Generalized Nonhomogeneous Abstract
Degenerate Cauchy Problem

Susilo Hariyanto
Department of Mathematics
Gadjah Mada University, Yogyakarta, Indonesia
sus2_hariyanto@yahoo.co.id

Lina Aryati
Department of Mathematics
Gadjah Mada University, Yogyakarta, Indonesia
lina@ugm.ac.id

Widodo
Department of Mathematics
Gadjah Mada University, Yogyakarta, Indonesia
widodo_mathugm@yahoo.com

Copyright © 2013 Susilo Hariyanto et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

We will discuss about how to solve the generalized nonhomogeneous abstract degenerate Cauchy problem by factorization method. Here, generalization is meant the addition of linear operator $B$ on the non-homogeneous term’s. The problem is formulated in a Hilbert space which can be written as an orthogonal direct sum of $\text{Ker} M$ and $\text{Ran} M^*$. In this paper we will solve the problem in which $B$ is a bounded or closed operator. By using certain assumption, the generalized non-homogeneous abstract degenerate Cauchy problem can be reduced to...
nondegenerate problem, which is easier, because we can transform it to a common normal form that its solution is understood completely. Finally, by using a specific operator, the solution of nondegenerate Cauchy problem can be mapped to the solution of generalization problem.

**Keywords:** Degenerate Cauchy problem, nondegenerate Cauchy problem, factorization method

1 Introduction

We are going to investigate how to solve the generalized nonhomogeneous abstract degenerate Cauchy problem. The generalization is meant the addition of linear operator $B$ on the nonhomogeneous term’s. The generalized nonhomogeneous abstract degenerate Cauchy problem have been introduced by Thaller and Thaller [7], but they did not investigate completely. They only discussed problem (1) in special condition, when function $f(t)$ is a feedback control. In this case the problem (1) becomes homogeneous, where its solution is understood in [7]. In this paper, we will discuss it without assuming that $f(t)$ is a feedback control. Hence the problem (1) is interesting to discuss.

The generalize degenerate Cauchy problem is

$$\frac{d}{dt} Mz(t) = Az(t) + Bf(t),$$

where $M$, $A$ are linear operators from Hilbert space $\mathcal{H}$ to Hilbert space $\mathcal{W}$. The operator $B$ which is the generalization on nonhomogeneous term’s, is a linear operator from Hilbert space $\mathcal{U}$ to Hilbert space $\mathcal{W}$. If the operator $B$ is an identity operator, then we have nonhomogeneous abstract degenerate Cauchy problem:

$$\frac{d}{dt} Mz(t) = Az(t) + f(t), \quad z(0) = z_0$$

The operator $M$ is not invertible, so we call it the degenerate problem. If operator $M$ is invertible then we call it the nondegenerate. The solution of nonhomogeneous abstract degenerate Cauchy problem (2) have been discussed by Thaller and Thaller in [7].

his articles, the problem is treated under the assumption that the Banach space of the system can be written as a direct sum of suitable subspaces. Thaller and Thaller [7], [8] emphasize on the possibility of factorization and the relation of the factorized problem with the original degenerate system-without assuming parabolicity. Making use of the decomposition of the Hilbert space into a direct sum of $Ker\ M$ and $Ran\ M^*$ (resp. $Ker\ M^*$ and $Ran\ M$) they formulate the conditions which allow us to obtain an equivalent but nondegenerate Cauchy problem in the factor space $\mathcal{H}/Ker\ M$ and give the explicit form of generators of the factorized problems. The crucial assumptions is that the restriction of $A$ to a mapping from $Ker\ M$ to $Ker\ M^*$ is well defined and invertible. This allows to define a factorization operator $Z_A$ which maps the solutions of the factorized system to solutions of the original degenerate system. Hariyanto et. al. [5] have been discussed the solution of problem (2) by the alternative form on factorization method. The generalization of nonhomogeneous abstract degenerate Cauchy problem with bounded operator on nonhomogeneous term’s was solved in [6]. According the first paragraph, problem (1) have been introduced by Thaller and Thaller in their article [7], but they only discussed problem (1) in special condition when $f(t)$ is a feedback control. In this article we will discuss problem (1) without assuming that $f(t)$ is a feedback control. Here we will try to solve the generalization problem with bounded operator on nonhomogeneous term’s by the alternative form on factorization method. And then according the earlier research on the topics, we will also look for sufficient conditions, so the problem with $B$ closed operator can be solved by factorization method in [7] and alternative form on factorization method in [5]. Moreover, under the several assumptions we will give an example problem that can be solved by the method.

