The Stability of Cubic Functional Equations in 2-Banach Space

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Abstract
In this paper, we study the generalized Hyers-Ulam-Rassias stability of the following Cubic functional equations

\[ f(2x+y) + f(2x-y) = 2f(x+y) + 2f(x-y) + 12f(x) \]  \hspace{1cm} (0.1)

\[ f(3x+y) + f(3x-y) = 3f(x+y) + 3f(x-y) + 48f(x) \]  \hspace{1cm} (0.2)

for the mapping \( f \) from normed linear space in to 2-Banach spaces.

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1. Introduction & Preliminaries

In 1940, Stanislaw M. Ulam [10], triggered the study of stability problems for various functional equations. He presented a number of important unsolved problems. One of the interesting problem in the theory of non-linear analysis concerning the stability of homomorphism was as follows:

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with the metric \( d(.,.) \). Given \( \varepsilon > 0 \), does there exist a \( \delta > 0 \) such that if a mapping \( h : G_1 \rightarrow G_2 \) satisfies the inequality \( d(h(xy), h(x)h(y)) < \delta \), for all \( x, y \in G_1 \), then there is a homomorphism \( H : G_1 \rightarrow G_2 \) with \( d(h(x), H(x)) < \varepsilon \), for all \( x \in G_1 \)? If the answer is affirmative, we would say that equation of homomorphism \( H(xy) = H(x)H(y) \) is stable.


\[
f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)
\]

is called as cubic functional equation. Since the function \( f(x) = cx^3 \) is a solution of the above functional equation (1.1). Thus, it is called the cubic functional equation and every solution of the cubic functional equation (1.1) is said to be a cubic function. In 2002, K. W. Jun and H. M. Kim [5] established the general solution and proved the generalized Hyers-Ulam-Rassias stability of the cubic functional equation (1.1). The functional equation

\[
f(3x + y) + f(3x - y) = 3f(x + y) + 3f(x - y) + 48f(x)
\]

is also called as cubic functional equation. In 2004, K. H. Park and Y. S. Jung [4] introduced the cubic functional equation (1.2) and proved the generalized Hyers-Ulam-Rassias stability of the cubic functional equation (1.2) on abelian groups.

Recently, W. G. Park [13] investigated the stability of approximate additive mappings, approximate Jensen mappings and approximate quadratic mappings in
2-Banach spaces. This paper is organized as follows: In section 1, we adopt some usual terminology, notations and conventions which will be used later in the next sections. In Section 2 we establish the Hyers-Ulam-Rassias stability of the cubic functional equation (1.1) in 2-Banach space. Further in the last section we established the Hyers-Ulam-Rassias stability of the cubic functional equation (1.2) in 2-Banach space. In 1960s, S. Gahler [7, 8, 9] introduced the concept of linear 2-normed spaces.

**Definition 1.1.** Let $A$ be a linear space over $\mathbb{R}$ with $\dim A > 1$ and let $\|\cdot\|: A \times A \to \mathbb{R}$ be a function satisfying the following properties:

(a) $\|x, y\| = 0$ if and only if $x$ and $y$ are linearly dependent,

(b) $\|x, y\| = \|y, x\|$,

(c) $\|\lambda x, y\| = |\lambda| \|x, y\|$,

(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$,

for all $x, y, z \in A$ and $\lambda \in \mathbb{R}$. Then the function $\|\cdot\|$ is called a 2-norm on $A$ and the pair $(A, \|\cdot\|)$ is called a linear 2-normed space. Sometimes the condition (d) called the triangle inequality.


**Lemma 1.2.** Let $(A, \|\cdot\|)$ be a linear 2-normed space. If $\|x, y\| = 0$ for all $y \in A$, then $x = 0$.

In the 1960’s, S. Gahler and A. White [1, 2, 9] introduced the concept of 2-Banach spaces. In order to define completeness, the concepts of Cauchy sequences and convergence are required.

**Definition 1.3.** A sequence $\{x_n\}$ in a linear 2-normed space $A$ is called a Cauchy sequence if there are two points $y, z \in A$ such that $y$ and $z$ are linearly independent, $\lim_{n,m \to \infty} \|x_n - x_m, y\| = 0$ and $\lim_{n,m \to \infty} \|x_i - x_m, z\| = 0$. 

