Solving a Two Dimensional Unsteady-State Flow Problem by Meshless Method

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Abstract

A meshless method called smoothed particle hydrodynamics (SPH) was applied to a barotropic vorticity equation of the atmosphere to investigate its accuracy. The tensor product weight function for the SPH method was a quartic spline function. Results from the SPH approximation of the equation on a sphere showed that errors can occur near the boundary due to asymmetry of the weight function at the boundary. This problem was partially solved by using the ghost particle technique.

Keywords: Smoothed Particle Hydrodynamics, barotropic vorticity

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1 INTRODUCTION

Physical phenomena and engineering problems modeled by partial differential equations (PDE) have traditionally been solved by finite differences (FDM) or finite elements (FEM). Both methods are based on a set of points that are positioned in the problem domain and connected into a mesh. The meshless method is simple, accurate, and requires non-grid. The field variable $u$ at any point $\mathbf{x}$ within the problem domain is interpolated using the displacement at its nodes within the support domain of the point at $\mathbf{x}$, i.e. \(^1\)

$$ u(\mathbf{x}) = \sum_{j=1}^{n} \varphi_j(\mathbf{x})u_j = \Phi(\mathbf{x})\mathbf{U}_s $$

for $\mathbf{x} = (x_1, x_2, x_3, \ldots, x_n)$, where $n$ is the number of nodes include in a “small local domain” of the point at $\mathbf{x}$. The local domain means the interpolation area which is represented by point $\mathbf{x}$, $u_j$ is the nodal field variable at the $j$th node in the small local domain, $\mathbf{U}_s$ is the vector that collects all the field variables at these nodes, and $\varphi_j(\mathbf{x})$ is the shape function of the $j$th node determined using the nodes that are included in the small domain of $\mathbf{x}$. This small local domain is called the support domain of $\mathbf{x}$. Figure 1 shows the concept of a support domain.

![Figure 1. The support domain of a particle at $\mathbf{x}$](image)

There are several meshfree methods. One of these is called the smoothed particle hydrodynamics (SPH) method. The objective of this paper is to apply SPH to a barotropic vorticity equation of the atmosphere and propose an improvement in the attempt to decrease the errors on the boundary.

2 METHODS

2.1 Smoothed Particle Hydrodynamics (SPH)

In the SPH method, the equation to be approximated is represented by a collection of nodes which are under the influence of hydrodynamics \(^1\).
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\[ u^h(x) \approx \sum_{j=1}^{N} \varphi_j(x)u_j \]  \hspace{1cm} (2)

where \( \varphi_j(x) = \tilde{W}(x-x_j,h)\Delta V_j \), \( \tilde{W}(x-x_j,h) \) is a weight or smoothing function, \( h \) is the smoothing length and \( \Delta V_j \) is the volume element carried by particle \( j \).

The weight function is constructed using the information on all particles within the support domain, and the support domain size is defined by the smoothing length. The functions \( \varphi_j \) are the SPH shape functions of the approximation and the derivative can be approximated from (1) as\[1\]

\[ \frac{\partial u^h(x)}{\partial x} \approx \sum_{j=1}^{N} u_j \frac{\partial \tilde{W}(x-x_j,h)}{\partial x} \Delta V_j \]  \hspace{1cm} (3)

A summation representation is valid and converges when the weight function satisfies certain positivity, compact, unity, decay and direct delta function conditions\[1\].

2.2 The Barotropic Vorticity Equation

In fluid dynamics, vorticity is the curl of the fluid velocity. It can also be considered as the circulation per unit area at a point in a fluid flow field. It is a vector quantity, whose direction is along the axis of the fluid’s rotation. The barotropic vorticity equation assumes the atmosphere is nearly barotropic, which means that the direction and speed of the geostrophic wind are independent of height. That is given by the expression\[3\],

\[ \frac{\partial \zeta}{\partial t} = -\left( u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial y} + \beta v \right) \]  \hspace{1cm} (4)

where \( u, v \) are velocity component in \( x \) direction and \( y \) direction, respectively. \( \zeta \) is the relative vorticity.