2 Main Results

We will investigate how to solve the generalized nonhomogeneous abstract degenerate Cauchy problem by factorization method which was discussed by Thaler and Thaler [7]. Hence, we use again several notations and assumptions in [7]. Here, $\mathcal{D}(T)$ is represented domain of $T$, where $T$ is an operator. According of [4], in this article the problem is treated also under the assumption that the Hilbert spaces of the system can be written as a direct sum of the kernel of $M$ ($Ker\ M$) and the range of adjoint $M$ ($Ran\ M^*$). In order to write Hilbert space $\mathcal{H}$ as an orthogonal direct sum of $Ker\ M$ and $Ran\ M^*$, they have assumed that operator $M : \mathcal{D}(M) \subset \mathcal{H} \to \mathcal{W}$ and $A : \mathcal{D}(A) \subset \mathcal{H} \to \mathcal{W}$ are closed
linear operators and densely defined. So, we define the orthogonal projection operator $P$ on Hilbert space $H$ onto $\text{Ker } M$ and operator $Q$ on Hilbert space $W$ onto $\text{Ker } M^*$. In order to transform the problem (2) to nondegenerate problem, they restricted $\mathcal{D}(M)$ to $\mathcal{D}(M_r)$. Hence, operator $M$ can be reduced to invertible operator $M_r = M|_{\mathcal{D}(M_r)}$ where $\mathcal{D}(M_r) = (\text{Ker } M)^\perp \cap \mathcal{D}(M)$ and $(\text{Ker } M)^\perp$ denotes orthogonal complement of Kernel $M$.

Before we discuss assumptions, lemmas, and theorems for solving the problem (1), we introduce the definition strict solution of problem (1).

**Definition 2.1.** Function $z : [0, \infty) \to H$ is a strict solution of nonhomogeneous abstract degenerate Cauchy problem with the linear operator on the nonhomogeneous term if $z(t) \in \mathcal{D}(A) \cap \mathcal{D}(M)$ for all $t \geq 0$, $Mz$ is continuously differentiable, and (1) holds.

Any strict solution $z(t)$ of the degenerate Cauchy problem (1) clearly satisfies $z(t) \in \mathcal{D}_{A,B}$ for all $t \geq 0$, where

$$\mathcal{D}_{A,B} = \left\{ z \in \mathcal{D}(A) | A z + B f(t) \in \overline{\text{Ran } M} \right\},$$

and $\overline{\text{Ran } M}$ denotes the closure of range of $M$. Next, we define an operator $A_0$ which is restriction $A$ on $(\text{Ker } M)^\perp$,

$$A_0 x(t) = A \left\{ (P^T)^{-1} \{ x(t) \} \cap \mathcal{D}_{A,B} \right\} \text{ for every } x(t) \in \mathcal{D}(A_0)$$

where $\mathcal{D}(A_0) = \left\{ x(t) \in (\text{Ker } M)^\perp | (P^T)^{-1} \{ x(t) \} \cap \mathcal{D}_{A,B} \neq \emptyset \right\}$. In this discussion, $(P^T)^{-1} \{ x(t) \}$ is inverse image of $x(t)$ under projection $P^T$, so we define it by $(P^T)^{-1} \{ x(t) \} = \{ x(t) + y(t) | y(t) \in \text{Ker } M \}$, for every $x(t) \in (\text{Ker } M)^\perp$.

The operator $A$, however, will become a multi valued operator $A_0$ on $(\text{Ker } M)^\perp$. If \(\left\{ (P^T)^{-1} \{ x(t) \} \cap \mathcal{D}_{A,B} \right\}\) is singleton, then operator $A_0$ on $(\text{Ker } M)^\perp$ become a single valued operator. So, we need the following Assumption 2.2, and 2.3.

**Assumption 2.2.** The operator $B : \mathcal{D}(B) \subset \mathcal{U} \to W$ is a linear operator which is $\text{Ran } B \subset \overline{\text{Ran } M}$.