**Definition 1.4.** A sequence \( \{ x_n \} \) in a linear 2-normed space \( A \) is called a convergent sequence if there is an \( x \in A \) such that \( \lim_{n \to \infty} \| x_n - x \| = 0 \) for all \( y \in A \). If \( \{ x_n \} \) converges to \( x \), write \( x_n \to x \) as \( n \to \infty \) and call \( x \) the limit of \( \{ x_n \} \). In this case, we also write \( \lim_{n \to \infty} x_n = x \).

**Lemma 1.5.** For a convergent sequence \( \{ x_n \} \) in a linear 2-normed space \( A \),
\[
\lim_{n \to \infty} \| x_n, y \| = \| \lim_{n \to \infty} x_n, y \| \quad \text{for all} \quad y \in A.
\]

**Definition 1.6.** A linear 2-normed space in which every Cauchy sequence is a convergent sequence is called a 2-Banach space.

Throughout this paper, let \( A \) be a normed linear space and \( B \) a 2-Banach space.

### 2. Stability of the cubic functional equation (1.1)

**Theorem 2.1.** Let \( A \) be a linear space, \( B \) be a 2-Banach space and \( f : A \to B \) be a mapping with \( 0 \leq \mu < \infty \) and \( 0 < p < 3 \) satisfying the inequality
\[
\| f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), z \| \leq \mu(\| x \|^p + \| y \|^p) \quad (2.1)
\]
for all \( x, y \in A \) and \( z \in B \). Then, there exists a unique cubic mapping \( C : A \to B \) such that
\[
\| f(x) - C(x), y \| \leq \frac{8^{(l+1)(m-1)} - 2^{p(m-1)}}{2(8 - 2^p)} \mu \| x \|^p \quad (2.2)
\]
for all \( x \in A \) and \( y \in B \).

**Proof:** Let \( y = 0 \) in (2.1), we get
\[
\| 2f(2x) - 16f(x), z \| \leq \mu \| x \|^p
\]
\[
\| f(2x) - f(x), z \| \leq \frac{1}{16} \mu \| x \|^p \quad (2.3)
\]
for all \( x \in A \) and \( z \in B \). Substituting \( 2x \) at the place of \( x \) and dividing by \( 8 \) in the above inequality (2.3), we get
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\[ \left\| \frac{f(2x) - f(2x)}{8} \right\| \leq \frac{1}{2.8^2} \|2x\|^p \]  \hspace{1cm} (2.4)

for all \( x \in A \) and \( z \in B \). Again replacing \( x \) with \( 2^i x \) and dividing by \( 8^i \) in inequality (2.3), we have

\[ \left\| \frac{f(2^{i+1}x) - f(2^i x)}{8^i} \right\| \leq \frac{1}{2.8^{i+1}} \|2^i x\|^p \leq \frac{1}{2.8^{i+1}} 2^{i p} \|x\|^p \]  \hspace{1cm} (2.5)

for all \( x \in A, \ z \in B \) and \( i \geq 0 \). Now, in order to prove that the sequence \( \left\{ \frac{f(2^n x)}{8^n} \right\} \) is a convergent sequence, let us consider \( l, m \) be two positive number with \( l < m \), such that

\[ \left\| \frac{f(2^m x) - f(2^l x)}{8^m} \right\| \leq \frac{1}{2} \sum_{i=l}^{m-1} \frac{2^i}{8^{i+1}} \|x\|^p \]  \hspace{1cm} (2.6)

for all \( x \in A, \ z \in B \) and \( i \geq 0 \). Taking limit on both sides of (2.6), we get

\[ \lim_{m, l \to \infty} \left\| \frac{f(2^n x)}{8^n} \right\| = 0 \]

for all \( x \in A \) and all \( z \in B \). Which implies that the sequence \( \left\{ \frac{f(2^n x)}{8^n} \right\} \) is a Cauchy sequence in \( B \). Since the space \( B \) is 2- Banach space, the sequence \( \left\{ \frac{f(2^n x)}{8^n} \right\} \) is convergent also. Therefore, we may define a cubic mapping \( C : A \to B \) defined by

