Mathematical Model of Magnetometric Resistivity

Sounding for a Conductive Host

with a Bulge Overburden

Teerasak Chaladgarn

Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom 73000, Thailand

Suabsagun Yooyuanyong

Department of Mathematics
Faculty of Science
Silpakorn University
Nakhon Pathom 73000, Thailand
Corresponding author: suabkul@su.ac.th

Abstract

In this paper, the solution of magnetic field response from DC source located on a two layered uniform conductive host medium is formulated. For the first layer, the conductivity of overburden is a function of depth(z) and denoted by \( \sigma(z) = \sigma_0 e^{l(z-h)^2/l} \), where \( b \) is constant, \( l \) is positive which is used to locate the peak of the bulge, \( h \) is the thickness of overburden and \( \sigma_0 \) is positive constant. The second layer, the conductivity of host medium, \( z > h \), is constant and
is given by $\sigma_2(z) = \sigma_0$. The Hankel transforms are introduced to solve the magnetic fields which are expressed in the form of integral expressions. In order to determine the magnetic fields, numerical solutions are computed to show the behavior of the field while some parameters are given approximately. The graphs of computation results perform the same relationship to the conductivity profile of the ground.

**Mathematics Subject Classification:** 86A25

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### 1. Introduction

The magnetometric resistivity method has recently become an additional electrical prospecting technique used for finding underground resources. This technique is based on the measurement of low-level, low-frequency magnetic fields associated with non-inductive current flow from the electrodes in the ground. Chen and Oldenburg [4] derived the magnetic field directly from solving a boundary value problems which was similar to the approach used by Edward [5] and then discussed in a homogeneous and a 2-layered earth model. Yooyuanyong and Sripanya [10] derived and performed the solutions of the steady state magnetic field due to a DC current source in three types of heterogeneous earth models. These solutions are critical to interpret the magnetometric resistivity (MMR) data.

In this paper, a 2-layered conductive earth model is considered similar to Chen and Oldenburg [4], but it is different in the conductivity profile. For the first layer, the conductivity of overburden is denoted by $\sigma_1(z) = \sigma_0 e^{-b(z-l)^2/h^2}$, $0 \leq z \leq h$, where $b$ is constant, $l$ is positive which is used to locate the peak of the bulge, $h$ is the thickness of overburden and $\sigma_0$ is positive constant. The second layer, the conductivity of host medium, $z > h$, is constant and is given by $\sigma_2(z) = \sigma_0$. The objective of this paper is to show the behavior of the field while some parameters are given approximately.
2. Formulation of the Problem

The general steady state Maxwell’s equation in the frequency domain [4] can be used to determine the magnetic field for this problem, namely

\[ \nabla \times \vec{E} = \hat{0} \quad (1) \]

and

\[ \nabla \times \vec{H} = \sigma \vec{E}, \quad (2) \]

where \( \vec{E} \) is the vector electric field, \( \vec{H} \) is the vector magnetic field, \( \sigma \) is the conductivity of the medium in Siemens per meter \( (S/m) \) which is assumed to be a function of \( z \) only and \( \nabla \) is the del operator. Eliminating \( \vec{E} \) from equations (1) and (2), we obtain

\[ \nabla \times \frac{1}{\sigma} \nabla \times \vec{H} = \hat{0}. \quad (3) \]