**Assumption 2.3.** $PD_{A,B} \subset \mathcal{D}(A)$ dan operator $(QAP)|PD_{A,B}$ has a bounded inverse.

**Lemma 2.4.** Under the Assumption 2.2, and 2.3 a vector $z(t) \in H$ is in the subspace $\mathcal{D}_{A,B}$ if and only if $z(t) \in \mathcal{D}(A)$ and $Pz(t) = -(QAP)^{-1}QAP^Tz(t)$. 
Proof. By definition (3) and Assumption 2.2, \( z(t) \in \mathcal{D}_{A,B} \) if and only if \( z(t) \in \mathcal{D}(A) \) and \( Q(Az(t) + Bf(t)) = QAz(t) = 0 \). By Assumption 2.3, the vector \( Pz(t) \) and \( P^Tz(t) \) are in \( \mathcal{D}(A) \). Writing \( z(t) = Pz(t) + P^Tz(t) \) gives \( QAPz(t) + QAP^Tz(t) = 0 \) from which the result follows immediately, since \( QAP \) is invertible on the range of the projection \( Q \). Therefore any \( x(t) \in P^T\mathcal{D}_{A,B} \subset (\text{Ker} M)^\perp \) uniquely determines \( z(t) \in \mathcal{D}_{A,B} \) such that \( x(t) = P^Tz(t) \) and \( z(t) = (1 - (QAP)^{-1}QA)x(t) \). Hence the set \( (P^\perp)^{-1}x(t) \cap \mathcal{D}_{A,B} \) contains precisely one element.

According to Lemma 2.4, we can define the operator \( Z_A \) by
\[
Z_A = P^T - (QAP)^{-1}QAP^T
\]
which is defined on \( \mathcal{D}(Z_A) \subset P^T\mathcal{D}_{A,B} \). The restriction \( Z_A|P^T\mathcal{D}_{A,B} \) is \( 1 - (QAP)^{-1}QA \) on \( P^T\mathcal{D}_{A,B} \). This is the inverse of the projection \( P^T|\mathcal{D}_{A,B} \). Hence, the operator \( A_0 \) can be determined by
\[
A_0 = AZ_A, \quad \text{on } P^T\mathcal{D}_{A,B}.
\]
For every \( z(t) \in \mathcal{D}_{A,B} \), we can write \( A_0x(t) = Az(t) \) where \( x(t) = P^Tz(t) \).
Since \( Az(t) = Q^TAz(t) \) for all \( z(t) \in \mathcal{D}_{A,B} \), the operator \( A_0 \) can be written in the more symmetric form:
\[
A_0 = Q^TAP^T - Q^TAP(QAP)^{-1}QAP^T.
\]
To factorize \( A_0 \), we define an operator \( Y_A = Q^T - Q^TAP(QAP)^{-1}Q \). Therefore \( Y_AAP = 0 \), then \( Y_AAP^T = Y_AA \) and \( A_0 = Y_AA \) on \( \mathcal{D}(A_0) = P^T\mathcal{D}_{A,B} \).

Assumption 2.5. The operator \( A \) has a bounded inverse and \( \mathcal{D}_{A,B} \subset \mathcal{D}(M) \).

This Assumption implies that \( A|\mathcal{D}_{A,B} \) has a bounded inverse.
\[
A|\mathcal{D}_{A,B} : \mathcal{D}_{A,B} \rightarrow Q^T\mathcal{W} \quad \text{and} \quad (A|\mathcal{D}_{A,B})^{-1} : Q^T\mathcal{W} \rightarrow \mathcal{D}_{A,B}.
\]
Hence, the operator \( A_0^{-1} = P^T(AZ_A)^{-1} = P^TA^{-1}Q^T\mathcal{W} \) is bounded and densely defined on \( Q^T\mathcal{W} \). According to Assumption 2.2, we have:
\[
Bf(t) = (Q^T - Q^TAP(QAP)^{-1}Q)Bf(t) = Y_ABf(t),
\]
for every \( f(t) \in \mathcal{D}(B) \). Finally by Assumptions 2.2, 2.3, 2.5 and for every \( z(t) \in \mathcal{D}_{A,B} \cap \mathcal{D}(M) \), the degenerate Cauchy problem (1) can be reduced to nondegenerate problem:
\[
\frac{d}{dt}M_r x(t) = A_0x(t) + Y_Af(t), \quad x(t) = P^Tz_0
\]
where $M_r$ is invertible. The nondegenerate Cauchy problem (5) can be written in two types of normal form:

\[
\frac{d}{dt}x(t) = A_1 x(t) + (M_r)^{-1} Y_A B f(t), \quad \text{where} \quad A_1 = (M_r)^{-1} A_0. \quad (6)
\]

\[
\frac{d}{dt}y(t) = A_2 y(t) + Y_A B f(t), \quad \text{where} \quad A_2 = A_0 (M_r)^{-1}. \quad (7)
\]

The next step depends on the assumption of operator $B$ and $M$.

**Assumption 2.6.** Operator $B$ is a bounded operator.

Solution of problem (1) with $B$ bounded operator via normal form (6) has been discussed completely in [6]. The Assumption 2.7.a. bellow has given in [6], when we solve problem (1) via (6). In solving the problem (1) via normal form (7), we need the following Assumption 2.7.b.

**Assumption 2.7.**

a. Operator $M$ has a closed domain.

b. Operator $M$ has a closed range.

Operator $M$ is closed and densely defined, then by using Closed Graph Theorem, $(M_r)^{-1}$ is bounded and defined on all of $Q^T W$. If the Assumption 2.7.b. is satisfied, operator $A_2 = A_0 (M_r)^{-1}$ is closed on the domain,

\[
\mathcal{D}(A_2) = \{ y \in Q^T W | (M_r)^{-1} y \in \mathcal{D}(A_0) \} = M_r P^T \mathcal{D}_{A,B} = MD_{A,B}.
\]

It is closed because the product of a closed operator $A_0$ and a bounded operator $(M_r)^{-1}$. It is also densely defined in the Hilbert space $W_0 = (MD_{A,B})$. By Assumption 2.2, 2.3, 2.5, we can define:

\[
k(t) = M_r P^T A^{-1} B f(t) = MA^{-1} B f(t),
\]

and then (7) can be written in the following form:

\[
\frac{d}{dt}y(t) = A_2(y(t) + k(t)).
\]

**Assumption 2.8.** $A_2$ generates a strongly continuous semigroup in $W_0$.

By Assumptions 2.2, 2.3, 2.5, 2.6, 2.7.b, 2.8, and whenever $k(t)$ is in $\mathcal{D}(A_2) = \cdots$
Nonhomogeneous abstract degenerate Cauchy problem

$MD_{A,B}$, the solution of (7) is given by:

$$y(t) = e^{Az}y(0) + A_2 \int e^{A_2(t-s)}k(s)ds, \quad \text{where } k(t) = MA^{-1}Bf(t). \quad (8)$$

**Lemma 2.9.** If $z(t)$ is a solution of (1), then

$$z(t) = ZAM^{-1}y(t) - (QAP)^{-1}QBf(t), \quad \text{for all } t \geq 0$$

**Proof.** We will prove this Lemma by reductio ad absurdum. If $ZAM^{-1}y(t) - (QAP)^{-1}QBf(t) = r(t)$ and $z(t)$ is a solution, then we have:

$$\frac{d}{dt}Mr(t) = \frac{d}{dt}M[ZAM^{-1}y(t) - (QAP)^{-1}QBf(t)] = [MZAM^{-1}y(t) - M(QAP)^{-1}QBf(t)] = \frac{d}{dt}MZAx(t) = \frac{d}{dt}Mz(t) = Az(t) + Bf(t), \quad \text{for } z(t) = r(t).$$

Moreover, if we assume $z(t) \neq r(t)$, we have:

$$\frac{d}{dt}Mz(t) \neq \frac{d}{dt}Mr(t) \neq Az(t) + Bf(t).$$

It is a contradiction, therefore

$$z(t) = ZAM^{-1}y(t) - (QAP)^{-1}QBf(t). \quad \blacksquare$$

**Theorem 2.10.** Let $f(t)$ be a continuously differentiable function with values in Ran $M$. Under Assumption 2.2, 2.3, 2.5, 2.6, 2.7.b, and 2.8, the degenerate Cauchy problem (1) for each initial value $z_0 \in D_{A,B}$ has a unique strict solution:

$$z(t) = ZAM^{-1}y(t) - (QAP)^{-1}QBf(t),$$

where $y(t) = e^{Az}y(0) + A_2 \int e^{A_2(t-s)}k(s)ds$, and $k(t) = MA^{-1}Bf(t)$.