2.3 The Barotropic Vorticity Equation in SPH Form

From (2) and (3) the barotropic vorticity equation (4) can be written in the form of SPH as,

\[ \zeta^{n+1} = \zeta^n - \Delta t \left[ \sum_{j=1}^{N} \tilde{u}_j w_j(\tilde{x}_j,h)\Delta V_j \right] \left[ \sum_{j=1}^{N} \tilde{\zeta}_y \frac{\partial w_j(\tilde{x}_j,h)}{\partial \tilde{x}_j} \Delta V_j \right] + \ldots \]

\[ \left[ \sum_{j=1}^{N} \tilde{v}_j w_j(\tilde{x}_j,h)\Delta V_j \right] \left[ \sum_{j=1}^{N} \tilde{\zeta}_y \frac{\partial w_j(\tilde{x}_j,h)}{\partial \tilde{x}_y} \Delta V_j \right] + \beta \left( \sum_{j=1}^{N} \tilde{v}_j w_j(\tilde{x}_j,h)\Delta V_j \right) \]  \hspace{1cm} (5)

where \( \tilde{\zeta}_y = (\tilde{\zeta}_x - \tilde{\zeta}_y) \) and \( \tilde{\zeta}_x \) is the null term\[1\]. The forward time difference is used for time derivative.
In two dimensions a node’s domain of influence covers an area in the domain. The choice of the shape of this domain is arbitrary. However, circular domains or square domains have been used most frequently. The weight function with a circular support is called isotropic weight function\cite{4}:
\[
\tilde{W}_i(S) = \tilde{W} \left( \frac{|x-x_i|}{r_i} \right)
\]
(6)
where \( r_i \) is the radius of the domain of influence of the node \( i \).

The isotropic domain should be useful in many problems but have not yet been studied extensively. The domain with rectangular support is called tensor-product weight function. That function at any given point is given by\cite{4}
\[
\tilde{W}_i(S) = \tilde{W}_i(S_x) \cdot \tilde{W}_i(S_y) = \tilde{W}(x-x_i) \cdot \tilde{W}(y-y_i)
\]
(7)
where \( \tilde{W}(S_x) \) and \( \tilde{W}(S_y) \) are any of the standard 1D weight functions in \( x \) and \( y \) directions, respectively. In order to compute the derivatives of shape functions, it is necessary to compute derivatives of weight functions. The first and second order derivatives of the weight functions can be easily obtained by chain rule
\[
\frac{d\tilde{W}(S)}{dx} = \frac{d\tilde{W}(S)}{dS} \cdot \frac{dS}{dx}
\]
where \( \frac{dS}{dx} = \frac{\text{sign}(x-x_i)}{d_i} \)

or
\[
\frac{d\tilde{W}(S)}{dy} = \frac{d\tilde{W}(S)}{dS} \cdot \frac{dS}{dy}
\]
where \( \frac{dS}{dy} = \frac{\text{sign}(y-y_i)}{d_j} \)

For the weight function which is used in this research is the quartic spline\cite{2}
\[
\tilde{W}(S) = \begin{cases} 1 - 6S^2 + 8S^3 - 3S^4 & , S \leq 1 \\ 0 & , S > 1 \end{cases}
\]
(8)
where \( S = \frac{d_s}{d} \)

\( d_s \) is the nodal spacing near the point at \( x_0 \) and \( d_s \) is the size of the support domain for the weight function.

### 2.4 The Barotropic Vorticity Equation in FDM

The barotropic vorticity equation is solved by two methods; SPH and finite difference methods. The results from SPH method are compared to the results from finite different method (FDM). The center difference can be used only for the interior grid points \cite{3}. Thus, the finite difference form of (4) is
\[
\tilde{\zeta}_{m,n}^{t+1} = \tilde{\zeta}_{m,n}^{t} - \Delta t \left[ \frac{1}{2d} \left( (u_{m-1,n}^{t} - u_{m-1,n}^{t-1}) + (v_{m,n+1}^{t} - v_{m,n+1}^{t-1}) \right) + \beta v_{m,n}^{t} \right]
\]
(9)
At the top and bottom edges, forward space differences are used\cite{3}
\[
\tilde{\zeta}_{m,n}^{t+1} = \tilde{\zeta}_{m,n}^{t} - \Delta t \left[ \frac{1}{2d} \left( (u_{m+1,n}^{t} - u_{m+1,n}^{t-1}) + (v_{m,n}^{t} - v_{m,n}^{t-1}) \right) + \beta v_{m,n}^{t} \right]
\]
(10)
Cyclic boundary condition is applied for the left- and right boundaries. This
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model is used to compare with the SPH model (5).

