we establish a new fixed point theorem for such mappings. We give an application to ordinary differential equations.

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1 Introduction and preliminaries

Banach’s contraction principle [6] is one of the pivotal results of analysis. Because of its importance in mathematical theory, many authors gave generalizations and extensions of it in many directions (see [2, 3, 4, 7, 10, 14, 18, 19, 22, 23, 24, 25, 26, 29] and reference therein).

The authors [1] introduced the notion of weakly contractive mappings in Hilbert spaces and proved that any weakly contractive mapping defined on complete Hilbert spaces has a unique fixed point.

In [27], Rhoades extends some of results in [1] to Banach spaces.

We denote by \( \Psi \) the family of all nondecreasing functions \( \psi : [0, \infty) \to [0, \infty) \) such that \( \psi \) is positive on \((0, \infty)\), \( \psi(0) = 0 \) and \( \lim_{t \to \infty} \psi(t) = \infty \).

Let \((X, d)\) be a metric space, and let \( \psi \in \Psi \) and \( \psi \) be continuous.
A mapping $T : X \to X$ is called weakly contractive [27] if
\[
d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))
\]
for all $x, y \in X$.

In [27], the following theorem is proved.

**Theorem 1.1.** Let $(X, d)$ be a complete metric space. If a mapping $T : X \to X$ is weakly contractive, then $T$ has a unique fixed point in $X$.

In [16], Harjani and Sadarangani extended above theorem to the case of partially ordered metric spaces. In [16], the following theorem is proved.

**Theorem 1.2.** Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric $d$ on $X$ such that $(X, d)$ is a complete metric space. Assume that a mapping $T : X \to X$ satisfies the following conditions:

1. $d(Tx, Ty) \leq d(x, y) - \psi(d(x, y))$, where $\psi \in \Psi$ and $\psi$ is continuous; for all $x, y \in X$ with $y \preceq x$;

2. there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

3. $T$ is nondecreasing;

4. either $T$ is continuous or if $\{x_n\}$ is increasing sequence with $\lim_{n \to \infty} x_n = x$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then $T$ has a fixed point in $X$. Further if, for any $x, y \in X$, there exists $z \in X$ such that $z$ is comparable to $x$ and $y$, then $T$ has a unique fixed point in $X$.

Recently, the authors [28] introduced the notion of $\alpha$-$\phi$-contractive mapping in metric spaces:

Let $(X, d)$ be a metric space. A mapping $T : X \to X$ is called $\alpha$-$\phi$-contractive if there exist two function $\alpha : X \times X \to [0, \infty)$ and $\phi : [0, \infty) \to [0, \infty)$, where $\phi$ is nondecreasing and $\sum_{n=1}^{\infty} \phi^n(t) < \infty$ for each $t > 0$ and $\phi^n$ is the $n$-th iteration of $\phi$, such that
\[
\alpha(x, y)d(Tx, Ty) \leq \phi(d(x, y))
\]
for all $x, y \in X$.

In [28], the following theorem is proved.

**Theorem 1.3.** Let $(X, d)$ be a complete metric space. Suppose that an $\alpha$-$\phi$-contractive mapping $T : X \to X$ satisfies the following conditions:
(1) $T$ is $\alpha$-admissible, i.e. for all $x, y \in X$, $\alpha(x, y) \geq 1$ implies $\alpha(Tx, Ty) \geq 1$;

(2) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;

(3) either $T$ is continuous or if $\{x_n\}$ is a sequence with $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} x_n = x$, then $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N}$.

Then $T$ has a fixed point in $X$. Further if, for any $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then $T$ has a unique fixed point in $X$.

In this paper, we introduce a concept of weakly $\alpha$-contractive mappings in metric spaces and establish a new fixed point theorem for such mappings, which are generalizations of the results in [27] and [16].

Finally, we give an application of our result for weakly $\alpha$-contractive mappings to ordinary differential equations.

## 2 Fixed point theorems

Let $(X, d)$ be a metric space.