This can be expressed in cylindrical coordinates \( (r, \phi, z) \) as

\[
\begin{align*}
\left( \frac{1}{r} \frac{\partial}{\partial \phi} \left( \frac{1}{r} \frac{\partial}{\partial \phi} (rH_\phi) \right) - \frac{1}{\sigma} \frac{\partial}{\partial \phi} \left( \frac{\partial H_r}{\partial z} \right) - \frac{1}{\sigma} \frac{\partial}{\partial \phi} \left( \frac{\partial H_z}{\partial r} \right) \right) e_r & \\
+ \left( \frac{1}{\sigma} \frac{\partial}{\partial \phi} \frac{\partial}{\partial \phi} (rH_\phi) - \frac{\partial}{\partial r} (rH_\phi) - \frac{1}{\sigma} \frac{\partial}{\partial \phi} \left( \frac{\partial H_r}{\partial z} \right) - \frac{1}{\sigma} \frac{\partial}{\partial \phi} \left( \frac{\partial H_z}{\partial r} \right) \right) e_\phi & \\
+ \left( \frac{1}{\sigma} \frac{\partial}{\partial r} \frac{\partial}{\partial r} (rH_r) - \frac{\partial}{\partial \phi} (rH_r) - \frac{1}{\sigma} \frac{\partial}{\partial r} \left( \frac{\partial H_\phi}{\partial z} \right) - \frac{1}{\sigma} \frac{\partial}{\partial \phi} \left( \frac{\partial H_z}{\partial r} \right) \right) e_z = 0 \\
\end{align*}
\]

(4)

where \( H_r, \ H_\phi \) and \( H_z \) are the components of \( \vec{H} \) in \( e_r, \ e_\phi \) and \( e_z \) directions, respectively. Since the problem is axisymmetric and \( \vec{H} \) has only the azimuthal component in cylindrical coordinates, for simplicity, we use \( H \) to represent the azimuthal component \( H_\phi \) in the following derivations. Simplifying equation (4) yields

\[
\frac{1}{\sigma} \frac{\partial^2 H_\phi}{\partial z^2} + \frac{\partial}{\partial z} \left( \frac{1}{\sigma} \frac{\partial H_\phi}{\partial z} \right) + \frac{1}{\sigma} \frac{\partial^2 (rH)}{\partial r^2} + \frac{\partial}{\partial r} \left( \frac{1}{\sigma} \frac{\partial (rH)}{\partial r} \right) = 0.
\]
In our study, we denote $\sigma$ as a function of only depth $z$, and we now have

$$\frac{\partial^2 H}{\partial z^2} + \sigma \frac{\partial}{\partial z} \left( \frac{1}{\sigma} \right) \frac{\partial H}{\partial z} + \frac{\partial^2 H}{\partial r^2} + \frac{1}{r} \frac{\partial H}{\partial r} - \frac{1}{r^2} H = 0. \quad (5)$$

The Hankel transform [1] is introduced and defined by

$$\tilde{H}(\lambda, z) = \int_0^\infty r H(r, z) J_0(\lambda r) dr \quad (6)$$

and

$$H(r, z) = \int_0^\infty \lambda \tilde{H}(\lambda, z) J_1(\lambda r) d\lambda, \quad (7)$$

where $J_1$ is the Bessel function of the first kind of order one and $\lambda$ is the Hankel variable. Taking the transformation on both sides of equation (5), we obtain

$$\frac{\partial^2 \tilde{H}}{\partial z^2} + \sigma \frac{\partial}{\partial z} \left( \frac{1}{\sigma} \right) \frac{\partial \tilde{H}}{\partial z} - \lambda^2 \tilde{H} = 0. \quad (8)$$

Since the electrode is in the overburden which the end of the electrode is positioned at $z = h$, so that a magnetic field will be separated into two parts. The magnetic field is come from the ground layer that can be described by the general solution of equation (8) and the magnetic field arising from probe sources $H_0$, which is only one element. It can be explained by the Ampere’s law [7, 11], as

$$H_0(r, z) = \frac{1}{2\pi r},$$

where $I$ is the current at the probe on the ground surface.

From equation (6), we have

$$\tilde{H}_0(\lambda, z) = \int_0^\infty r \left( \frac{1}{2\pi r} \right) J_1(\lambda r) dr = \frac{1}{2\pi\lambda}.$$

Therefore, the magnetic field in each layer can be obtained by taking the inverse
Hankel transform to the solution of equation (8), which satisfies the following boundary conditions [2, 8]:

1. The vertical component of the current density must be zero at the ground surface \( z = 0 \),

\[
\sigma_1(z) E^z_z (r, z) \bigg|_{z=0} = 0, \tag{9}
\]

where \( E^z_z \) is the vertical component of the electric field in overburden.