**Proof.** We know that $y(t) = e^{Az}y(0) + A_2 \int e^{A_2(t-s)}k(s)ds$,

where $k(t) = M_1P^T A^{-1}Bf(t)$ is a solution of nonhomogeneous abstract degenerate Cauchy problem with bounded operator on the nonhomogeneous term (7). By Assumption 2.8, the function $t \rightarrow y(t)$ is continuously differentiable and contained in $D(A_2)$ for all $t \geq 0$. Therefore $z(t) = ZAM^{-1}y(t)$ is well defined. Using the continuity of $M^{-1}$ we find $Mz(t) = M_1x(t)$ continuously
differentiable with
\[
\frac{d}{dt}Mz(t) = \frac{d}{dt}M_r x(t) = M_r \frac{d}{dt}x(t) = M_r [A_1 x(t) + (M_r)^{-1}YA_f(t)] \\
= A_0 x(t) + YA_r f(t) =Az(t) + f(t).
\]

Hence \(z(t)\) is a strict solution of degenerate Cauchy problem (1) with \(B\) bounded operator. ■

Next, we will investigate problem (1) with \(B\) closed operator. In order to solve of problem (1) with \(B\) closed operator on the nonhomogeneous term, we will replace Assumption 2.6 by Assumption 2.11:

**Assumption 2.11.** *Operator \(B\) is a closed operator.*

Operator \(B\) is closed, so we have a closed subspace \(\text{Ker} B\) in \(W\). Let \(R\) be orthogonal projection on \(\text{Ker} B\), then \(R^T = 1 - R\) is also orthogonal projection on \((\text{Ker} B)^\perp\). Hence we have:

\[
RW = \text{Ker} B, R^T W = (\text{Ker} B)^\perp,
\]

and the restriction operator \(B\) to \((\text{Ker} M)^\perp \cap \mathcal{D}(B)\) is defined by:

\[
B_r = B|_{\mathcal{D}(B_r)}, \text{ dengan } \mathcal{D}(B_r) = (\text{Ker} B)^\perp \cap \mathcal{D}(B).
\]

**Lemma 2.12.** *Operator \(B|_{\mathcal{D}(B_r)} = B_r\) is bounded.*

**Assumption 2.13.** *Function \(f(t) \in U\) is contained in \((\text{Ker} B)^\perp\).*

According to Assumption 2.13, we have:

\[
B f(t) = Q^T BR f(t) = (Q^T - Q^T AP(QAP)^{-1}Q)BR f(t) = YA_r f(t),
\]

for every \(f(t) \in (\text{Ker} B)^\perp \cap \mathcal{D}(B)\). And then by using (11) and for every \(z(t) \in \mathcal{D}_{A,B}\), the right hand side on the original problem can be reduced to

\[
Az(t) + B f(t) = A_0 x(t) + YA_r f(t) \text{ where } x(t) = P^T z(t).
\]

Hence the generalization of nonhomogeneous abstract degenerate Cauchy problem on nonhomogeneous term’s can be reduced to

\[
\frac{d}{dt}M_r x(t) = A_0 x(t) + YA_r f(t), x(0) = P^T z_0
\]
where \( M_r \) is invertible. Similarly with (6) and (7), we can transform (13) to normal form:

\[
\frac{d}{dt} x(t) = A_1 x(t) + (M_r)^{-1} Y_A B_r f(t), \quad \text{or} \quad \tag{14}
\]

\[
\frac{d}{dt} y(t) = A_2 y(t) + Y_A B_r f(t), \quad \tag{15}
\]

where \( A_1 = (M_r)^{-1} A_0 \) or \( A_2 = A_0 (M_r)^{-1} \) is a generator of strongly continuous semigroup in \( \mathcal{H}_0 \) or \( \mathcal{W}_0 \). Under the Assumption 2.5, we can define