\[ C(x) = \lim_{n \to \infty} \frac{f(2^n x)}{8^n} \] for all \( x \in A \). Now, to prove that the mapping \( C : A \to B \) also satisfies the functional equation (1.1), by using Lemma 1.5 and the inequality (2.1), we have

\[ \|C(2x+y) + C(2x-y) - 2C(x+y) - 2C(x-y) - 12C(x), z\| \]

\[ = \lim_{i \to \infty} \frac{1}{8} \left\| f(2^{i+1} x + 2^i y) + f(2^{i+1} x - 2^i y) - 2 f(2^i x + 2^i y) - 2 f(2^i x - 2^i y) - 12 f(2^i x), z\| \]

\[ \leq \lim_{i \to \infty} \frac{1}{8} \mu \|2^i x\|^p + \|2^i y\|^p \] \leq \lim_{i \to \infty} \frac{2^{i p}}{8^i} \mu \|x\|^p + \lim_{i \to \infty} \frac{2^{i p}}{8^i} \|y\|^p = 0 \]
\[ \|C(2x + y) + C(2x - y) - 2C(x + y) - 2C(x - y) - 12C(x), z\| = 0 \]

for all \( x, y \in A \) and all \( z \in B \). In order to prove the inequality (2.2) that is the main result of theorem 2.1, by using (2.6), we have

\[
\left\| f(x) - C(x), y \right\| = \frac{1}{8^n} \left\| C(2^n x) - C(2^n x), y \right\| \leq \frac{8^{(l+1)(m-1)} - 2^{l(p(m-1))}}{2(8 - 2^p)} \mu \|x\|^p
\]

for all \( x \in A \) and all \( y \in B \). Now, to prove the uniqueness of the cubic mapping \( C : A \to B \), let us consider another cubic mapping \( C' : A \to B \) which satisfies the inequality (1.1), we have

\[
\left\| C(x) - C'(x), y \right\| = \frac{1}{8^n} \left\| C(2^n x) - C'(2^n x), y \right\| \leq \frac{1}{8^n} \left( \left\| C(2^n x) - f(2^n x), y \right\| + \left\| C'(2^n x) - f(2^n x), y \right\| \right)
\]

which tends to zero as \( n \to \infty \) for all \( x \in A \) and all \( y \in B \). Hence by Lemma 1.2 we conclude that \( C(x) = C'(x) \) for all \( x \in A \). Which proves the uniqueness of \( C : A \to B \). Hence proved the theorem.

**Theorem 2.2.** Let \( A \) be a linear space, \( B \) be a 2-Banach space and \( f : A \to B \) be a mapping with \( 0 \leq \mu < \infty \) and \( p > 3 \) satisfying the inequality

\[
\left\| f(2x + y) + f(2x - y) - 2f(x + y) - 2f(x - y) - 12f(x), z\right\| \leq \mu(\|x\|^p + \|y\|^p) \quad (2.7)
\]

for all \( x, y \in A \) and \( z \in B \). Then, there exists a unique cubic mapping \( C : A \to B \) such that

\[
\left\| f(x) - C(x), y \right\| \leq \frac{2^{l(p(m-1))} - 8^{(l)(m-1)}}{2(2^p - 8)} \mu \|x\|^p \quad (2.8)
\]

for all \( x \in A \) and \( y \in B \).

Proof: Putting \( y = 0 \) in (2.7), we get

\[
\left\| 2f(2x) - 16f(x), z \right\| \leq \mu \|x\|^p
\]

\[
\left\| f(x) - 8f\left(\frac{x}{2}\right), z \right\| \leq \frac{1}{2} \mu \left\|\frac{x}{2}\right\|^p \quad (2.9)
\]

for all \( x \in A \) and \( z \in B \). Substituting \( x/2 \) at the place of \( x \) and multiplying by 8 in the above inequality (2.9), we get
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\[ \left\| 8f\left(\frac{x}{2}\right) - 8^2 f\left(\frac{x}{2^2}\right), z \right\| \leq \frac{1}{2} 8\mu \left\| \frac{x}{2^2} \right\|^p \]

for all \( x \in A \) and \( z \in B \). Again replacing \( x \) with \( x/2 \) and multiplying by \( 8^i \) in inequality (2.9), we have