2. The azimuthal component of the magnetic field needs to be continuous on each of the boundary planes in the earth,

\[
\lim_{z \to h} \tilde{H}_1(\lambda, z) = \lim_{z \to h} \tilde{H}_2(\lambda, z), \tag{10}
\]

where \( H_1 \) and \( H_2 \) are magnetic fields in the first and second layer, respectively.

3. The radial component of the electric field, denoted by \( E^r_r \), needs to be continuous on each of the boundary planes in the earth,

\[
\lim_{z \to h} \tilde{E}^r_1(\lambda, z) = \lim_{z \to h} \tilde{E}^r_2(\lambda, z), \tag{11}
\]

where \( E^r_1 \) and \( E^r_2 \) are the radial component of electric fields in the first and second layer, respectively. To determine the radial and vertical components of the electric field related to the azimuthal component of the magnetic field, we expand equation (2) and obtain

\[
E = \left( \frac{1}{\sigma r} \frac{\partial H_z}{\partial \phi} - \frac{1}{\sigma} \frac{\partial H_\phi}{\partial z} \right) e_r + \left( \frac{1}{\sigma} \frac{\partial H_r}{\partial z} - \frac{1}{\sigma r} \frac{\partial H_\phi}{\partial r} \right) e_\phi + \left( \frac{1}{\sigma r} \frac{\partial H_\phi}{\partial r} - \frac{1}{\sigma} \frac{\partial H_r}{\partial \phi} \right) e_z.
\]

Since the problem is axisymmetric and \( \tilde{H} \) has only the azimuthal component in cylindrical coordinates, for simplicity as equation (4), we use \( H \) to represent the azimuthal component \( H_\phi \) in the above derivations. This yield

\[
E^r = -\frac{1}{\sigma} \frac{\partial H}{\partial z}, \quad E^z = \frac{1}{\sigma r} \frac{\partial}{\partial r} (rH).
\]

By using equation (6), we have
3. A Geometric 2-Layered Earth Model

In our geometric model, a two-layered earth model is considered which the interface between the layers is a plane parallel to the ground surface. A point source of direct current 1 is located into the overburden which the end of the electrode is positioned at \( z = h \). For the first layer, the conductivity of overburden is denoted by \( \sigma_1(z) = \sigma_0 e^{-b(z-l)^2/2} \), \( 0 \leq z \leq h \), where \( b \) is constant, \( l \) is positive and it is used to locate the peak of the bulge, \( h \) is the thickness of overburden and \( \sigma_0 \) is positive constant. The second layer, the conductivity of host medium, \( z > h \), is constant and is given by \( \sigma_2(z) = \sigma_0 \).

![Figure 1: A geometric 2-layered conductive earth model.](image)

4. Solution of Magnetic Field for a 2-Layered Earth Model

An overburden has a variation of conductivity \( \sigma_1(z) \) with thickness \( h \) over a conductive host medium having constant conductivity \( \sigma_2(z) \). Hence, the equation for the magnetic field in overburden and conductive host medium can be simplified by substituting \( \sigma_1(z) \) and \( \sigma_2(z) \) into equation (8), thus, we obtain
The power series method and auxiliary equation are used to find the magnetic field formulation in an overburden, denoted by $H_1$, and conductive host medium, denoted by $H_2$, respectively. Therefore, the solutions of the equation (12) and (13) are written by [6, 9]

$$H_1(\lambda, z) = \frac{I}{2\pi \lambda} (1 + \lambda^2 z^2) + A_1 \left( 1 + \frac{\lambda^2 z^2}{2} \right) + A_2 \left( z + \frac{blz^2}{2} \right),$$

and

$$H_2(\lambda, z) = A_3 e^{-\lambda(z-h)} + A_4 e^{\lambda(z-h)},$$

respectively, where $A_1, A_2, A_3$ and $A_4$ are arbitrary constants, which can be determined by using the boundary conditions. The condition at $z \to \infty$, the magnetic field tends to zero, that leads $H_2$ to

$$H_2(\lambda, z) = A_3 e^{-\lambda(z-h)}.$$ 

For the first boundary condition in equation (9), we obtain

$$\sigma_1(z) \left( \frac{1}{r \sigma_1(z)} \frac{\partial}{\partial r} (rH_1(r,z)) \right) = 0.$$ 