From now on, let $\alpha : X \times X \to [0, \infty)$ be a function and $\eta \in \Psi$.

A mapping $T : X \to X$ is called weak $\alpha$-contractive if

$$\alpha(x, y)d(Tx, Ty) \leq d(x, y) - \eta(d(x, y))$$

for all $x, y \in X$.

**Theorem 2.1.** Let $(X, d)$ be a complete metric space. Suppose that a weak $\alpha$-contraction mapping $T : X \to X$ satisfies the following:

(1) for each $x, y, z \in X$, $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$;

(2) $T$ is $\alpha$-admissible;

(3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$;

(4) either $T$ is continuous or

$$\lim_{n \to \infty} \inf \alpha(T^n x_1, x) > 0 \quad (2.0)$$

for any cluster point $x$ of $\{T^n x_1\}$.

Then $T$ has a fixed point in $X$. Further if, for all $x, y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$ and $\alpha(y, z) \geq 1$, then $T$ has a unique fixed point.
Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\} \subset X$ by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$.

If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $x_n$ is a fixed point of $T$, and the proof is finished.

Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.

Since $\alpha(x_1, x_2) = \alpha(x_1, Tx_1) \geq 1$, from (2) we have $\alpha(x_2, x_3) = \alpha(Tx_1, Tx_2) \geq 1$. Again, from (2) we have $\alpha(x_3, x_4) = \alpha(Tx_2, Tx_3) \geq 1$.

By induction, we obtain

$$\alpha(x_n, x_{n+1}) \geq 1$$

for all $n \in \mathbb{N}$.

Then we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1})$$
$$\leq \alpha(x_n, x_{n+1})d(Tx_n, Tx_{n+1})$$
$$\leq d(x_n, x_{n+1}) - \eta(d(x_n, x_{n+1}))$$

for all $n \in \mathbb{N}$.

Thus, $d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, and so the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing. Hence there exists $r \geq 0$ such that $\lim_{n \to \infty} d(x_n, x_{n+1}) = r$. So $r \leq d(x_n, x_{n+1})$. Since $\eta$ in nondecreasing, $\eta(r) \leq \eta(d(x_n, x_{n+1}))$, and so $\eta(r) \leq \lim_{n \to \infty} \eta(d(x_n, x_{n+1}))$.

Letting $n \to \infty$ in the inequality (2.2), we have

$$r \leq r - \lim_{n \to \infty} \eta(d(x_n, x_{n+1})) \leq r - \eta(r).$$

Hence $\eta(r) = 0$, and hence $r = 0$. Thus,

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = 0.$$

We now show that $\{x_n\}$ is a Cauchy sequence.

Suppose that $\{x_n\}$ is not a Cauchy sequence.

Then there exists $\epsilon > 0$ such that, for all $k > 0$, there exist $m(k) > n(k) > k$ with

$$d(x_{m(k)}, x_{n(k)}) \geq \epsilon$$
and $$d(x_{m(k)-1}, x_{n(k)}) < \epsilon.$$

Then we have

$$\epsilon$$
$$\leq d(x_{m(k)}, x_{n(k)})$$
$$\leq d(x_{m(k)}, x_{m(k)-1}) + d(x_{m(k)-1}, x_{n(k)})$$
$$< d(x_{m(k)}, x_{m(k)-1}) + \epsilon.$$
Letting $k \to \infty$ in above inequality, we have

$$\lim_{k \to \infty} d(x_{m(k)}, x_{n(k)}) = \epsilon. \quad (2.4)$$

By using (2.3) and (2.4), we obtain $\lim_{k \to \infty} d(x_{m(k)-1}, x_{n(k)-1}) = \epsilon$. Thus, we have $d(x_{m(k)-1}, x_{n(k)-1}) > \frac{\epsilon}{2}$ for sufficiently large $k$. Since $\eta$ is nondecreasing, $\eta(d(x_{m(k)-1}, x_{n(k)-1})) > \eta(\frac{\epsilon}{2})$ for sufficiently large $k$.