\[
l(t) = P T A_1^{-1} B_r f(t) = P T A_1^{-1} B R^T f(t) = P T A_1^{-1} B f(t), \quad \text{and then} \quad (14) \text{can be written in the following form:}
\]

\[
\frac{d}{dt} x(t) = A_1 x(t) + (M_r)^{-1} Y_A B_r f(t) \\
= A_1 \left( x(t) + (A_1)^{-1} (M_r)^{-1} Y_A B_r f(t) \right) \\
= A_1 \left( x(t) + (A_0)^{-1} Y_A B_r f(t) \right) \\
= A_1 \left( x(t) + A_1 P T A_1^{-1} B_r f(t) \right) \\
= A_1 \left\{ x(t) + l(t) \right\}. 
\]

If Assumption 2.7.a. is satisfied, the operator \( A_1 = (M_r)^{-1} A_0 \) is closed on the domain \( \mathcal{D}(A_1) = \left\{ x \in P T \mathcal{D}_{A,B} \mid A_0 x \in \text{Ran} M \right\} = A_0^{-1} \text{Ran} M = P T \mathcal{D}_{A,B} \) because it is the product of a boundedly invertible operator \( (M_r)^{-1} \) and a closed operator \( A_0 \). It is also densely defined in the Hilbert space \( \mathcal{H}_0 = (P T \mathcal{D}_{A,B}) \).

Let \( l(t) \in \mathcal{D}(A_1) \) and be continuously differentiable, the solution of (14) is:

\[
x(t) = e^{A_1 t} P T z_0 + \int_0^t e^{A_1(t-s)} A_1 l(s) ds \\
= e^{A_1 t} P T z_0 + A_1 \int_0^t e^{A_1(t-s)} l(s) ds. \tag{16}
\]

Solving problem (1) via (15) is analogue, we can define

\[
m(t) = M_r P T A_1^{-1} B_r f(t) = M_r P T A_1^{-1} B R^T f(t) = M_r P T A_1^{-1} B f(t), \quad \text{and (16) can be written in the following form:}
\]

\[
\frac{d}{dt} y(t) = A_2 (y(t) + m(t)). \tag{17}
\]
Let \( m(t) \in \mathcal{D}(A_2) \) and be continuously differentiable, the solution of (17) is:

\[
y(t) = e^{A_2 t} P^T z_0 + \int_0^t e^{A_2 (t-s)} A_2 m(s) ds
= e^{A_2 t} P^T z_0 + A_2 \int_0^t e^{A_2 (t-s)} m(s) ds.
\]  

(18)

Finally, we have the following theorem for generalization of nonhomogeneous abstract degenerate Cauchy problem with closed operator on nonhomogeneous term.

**Theorem 2.14.**

i. Let \( f(t) \) be a continuously differentiable function with value \( Bf(t) \) in \( \text{Ran } M \). The problem (1) which is satisfied Assumption 2.2, 2.3, 2.5, 2.6, 2.7.a, 2.11, and 2.13, has a unique strict solution, \( z(t) = Z_A x(t) - (QAP)^{-1} QBf(t) \) for each initial value \( z_0 \in \mathcal{D}_{A,B} \), where \( x(t) = e^{A_1 t} P^T z_0 + A_1 \int_0^t e^{A_1 (t-s)} l(s) ds \) and \( l(t) = P^T A^{-1} B_f(t) \).

ii. Let \( f(t) \) be a continuously differentiable function with value \( Bf(t) \) in \( \text{Ran } M \). The problem (1) which is satisfied Assumptions 2.2, 2.3, 2.5, 2.6, 2.7.b, 2.11, and 2.13, has a unique strict solution, \( z(t) = Z_M^{-1} r(t) - (QAP)^{-1} QBf(t) \) for each initial value \( z_0 \in \mathcal{D}_{A,B} \), where \( y(t) = e^{A_2 t} P^T z_0 + A_2 \int_0^t e^{A_2 (t-s)} m(s) ds \) and \( m(t) = M_r P^T A^{-1} B_f(t) \).