\[ \left\| 8^i f\left(\frac{x}{2^i}\right) - 8^{i+1} f\left(\frac{x}{2^{i+1}}\right), z \right\| \leq \frac{1}{2} 8^i \mu \left\| \frac{x}{2^{i+1}} \right\|^p \]

for all \( x \in A \), \( z \in B \) and \( i \geq 0 \). Now in order to prove that the sequence \( \left\{ 8^n f\left(\frac{x}{2^n}\right) \right\} \) is a convergent sequence, let us consider \( l, m \) be two positive number with \( l < m \), such that

\[ \left\| 8^m f\left(\frac{x}{2^m}\right) - 8^l f\left(\frac{x}{2^l}\right), z \right\| \leq \frac{\mu}{2} \sum_{i=l}^{m-1} 8^i \left\| x \right\|^p \]  \hspace{1cm} (2.10)

for all \( x \in A \), \( z \in B \) and \( i \geq 0 \). Taking limit on both sides of (2.10), we get

\[ \lim_{m,l \to \infty} \left\| 8^m f\left(\frac{x}{2^m}\right) - 8^l f\left(\frac{x}{2^l}\right), z \right\| = 0 \]

for all \( x \in A \) and all \( z \in B \). Which implies that the sequence \( \left\{ 8^n f\left(\frac{x}{2^n}\right) \right\} \) is a Cauchy sequence in \( B \). Since the space \( B \) is 2-Banach space, the sequence \( \left\{ 8^n f\left(\frac{x}{2^n}\right) \right\} \) is convergent also. Therefore, we may define a cubic mapping \( C : A \to B \) defined by

\[ C(x) = \lim_{n \to \infty} 8^n f\left(\frac{x}{2^n}\right) \]

for all \( x \in A \). Further, the remaining proof of this theorem is similar as the proof of Theorem 2.1.

3. Stability of the cubic functional equation (1.2)

**Theorem 3.1.** Let \( A \) be a linear space, \( B \) be a 2-Banach space and \( f : A \to B \) be a mapping with \( 0 \leq \mu < \infty \) and \( 0 < p < 3 \) satisfying the inequality

\[ \left\| f(3x + y) + f(3x - y) - 3f(x + y) - 3f(x - y) - 48f(x), z \right\| \leq \mu (\| x \|^p + \| y \|^p) \]  \hspace{1cm} (3.1)
for all $x, y \in A$ and $z \in B$. Then, there exists a unique cubic mapping $C: A \rightarrow B$ such that

$$
\|f(x) - C(x), y\| \leq \frac{27^{(l+1)(m-l)-3^{p(m-l)}}}{2(27-3^p)} \mu \|x\|^p
$$

(3.2)

for all $x \in A$ and $y \in B$.

Proof: Taking $y = 0$ in (3.1), we get

$$
\|2f(3x) - 54f(x), z\| \leq \mu \|x\|^p
$$

(3.3)

for all $x \in A$ and $z \in B$. Substituting $3x$ at the place of $x$ and dividing by $27$ in the above inequality (3.3), we get

$$
\left\| \frac{f(3^2x)}{27^2} - \frac{f(3x)}{27}, z \right\| \leq \frac{1}{2.27^2} \mu \|3x\|^p
$$

(3.4)

for all $x \in A$ and $z \in B$. Again replacing $x$ with $3^i x$ and dividing by $27^i$ in inequality (3.3), we have

$$
\left\| \frac{f(3^{i+1}x)}{27^{i+1}} - \frac{f(3^ix)}{27^i}, z \right\| \leq \frac{1}{2.27^{i+1}} \mu \|3^ix\|^p \leq \frac{1}{2.27^{i+1}} 3^{ip} \mu \|x\|^p
$$

(3.5)

for all $x \in A$, $z \in B$ and $i \geq 0$. Now in order to prove that the sequence $\left\{ \frac{f(3^nx)}{27^n} \right\}$ is a convergent sequence, let us consider $l, m$ be two positive number with $l < m$, such that

$$
\left\| \frac{f(3^nx)}{27^n} - \frac{f(3^lx)}{27^l}, z \right\| \leq \frac{\mu}{2} \sum_{i=l}^{m-1} \frac{3^{ip}}{27^{i+1}}, \|x\|^p
$$