From (1) and (2.1) we have $\alpha(x_{n(k)-1}, x_{m(k)-1}) \geq 1$. Thus we have

$$d(x_{m(k)}, x_{n(k)}) = d(Tx_{m(k)-1}, Tx_{n(k)-1})$$

$$\leq \alpha(x_{n(k)-1}, x_{m(k)-1})d(Tx_{n(k)-1}, Tx_{m(k)-1})$$

$$\leq d(x_{n(k)-1}, x_{m(k)-1}) - \eta(d(x_{n(k)-1}, x_{m(k)-1})).$$

Hence, we have $\epsilon \leq \epsilon - \lim_{k \to \infty} \eta(d(x_{m(k)-1}, x_{n(k)-1})) \leq \epsilon - \eta(\frac{\epsilon}{2})$. So $\eta(\frac{\epsilon}{2}) = 0$, and $\epsilon = 0$, which is a contradiction. Thus $\{x_n\}$ is a Cauchy sequence.

It follows from the completeness of $X$ that there exists

$$x_* = \lim_{n \to \infty} x_n \in X. \quad (2.5)$$

If $T$ is continuous, then $\lim_{n \to \infty} x_n = Tx_*$, and so $x_* = Tx_*$. Assume that $\lim_{n \to \infty} \inf \alpha(T^n x_1, x) > 0$ for any cluster point $x$ of $\{T^n x_1\}$.

Then, $p := \lim_{n \to \infty} \inf \alpha(x_n, x_*) > 0$. Hence there exists $N \in \mathbb{N}$ such that $\alpha(x_n, x_*) > 0$ for all $n > N$.

Thus we have

$$d(x_{n+1}, Tx_*) = d(Tx_n, Tx_*)$$

$$\leq \frac{1}{\alpha(x_n, x_*)} \left[ d(x_n, x_*) - \eta(d(x_n, x_*)) \right]$$

$$\leq \frac{1}{\alpha(x_n, x_*)} d(x_n, x_*)$$

for all $n > N$.

Hence, we obtain

$$d(x_*, Tx_*) = \lim_{n \to \infty} \sup_{n \to \infty} d(x_{n+1}, Tx_*)$$

$$\leq \frac{1}{p} \lim_{n \to \infty} \sup_{n \to \infty} d(x_n, x_*)$$

$$= 0.$$
Hence, \( x_\ast \) is a fixed point of \( T \).

We now show that the fixed point of \( T \) is unique under assumption that, for all \( x, y \in X \), there exists \( z \in X \) such that \( \alpha(x, z) \geq 1 \) and \( \alpha(y, z) \geq 1 \).

Let \( y_\ast \in X \) be another fixed point of \( T \).

Then by assumption, there exists \( z \in X \) such that \( \alpha(x_\ast, z) \geq 1 \) and \( \alpha(y_\ast, z) \geq 1 \). From (2) we obtain \( \alpha(x_\ast, T^n z) \geq 1 \) and \( \alpha(y_\ast, T^n z) \geq 1 \) for all \( n \in \mathbb{N} \).

Hence we have

\[
\begin{align*}
    d(x_\ast, T^n z) & \leq \alpha(x_\ast, T^{n-1} z) d(T x_\ast, TT^{n-1} z) \\
                     & \leq d(x_\ast, T^{n-1} z) - \eta(d(x_\ast, T^{n-1} z)) \\
                     & < d(x_\ast, T^{n-1} z)
\end{align*}
\]

for all \( n \in \mathbb{N} \).

Thus, the sequence \( \{d(x_\ast, T^n z)\} \) is nonincreasing, and so there exists \( u \geq 0 \) such that \( \lim_{n \to \infty} d(x_\ast, T^n z) = u \). Since \( u \leq d(x_\ast, T^n z) \) and \( \eta \) is nondecreasing, \( \eta(u) \leq \eta(d(x_\ast, T^n z)) \). Hence \( \eta(u) \leq \lim_{n \to \infty} \eta(d(x_\ast, T^n z)) \).