3 NUMERICAL RESULTS

Initial stream function can be obtained from the relationship between vorticity ($\zeta$) and stream function ($\psi$)

$$\nabla^2 \psi = \zeta$$

The finite differences produces a system of equations which can be solved for the stream function field by standard methods of matrix inversion $[3]$. With the stream function filed, the velocity component $u$ and $v$ can be calculated from

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

Figure 2 Vorticity simulation from SPH method.

The cyclic boundary condition is applied in this study. In the experiments on the barotropic model, the number of node in the SPH model is set to 2,145 which is equal to the number of grid point in the finite difference model that is used for the purpose of comparison. The number of particle used here is 8,580. The initial configuration which are used to predict the moving of vorticity field in this study are; line-spaces in direction $x = 6000$ km, line-spaces in direction $y = 3000$ km and $d = \Delta x = \Delta y = 93,750$ m. The localized initial vorticity in this study is defined as,
\begin{equation}
\zeta = 10^{-4} e^{-2(\varepsilon^2 + m^2 \varepsilon^2)}
\end{equation}

where \( k = \frac{2\pi}{6 \times 10^6 \text{m}} \), and \( m = \frac{\pi}{3 \times 10^6 \text{m}} \).

In this study, the model is run for 10 days with the time step \( dt = 900 \text{s} \). So, the end of time-step is at \( 24 \times 3,600 \times 10 = 864,000 \text{s} \).

The vorticity results from SPH method for up to 42 hours forecast are shown in Figure 2, with different forecast intervals. Corresponding results from finite difference method are also shown for comparison in Figure 3. The vorticity field is in the basic flow with weak wind speed in the upper part of the domain and strong wind speed in the lower part, as can be seen from the gradient of stream function. Thus, the lower part of the vorticity moves a longer distance than the movement in the upper part. This results in deformation of the vorticity with time.

![Figure 3](image_url)

**Figure 3** Vorticity simulation from finite difference method.

When the vorticity is advected close to the east boundary (when time = 30 and 42 hours), large error occurs. The main cause of error is due to asymmetric weight function on the boundary\[^{[1]}\]. The unbalanced particles contribute to the discretized summation that is not satisfy the unity condition of weight function. The summation of weight function is smaller than one which causes accumulation error.
It is known that one of the disadvantages of the SPH method is the approximation near boundaries of the domain. SPH interpolant is unable to accommodate boundary interpolation without special treatment. Extra care has to be taken in order to enforce the essential boundary conditions. In the following, a so-called “ghost particle” [5] approach is described, which has been often used in practical computation. Suppose particle $I$ is a boundary particle. All the other particles within support domain, $N(I)$, can be divided into three subsets (see Figure 4). For $I$ ($I$) represents all the interior points that are the neighbors of $I$, $B(I)$ represents all the boundary points that are the neighbors of $I$ and $G(I)$ represents all the exterior points that are the neighbors of $I$, i.e. all the ghost particles. Therefore $N(I) = I(I) \cup B(I) \cup G(I)$.

Figure 4 Concept of “ghost particle” [5].

When the ghost particle is used to improve weight function, the results at time = 30 and 42 hours seem to be less noisy as shown in Figure 5. The contours of vorticity from the approximation with ghost particle are now close to the finite difference solution. Some error still remain because the weight function of point near the boundary is also not symmetric which results in the unity condition is not valid at these points. In this experiment FDM seems to have less wavy pattern in the shape of vorticity than that of SPH. This could be a consequence of stronger smoothing effect in FDM.

Figure 5 Vorticity simulation from SPH method with ghost particle.
4 SUMMARY

Smooth particle hydrodynamics (SPH) is a meshless method which approximates differential equations based on a set of nodes without mesh. In this research, SPH method is applied to the two dimensional unsteady-state problem that is the barotropic vorticity equation for the atmosphere. The results show that with simple basic flows and cyclic boundary condition the SPH method performed well. When the basic flow had a complex pattern, large error occurred near the boundary due to asymmetry of weight function at the boundary. However, this can be partially solved by using ghost particle technique.

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