**Proof.**

i. For each initial value \( z_0(t) \in \mathcal{D}_A, P^T z_0(t) \in \mathcal{D}(A_1) = P^T \mathcal{D}_{A,B} \), the function \( t \rightarrow x(t) \) is continuously differentiable, and \( x(t) \in \mathcal{D}(A_1) = P^T \mathcal{D}_{A,B} \) for all \( t \geq 0 \). Therefore \( z(t) = Z_A x(t) \) is well defined. Using the continuity of \( M_r \) we find \( Mz(t) = M_r x(t) \) continuously differentiable with:

\[
\frac{d}{dt} Mz(t) = M_r \frac{d}{dt} x(t)
= M_r (A_1 x(t) + M_r^{-1} Y_A B_f(t))
= A_0 x(t) + Y_A B_f(t)
= Az(t) + B_f(t).
\]
This shows that $Az(t) + Bf(t)$ is continuous. Hence $z(t)$ is a strict solution generalization of degenerate Cauchy problem with closed operator on nonhomogeneous term.

ii. For each initial value $z_0(t) \in D_{A,B}$, $Mz_0(t) \in D(A_2)$, the function $t \to y(t)$ is continuously differentiable, and $y(t) \in D(A_2) = M D_{A,B}$ for all $t \geq 0$. Therefore $(Mr)^{-1}$ is bounded, $x(t) = M r^{-1} y(t)$ is continuously differentiable and $z(t) = Z_A M r^{-1} y(t)$ is well defined. Hence, $Mz(t) = M r x(t) = y(t)$ is continuously differentiable, so:

$$\frac{d}{dt} y(t) = \frac{d}{dt} M r x(t) = \frac{d}{dt} M z(t) = Az(t) + Bf(t).$$

This shows that $Az(t) + Bf(t)$ is continuous, so $z(t)$ is continuous. Hence function $z(t)$ is a strict solution generalization of degenerate Cauchy problem with closed operator on nonhomogeneous term.

In the following we give an application of our approach on descriptor system.

**Example 2.15.** Consider the descriptor system on the Sobolev space $W^{2,2}(\mathbb{R})^3$:

$$\frac{d}{dt} Mz(t) = Az(t) + Bf(t), \quad z(0) = z_0 \quad (19)$$

$$y(t) = Cz(t) \quad (20)$$

where

$$M = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad A = \begin{pmatrix}
0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial x} \\
\frac{\partial}{\partial x} & a & 0 \\
\frac{\partial^2}{\partial x^2} & 0 & -\frac{\partial}{\partial x}
\end{pmatrix},$$

$$B = \begin{pmatrix}
\frac{\partial}{\partial x} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix}
0 & 0 & 1
\end{pmatrix}.$$
We choose orthogonal projection operators:

\[ P = Q = R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \]  
then \[ P^T = Q^T = R^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Therefore the problem (19) can be reduced to:

\[ \frac{d}{dt} M_r x(t) = A_0 x(t) + Y_A BR^T f(t), \quad x(0) = P^T z_0 \] on \( \mathcal{D}(A_0) = P^T \mathcal{D}_{A,B} \) \hspace{1cm} (21)

where

\[ M_r = MP^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad A_0 = Y_A A = \begin{pmatrix} \frac{a-1}{a} & \frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

According to (14), the problem (21) can be reduced to normal form:

\[ \frac{d}{dt} x(t) = A_1 x(t) + (M_r)^{-1} Y_A B_r f(t), \quad x(0) = P^T z_0 \]
on \( \mathcal{D}(A_1) = P^T \mathcal{D}_{A,B} \), where

\[ A_1 = \begin{pmatrix} \frac{a-1}{a} & \frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

The assumptions in Theorem 2.14.i. are satisfied then we get a solution of problem (19) in the following

\[ z(t) = Z_A x(t) - (QAP)^{-1} QBf(t) \] for each initial value \( z_0 \in \mathcal{D}_{A,B} \),

where \( x(t) = e^{A_1 t} P^T z_0 + A_1 \int_0^t e^{A_1(t-s)} l(s) ds, \quad l(s) = P^T A^{-1} BR^T f(s), \)
and

\[ Z_A = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \]

Hence, the solution of descriptor system (19-20) on \( P^T \mathcal{D}_{A,B} = (\text{Ker} \ M)^\perp \cap \mathcal{D}_{A,B} \) is
\( y(t) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} z(t). \)
References


Received: February 5, 2013