Taking limit of \( n \to \infty \) in (2.6), we obtain

\[
u \leq u - \lim_{n \to \infty} \eta(d(x_\ast, T^n z)) \leq u - \eta(u)
\]

which implies \( \eta(u) = 0 \). Hence \( u = 0 \), and hence \( \lim_{n \to \infty} T^n z = x_\ast \).

Similarly, we can show that \( \lim_{n \to \infty} T^n z = y_\ast \). Thus, we have \( x_\ast = y_\ast \).

**Example 2.1.** Let \( X = [0, \infty) \), and let \( d(x, y) = |x - y| \) for all \( x, y \in X \).

We define a mapping \( T : X \to X \) by

\[
T x = \begin{cases} 
\frac{1}{2} x & (0 \leq x \leq 1), \\
2 x & (x > 1).
\end{cases}
\]

Then, \( T \) is not weakly contractive. In fact, \( d(T 1, T 2) = 2 > d(2, 1) > d(2, 1) - \eta(d(2, 1)) \) for \( \eta \in \Psi \).

We define a function \( \alpha : X \times X \to [0, \infty) \) by

\[
\alpha(x, y) = \begin{cases} 
1 & (0 \leq x, y \leq 1), \\
0 & \text{otherwise}.
\end{cases}
\]

Obviously, condition (1) of Theorem 2.1 is satisfied. Condition (3) of Theorem 2.1 is satisfied with \( x_1 = 1 \). Condition (4) of Theorem 2.1 is satisfied with \( T^n x_1 = \frac{1}{n} \).

And \( T \) is a weak \( \alpha \)-contractive mapping with \( \eta(t) = \frac{1}{2} t \) for all \( t \geq 0 \).
Let \( x, y \in X \) such that \( \alpha(x, y) \geq 1 \).

Then \( x, y \in [0, 1] \), and so \( Tx \in [0, 1], Ty \in [0, 1] \) and \( \alpha(Tx, Ty) = 1 \). Hence \( T \) is \( \alpha \)-admissible. Thus, all hypotheses of Theorem 2.1 are satisfied, and \( T \) has a fixed point \( x^* = 0 \).

**Corollary 2.2.** Let \((X, d)\) be a complete metric space. Suppose that a weak \( \alpha \)-contraction mapping \( T : X \to X \) satisfies the following:

1. for each \( x, y, z \in X \), \( \alpha(x, y) \geq 1 \) and \( \alpha(y, z) \geq 1 \) implies \( \alpha(x, z) \geq 1 \);
2. \( T \) is \( \alpha \)-admissible;
3. there exists \( x_1 \in X \) such that \( \alpha(x_1, Tx_1) \geq 1 \);
4. either \( T \) is continuous or if \( \{x_n\} \) is a sequence with \( \alpha(x_n, x_{n+1}) \geq 1 \) for all \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} x_n = x \), then \( \alpha(x_n, x) \geq 1 \) for all \( n \in \mathbb{N} \).

Then \( T \) has a fixed point in \( X \). Further if, for all \( x, y \in X \), there exists \( z \in X \) such that \( \alpha(x, z) \geq 1 \) and \( \alpha(y, z) \geq 1 \), then \( T \) has a unique fixed point.

**Corollary 2.3.** Let \((X, d)\) be a complete metric space. Suppose that a mapping \( T : X \to X \) satisfies

\[
d(Tx, Ty) \leq d(x, y) - \eta(d(x, y))
\]

for all \( x, y \in X \).

Then \( T \) has a unique fixed point in \( X \).

**Proof.** Let \( \alpha : X \times X \to [0, \infty) \) be a function defined by \( \alpha(x, y) = 1 \) for all \( x, y \in X \).