(3.6)

for all $x \in A$, $z \in B$ and $i \geq 0$. Taking limit on both sides of (3.6), we get

$$
\lim_{m,l \rightarrow \infty} \left\| \frac{f(3^nx)}{27^m} - \frac{f(3^lx)}{27^l}, z \right\| = 0
$$

for all $x \in A$ and all $z \in B$. Which implies that the sequence $\left\{ \frac{f(3^nx)}{27^n} \right\}$ is a Cauchy sequence in $B$. Since the space $B$ is 2- Banach space, the sequence $\left\{ \frac{f(3^nx)}{27^n} \right\}$ is
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convergent also. Therefore, we may define a cubic mapping \( C : A \to B \) defined by
\[
C(x) = \lim_{n \to \infty} \frac{f(3^n x)}{27^n}
\]
for all \( x \in A \). Now, to prove that the mapping \( C : A \to B \) also satisfies the functional equation (1.2), by using Lemma 1.5 and the inequality (3.1), we get
\[
\|C(3x + y) + C(3x - y) - 3C(x + y) - 3C(x - y) - 48C(x), z\|
\]
\[
= \lim_{i \to \infty} \frac{1}{27} \|f(3^{i+1} x + 3^i y) + f(3^{i+1} x - 3^i y) - 3f(3^{i+1} x + 3^i y) - 3f(3^{i+1} x - 3^i y) - 48f(3^{i} x), z\|
\]
\[
\leq \lim_{i \to \infty} \frac{1}{27} \mu(\|3^i y\|^p + \|3^i y\|^p) \leq \lim_{i \to \infty} \frac{3^p}{27} \mu \|x\|^p + \lim_{i \to \infty} \frac{3^p}{27} \|y\|^p = 0
\]
\[
\|C(3x + y) + C(3x - y) - 3C(x + y) - 3C(x - y) - 48C(x), z\| = 0
\]
for all \( x, y \in A \) and all \( z \in B \). In order to prove the inequality (3.2) that is the main result of theorem 3.1, by using (3.6), we have
\[
\|f(x) - C(x), y\| = \lim_{m \to \infty} \left\|f(x) - \frac{f(3^m x)}{27^m}, y\right\|
\]
\[
\leq \frac{2^{(m+1)(m-1)} - 3^{2m-1}}{2(27-3^p)} \mu \|x\|^p
\]
for all \( x \in A \) and all \( y \in B \). Now, to prove the uniqueness of the cubic mapping \( C : A \to B \), let us consider another cubic mapping \( C' : A \to B \) which satisfies the inequality (1.2), we have
\[
\|C(x) - C'(x), y\| = \frac{1}{27^n} \|C(3^n x) - C'(3^n x), y\|
\]
\[
\leq \frac{1}{27^n} \left(\|C(3^n x) - f(3^n x), y\| + \|C'(3^n x) - f(3^n x), y\|\right)
\]
Which tends to zero as \( n \to \infty \) for all \( x \in A \) and all \( y \in B \). Hence by Lemma 1.2 we conclude that \( C(x) = C'(x) \) for all \( x \in A \). Which proves the uniqueness of \( C : A \to B \).
Hence proved the theorem.

**Theorem 3.2.** Let \( A \) be a linear space, \( B \) be a 2-Banach space and \( f : A \to B \) be a mapping with \( 0 \leq \mu < \infty \) and \( p > 3 \) satisfying the inequality
\[
\|f(3x + y) + f(3x - y) - 3f(x + y) - 3f(x - y) - 48f(x), z\| \leq \mu(\|x\|^p + \|y\|^p)
\]
(3.7)
for all \( x, y \in A \) and \( z \in B \). Then, there exists a unique cubic mapping \( C : A \to B \) such that
\[
\| f(x) - C(x), y \| \leq \frac{2^{p(m-1)} - 2^{q(1-x^2)}}{(3^p - 27)} \| x \|^p \quad (3.8)
\]
for all \( x \in A \) and \( y \in B \).

Proof: The proof this theorem is similar to the proof of Theorem 2.2.

References


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