Since $\sigma_1(z)$ and $r$ are not zero, then

$$\frac{\partial}{\partial r} (rH_1(r,z)) = 0,$$

or

$$rH_1(r,z) = \int_0^\infty \frac{\partial}{\partial r} (rH_1(r,z)) dr = \int_0^\infty (0) dr = 0.$$
Since \( r > 0 \), thus \( H_1(r, z) = 0 \), and
\[
\tilde{H}_1(\lambda, z) = \int_0^\infty rH_1(r, z)J_1(\lambda r)\,dr.
\]

Hence,
\[
\frac{I}{2\pi\lambda}(1 + \lambda^2 z^2) + A_1\left(1 + \frac{\lambda^2 z^2}{2}\right) + A_2\left(z + \frac{b\lambda^2}{2}\right) = \int_0^\infty r(0)J_1(\lambda r)\,dr = 0.
\]

Since no electric current across at the air-earth interface, \( I = 0 \), then
\[
\left[ A_1\left(1 + \frac{\lambda^2 z^2}{2}\right) + A_2\left(z + \frac{b\lambda^2}{2}\right) \right]_{z=0} = 0. \quad \text{Thus,} \quad A_1 = 0.
\]

By the second boundary condition, we obtain
\[
\lim_{z \to h^-} \left[ I(1 + \lambda^2 z^2) + A_2\left(z + \frac{b\lambda^2}{2}\right) \right] = \lim_{z \to h^-} A_3e^{-\lambda(z-h)}.
\]

Thus,
\[
A_3 = \frac{I}{2\pi\lambda}(1 + \lambda^2 h^2) + A_2\left(h + \frac{b\lambda h^2}{2}\right). \quad \text{(16)}
\]

Applying the third boundary condition, we have
\[
\lim_{z \to h^-} \left[ -\frac{1}{\sigma_1(z)} \frac{\partial}{\partial z} \tilde{H}_1(\lambda, z) \right] = \lim_{z \to h^-} \left[ -\frac{1}{\sigma_2(z)} \frac{\partial}{\partial z} \tilde{H}_2(\lambda, z) \right],
\]
or
\[
\lim_{z \to h^-} \left[ -\frac{1}{\sigma_0 e^{-b(z-h)^2/2}} \frac{\partial}{\partial z} \left( \frac{I}{2\pi\lambda}(1 + \lambda^2 z^2) + A_2\left(z + \frac{b\lambda^2}{2}\right) \right) \right] = \lim_{z \to h^-} \left[ -\frac{1}{\sigma_0} \frac{\partial}{\partial z} \left( A_3e^{-\lambda(z-h)} \right) \right].
\]

Hence,
\[
A_3 = -\frac{1}{\lambda e^{-b(h-h)^2/2}} \left( \frac{I\lambda h}{\pi} + A_2 \left(1 + b\lambda h\right) \right). \quad \text{(17)}
\]
From equation (16) and (17), we obtain

\[
\frac{I}{2\pi \lambda} \left( 1 + \lambda^2 h^2 \right) + A_2 \left( h + \frac{bhl^2}{2} \right) = -\frac{1}{\lambda e^{-bh-l\gamma^2/2}} \left( \frac{I\lambda h}{\pi} + A_2 \left( 1 + bhl \right) \right).
\]

Therefore,

\[
A_2 = -\frac{I}{\pi} \left\{ \frac{\left( 1 + \lambda^2 h^2 \right) \alpha_1 + 2\lambda h}{2h\lambda\alpha_1 + bhl\lambda^2 \alpha_1 + 2\left( 1 + bhl \right)} \right\},
\]

where \( \alpha_1 = e^{-bh-l\gamma^2/2} \).