Then all conditions of Corollary 2.2 are satisfied. By Corollary 2.2, \( T \) has a unique fixed point in \( X \). \( \square \)

**Corollary 2.4.** Let \((X, \preceq)\) be a partially ordered set and suppose that there exists a metric \( d \) on \( X \) such that \((X, d)\) is a complete metric space. Suppose that a mapping \( T : X \to X \) satisfies the following conditions:

1. \( d(Tx, Ty) \leq d(x, y) - \eta(d(x, y)) \)
   for all \( x, y \in X \) with \( y \preceq x \);
2. there exists \( x_1 \in X \) such that \( x_1 \preceq Tx_1 \);
3. \( T \) is nondecreasing;
4. either \( T \) is continuous or if \( \{x_n\} \) is nondecreasing sequence with \( \lim_{n \to \infty} x_n = x \), then \( x_n \preceq x \) for all \( n \in \mathbb{N} \).
Then $T$ has a fixed point in $X$. Further if, for any $x, y \in X$, there exists $z \in X$ such that $z$ is comparable to $x$ and $y$, then $T$ has a unique fixed point in $X$.

Proof. Define a function $\alpha : X \times X \rightarrow [0, \infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases}$$

Then, from (1) we have $\alpha(x, y)d(Ty, Tx) \leq d(y, x) - \eta(d(y, x))$ for all $x, y \in X$, and so $T$ is a weak $\alpha$-contraction mapping.

Obviously, condition (1) of Corollary 2.2 is satisfied. Since $T$ is nondecreasing, $\alpha(x, y) = 1$ implies $\alpha(Tx, Ty) = 1$ for all $x, y \in X$. Thus, the condition (2) of Corollary 2.2 is satisfied.

Condition (2) implies that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) = 1$, and so condition (3) of Corollary 2.2 is satisfied.

Thus, all conditions of Corollary 2.2 are satisfied. From Corollary 2.2 $T$ has a fixed point in $X$. \hfill \Box

Remark 2.1. (1) Without continuity of $\eta$, we have above Corollary 2.3 and Corollary 2.4.

(2) Corollary 2.3 (resp. Corollary 2.4) is a generalization of Theorem 1 in [27] (resp. Theorem 2 in [16]).

3 Application to ordinary differential equations

We consider the following two-point boundary value problem of second order differential equation:

$$\begin{cases} -\frac{d^2x}{dt^2} = f(t, x(t)), & t \in [0, 1] \\ x(0) = x(1) = 0, \end{cases} \tag{3.1}$$

where $f : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function.

The green function associated to (4.1) is given by

$$G(t, s) = \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1 \\ s(1-t), & 0 \leq s \leq t \leq 1. \end{cases}$$

Let $C(I)$ be the space of all continuous functions defined on $I$, where $I = [0, 1]$, and let $d(x, y) = \|x - y\|_\infty = \max_{t \in I} |x(t) - y(t)|$ for all $x, y \in C(I)$.

Then $(C(I), d)$ is a complete metric space.

We consider the following conditions:
(a) There exists a function $\xi : \mathbb{R}^2 \to \mathbb{R}$ such that for all $t \in I$, for all $a, b \in \mathbb{R}$ with $\xi(a, b) \geq 0$, we have

$$| f(t, a) - f(t, b) | \leq \ln(|a - b| + 1).$$

(b) There exists $x_1 \in C(I)$ such that for all $t \in I$

$$\xi(x_1(t), \int_0^1 G(t, s)f(s, x_1(s))ds) \geq 0.$$

(c) For all $t \in I$ and for all $x, y \in C(I)$,

$$\xi(x(t), y(t)) \geq 0 \text{ implies } \xi(\int_0^1 G(t, s)f(s, x(s))ds, \int_0^1 G(t, s)f(s, y(s))ds) \geq 0.$$

(d) If $\{x_n\}$ is a sequence in $C(I)$ such that $\lim_{n \to \infty} x_n = x \in C(I)$ and $\xi(x_n, x_{n+1}) \geq 0$ for all $n \in \mathbb{N}$, then $\xi(x_n, x) \geq 0$ for all $n \in \mathbb{N}$.

