Padovan $Q$-Matrix and the Generalized Relations

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Abstract

The Padovan numbers have the $Q$-matrix, $Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$. The $Q$-matrix is a $3 \times 3$ matrix that when raised to the $n^{th}$ power give a matrix whose entries are Padovan numbers. For which we established by mathematical induction that,

$$Q^n = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix}, \text{ for all } n \geq 3.$$

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1 Introduction

The Padovan sequence is named after Richard Padovan who attributed its discovery to Dutch architect Hans van der Laan in his 1994 essay Dom Hans van der Laan: Modern Primitive. The sequence was described by Ian Stewart in his Scientific American column Mathematical Recreations in June 1996. He also writes about it in one of his books, Math Hysteria: Fun Games With Mathematics.

In this paper, the Padovan sequence is the sequence of integers $P_n$ defined by the initial values $P_0 = 0, P_1 = 0, P_2 = 1$ and the recurrence relation

$$P_n = P_{n-2} + P_{n-3}, \text{ for all } n \geq 3.$$
The first few values of \( P_n \) are 0, 0, 1, 0, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, 37, 49, 65, 86, \ldots.

\section{Main Results}

In this study, we investigate the new property of Padovan numbers in relation with the Padovan \( Q \)-matrix, \( Q = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \). More generally, we have \( Q^n = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix} \), for all \( n \geq 3 \). This strategy allow us to obtain the new relations for the Padovan sequences.

**Theorem 2.1.** \( Q^n = \begin{bmatrix} P_{n-1} & P_{n+1} & P_n \\ P_n & P_{n+2} & P_{n+1} \\ P_{n+1} & P_{n+3} & P_{n+2} \end{bmatrix} \), for all \( n \geq 3 \).

**Proof.** Let use the principle of mathematical induction on \( n \).

For \( n = 3 \), it is easy to see that

\[
Q^3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} P_2 & P_4 & P_3 \\ P_3 & P_5 & P_4 \\ P_4 & P_6 & P_5 \end{bmatrix} = \begin{bmatrix} P_{3-1} & P_{3+1} & P_3 \\ P_3 & P_{3+2} & P_{3+1} \\ P_{3+1} & P_{3+3} & P_{3+2} \end{bmatrix}.
\]

Assume that it is true for all positive integer \( n = k \). That is,

\[
Q^k = \begin{bmatrix} P_{k-1} & P_{k+1} & P_k \\ P_k & P_{k+2} & P_{k+1} \\ P_{k+1} & P_{k+3} & P_{k+2} \end{bmatrix}.
\]

Therefore, we have to show that it is true for \( n = k + 1 \). By the laws of
Therefore, the result is true for every \( n \geq 3 \).

Let us generalize this observation using the Padovan \( Q \)-matrix.

**Proposition 2.2.** For all integers \( m, n \) such that \( 0 < m < n \), we have the following relations:

(a) \( P_n = P_{m-1} \cdot P_{n-m} + P_{m+1} \cdot P_{n-m+1} + P_m \cdot P_{n-m+2} \),

(b) \( P_n = P_m \cdot P_{n-m-1} + P_{m+2} \cdot P_{n-m} + P_{m+1} \cdot P_{n-m+1} \).

**Proof.** From Theorem 2.1 and the laws of exponent for the square matrix. We have,

\[
Q^n = Q^m Q^{n-m}
\]

yielding, upon equating corresponding elements. That is,

\[
P_n = P_{m-1} \cdot P_{n-m} + P_{m+1} \cdot P_{n-m+1} + P_m \cdot P_{n-m+2},
\]

and

\[
P_n = P_m \cdot P_{n-m-1} + P_{m+2} \cdot P_{n-m} + P_{m+1} \cdot P_{n-m+1}.
\]

**Remark 2.3.** In Proposition 2.2 (a), If \( m = 3 \), then

\[
P_n = P_2 \cdot P_{n-3} + P_4 \cdot P_{n-2} + P_3 \cdot P_{n-1},
\]

\[
= 1 \cdot P_{n-3} + 1 \cdot P_{n-2} + 0 \cdot P_{n-1}, \text{ (replaces } P_2 = P_4 = 1 \text{ and } P_3 = 0) = P_{n-2} + P_{n-3}.
\]

In Proposition 2.2 (b), If \( m = 2 \), then the result is similarly to the Proposition 2.2 (a). This results are the generalized relation of Padovan sequences.
References


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