Since \( A_3 = -\frac{1}{\lambda \alpha_1} \left( \frac{I\lambda h}{\pi} + A_2 \left( 1 + bhl \right) \right) \), then with the use of equation (18), we obtain

\[
A_3 = \frac{I}{\pi \lambda \alpha_1} \left[ \frac{\left( 1 + \lambda^2 h^2 \right) \alpha_1 + 2\lambda h \left( 1 + bhl \right)}{2\lambda h\alpha_1 + bhl\lambda^2 \alpha_1 + 2\left( 1 + bhl \right) - \lambda h} \right].
\]

Hence, with the use of inverse Hankel transforms, the magnetic field in overburden and conductive host medium are shown, respectively, as

\[
H_1(r,z) = \int_0^{\infty} \frac{I}{2\pi} \left( 1 + \lambda^2 z^2 \right) - \frac{I\lambda}{\pi} \left\{ \frac{\left( 1 + \lambda^2 h^2 \right) \alpha_1 + 2\lambda h}{2h\lambda\alpha_1 + bhl\lambda^2 \alpha_1 + 2\left( 1 + bhl \right)} \right\} \left( z + \frac{bhl^2}{2} \right) J_1(\lambda r) d\lambda,
\]

and

\[
H_2(\lambda,z) = \int_0^{\infty} \frac{I}{\pi \alpha_1} \left[ \left( 1 + \lambda^2 h^2 \right) \alpha_1 + 2\lambda h \left( 1 + bhl \right) \right] e^{-\lambda(z+bhl)} J_1(\lambda r) d\lambda.
\]
5. Numerical Experiments

In our numerical experiments, the magnetic field due to a direct current source on the ground surface of the model is calculated. Chave’s algorithm [3] is used for numerical calculating the inverse Hankel transform of the magnetic field solutions. The current of 1-Ampere is injected to the ground by the probe length of 1 and 3 meters perpendicular to the ground surface, $\sigma_0 = 2S/m$, $b = 0.005$. The results of magnetic field response are performed as the graphs in Figure 2, 3, 4 and 5. The graphs are shown the behavior of the magnetic field against source-receiver spacing ($r$) while the values of $h$, $l$ and $z$ are adjusted. As we fix the value of $h$, the magnetic field curves are quite different as $z$ and $l$ are varied between $h \geq l$ and $h \leq l$. The magnetic field intensities drop very fast when the value of $z$ close to the value of $h$. At large depth ($z$), the magnitude of magnetic fields tends to be small values as we expect. As the thickness of overburden is increased, the shape of graph is similar to the conductivity profile of the ground. This is the advantage of magnetic field that can be performed some relationship to the conductivity profile of the ground.

![Figure 2: The behavior of magnetic field against $r$ at different depth $z = 0.1, 0.2, ..., 0.7, 0.8m$. where $h = 1m$. and $l = 0.5m$.](image)
Mathematical model of magnetometric resistivity sounding

Figure 3: The behavior of magnetic field against $r$ at different depth $z = 0.2, 0.4, ..., 1.8, 2m.$ where $h = 3m.$ and $l = 1.5m.$

Figure 4: The behavior of magnetic field against $r$ at different depth $z = 1.1, 1.2, ..., 1.7, 1.8m.$ where $h = 1m.$ and $l = 1.5m.$
The behavior of magnetic field against \( r \) at different depth \( z = 3.2, 3.4, \ldots, 4.8, 5m \). where \( h = 3m \) and \( l = 4.5m \).

6. Conclusions

In this paper, the magnetic fields of an earth having the electrical conductivity 
\[
\sigma_1(z) = \sigma_0 e^{-h(z-\ell)/h} \quad \text{for the depth } 0 \leq z \leq h, \quad \text{and} \quad \sigma_2(z) = \sigma_0 \quad \text{for the depth } z > h
\]
are considered. The integral expressions are derived and computed the values of the magnetic field which is used to determine the behavior of the magnetic field against source-receiver spacing. The curves of the magnetic field against source-receiver spacing are plotted and shown the advantage in the ground exploration.

References