**Theorem 3.1.** Suppose that conditions (a)-(d) are satisfied. Then (3.1) has at least one solution $x^* \in C^2(I)$.

**Proof.** It is known that $x \in C^2(I)$ is a solution of (3.1) if and only if $x \in C(I)$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t, s)f(s, x(s))ds \text{ for all } t \in I.$$

We define $T : C(I) \to C(I)$ by

$$Tx(t) = \int_0^1 G(t, s)f(s, x(s))ds \text{ for all } t \in I.$$

Then the problem (3.1) is equivalent to find $x^* \in C(I)$ that is a fixed point of $T$.

Let $x, y \in C(I)$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in I$. 

From (a) we have
\[
|Tx(t) - Ty(t)| = |\int_0^1 G(t,s)[f(s, x(s)) - f(s, y(s))]ds| \\
\leq \int_0^1 G(t,s)|f(s, x(s)) - f(s, y(s))| \, ds \\
\leq \int_0^1 G(t,s) \ln(|x(s) - y(s)| + 1) \, ds \\
\leq \sup_{t \in I} \int_0^1 G(t,s)ds \ln(\|x(s) - y(s)\|_\infty + 1) \\
= \frac{1}{8} \ln(\|x(s) - y(s)\|_\infty + 1) \\
< \ln(\|x(s) - y(s)\|_\infty + 1) \\
\leq \|x - y\|_\infty - (\|x - y\|_\infty - \ln(\|x(s) - y(s)\|_\infty + 1)) \\
= \|x - y\|_\infty - \eta(\|x - y\|_\infty),
\]
where \(\eta(t) = t - \ln(t + 1).\) (Note that \(\eta \in \Psi.\))

Thus we have \(\|Tx - Ty\|_\infty \leq \|x - y\|_\infty - \eta(\|x - y\|_\infty)\) for all \(x, y \in C(I)\) such that \(\xi(x(t), y(t)) \geq 0\) for all \(t \in I.\)

We define \(\alpha : C(I) \times C(I) \to [0, \infty)\) by
\[
\alpha(x, y) = \begin{cases} 1, & \text{if } \xi(x(t), y(t)) \geq 0, t \in I, \\ 0, & \text{otherwise}. \end{cases}
\]

Then, for all \(x, y \in C(I),\) we have
\[
\alpha(x, y)\|Tx - Ty\|_\infty \leq \|x - y\|_\infty - \eta(\|x - y\|_\infty)
\]
and so \(T\) is a weak \(\alpha\)-contraction mapping defined on \((C(I), d).\)

Obviously, \(\alpha(x, y) \geq 1\) and \(\alpha(y, z) \geq 1\) implies \(\alpha(x, z) \geq 1\) for all \(x, y, z \in C(I).\)

If \(\alpha(x, y) \geq 1\) for all \(x, y \in C(I),\) then \(\xi(x(t), y(t)) \geq 0.\) From (c) we have \(\xi(Tx(t), Ty(t)) \geq 0,\) and so \(\alpha(Tx, Ty) \geq 1.\)

From (b) there exists \(x_1 \in C(I)\) such that \(\alpha(x_1, Tx_1) \geq 1.\)

From (d) we have that if \(\{x_n\}\) is a sequence in \(C(I)\) such that \(\lim_{n \to \infty} x_n = x \in C(I)\) and \(\alpha(x_n, x_{n+1}) \geq 1\) for all \(n \in \mathbb{N},\) then \(\alpha(x_n, x) \geq 1\) for all \(n \in \mathbb{N}.\)

By applying Corollary 2.2, \(T\) has a fixed point in \(C(I),\) i.e. there exists \(x_* \in C(I)\) such that \(Tx_* = x_*\), and \(x_*\) is a solution of (3.1).
References


[28] B. Samet, C. Vetro, P. Vetro, Fixed point theorems for $\alpha$-$\psi$-contractive type mappings, Nonlinear Analysis 75(2012) 2154-2165.


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