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Balanced Intuitionistic Fuzzy Graphs

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Abstract

In this paper, we introduce the notion of balanced intuitionistic fuzzy graphs and present some of their properties. We also prove that $G_1 \sqcap G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.

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1 Introduction

In 1736, Euler first introduced the notion of graph theory. In the history of mathematics, the solution given by Euler of the well known Königsberg bridge problem is considered to be the first theorem of graph theory. This has now become a subject generally regarded as a branch of combinatorics. The theory of graph is an extremely useful tool for solving combinatorial problems in different areas such as geometry, algebra, number theory, topology, operations research, optimization and computer science.

Density of a graph G $(D(G))$ is concerned with the patterns of connections of the entire networks. As the result of rapid increasing in the size of networks the graph problems become uncertain and we deal these aspects with the method of fuzzy logic. Graphs for which $D(H) \leq D(G)$ for all subgraph H of G are called balanced graph. balanced graph first arose in the study of random graphs and balanced IFG defined here is based on density functions. A graph with maximum density is complete and graph with minimum density is a null graph. There are several papers written on balanced extension of graph[10] which has tremendous applications in artificial intelligence, signal processing, robotics, computer networks and decision making.

Al-Hawary [1] introduced the concept of balanced fuzzy graphs and studied some operations of fuzzy graphs. Atanassov [6] introduced the concept of intuitionistic fuzzy relations and intuitionistic fuzzy graphs(IFGs). Parvathy and Karunambigai[8] introduced the concept of IFG elaborately and analysed its components. Articles[1, 6, 8] motivated us to analyze balanced IFGs and their properties. This paper deals with the significant properties of balanced IFG. The basic definition and theorems needed are discussed in section 2. The necessary condition for an IFG to be a Balanced IFG if the graph G is complete, strong, regular and self complementary IFG are discussed in section 3. We also discussed some properties of complementary and self complementary balanced IFGs. Section 4 deals with direct product, semi strong product and strong product of intuitionistic fuzzy graphs and their properties with suitable illustrations are given. The main theorem in this section is $G_1 \sqcap G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.

2 Preliminaries

An intuitionistic fuzzy graph (IFG) is of the form $G = \langle V, E \rangle$ said to be a Min-max IFG if

(i) $V = \{v_0, v_1, ..., v_n\}$ such that $\mu_1 : V \to [0, 1]$ and $\nu_1 : V \to [0, 1]$, denotes the degree of membership and non-membership of the element $v_i \in V$ respectively and $0 \leq \mu_1(v_i) + \nu_1(v_i) \leq 1$, for every $v_i \in V, (i = 1, 2, ..., n)$,

(ii) $E \subseteq V \times V$ where $\mu_2 : V\times \to [0,1]$ and $\nu_2 : V\times \to [0,1]$, are such that

$$\mu_2(v_i, v_j) \leq min[\mu_1(v_i), \mu_1(v_j)]$$
$$\nu_2(v_i, v_j) \leq max[\nu_1(v_i), \nu_1(v_j)],$$

denotes the degree of membership and non-membership of the edge $(v_i, v_j) \in E$ respectively, where $0 \leq \mu_2(v_i, v_j) + \nu_2(v_i, v_j) \leq 1$, for every $(v_i, v_j) \in E$.

An IFG $H = (V', E')$ is said to be an *IF subgraph*(IFSG) of $G = (V, E)$ if

(i) $V' \subseteq V$, where $\mu'_{1i} = \mu_{1i},\ \gamma'_{1i} = \gamma_{1i}$ for all $v_i \in V',\ i = 1, 2, 3 \dots, n$.

(ii) $E' \subseteq E$, where $\mu'_{2ij} = \mu_{2ij},\ \gamma'_{2ij} = \gamma_{2ij}$ for all $(v_i, v_j) \in E',\ i, j = 1, 2, \dots, n$.

An IFG, $G = (V, E)$ is said to be *complete IFG* if

$$\mu_{2ij} = \min(\mu_{1i}, \mu_{1j})\ and\ \gamma_{2ij} = \max(\gamma_{1i}, \gamma_{1j})$$

for every $v_i, v_j \in V$. An IFG, $G = (V, E)$ is said to be *strong IFG* if

$$\mu_{2ij} = \min(\mu_{1i}, \mu_{1j})\ and\ \gamma_{2ij} = \max(\gamma_{1i}, \gamma_{1j})$$

for every $(v_i, v_j) \in E$. The *complement* of an IFG, $G = (V, E)$ is an IFG, $\overline{G} = (\overline{V}, \overline{E})$, where

(i) $\overline{V} = V$,

(ii) $\overline{\mu_{1i}} = \mu_{1i}$ and $\overline{\gamma_{1i}} = \gamma_{1i}$, for all $i = 1, 2, \dots, n$,

(iii) $\overline{\mu_{2ij}} = \min(\mu_{1i}, \mu_{1j}) - \mu_{2ij}$ and $\overline{\gamma_{2ij}} = \max(\gamma_{1i}, \gamma_{1j}) - \gamma_{2ij}$ for all $i, j = 1, 2, \dots, n$.

An intuitionistic fuzzy graph $G = (V, E)$ is said to be *regular IFG* if all the vertices have the same closed neighborhood degree.

The *density* of a complete fuzzy graph $G = (\sigma, \mu)$ is

$$D(G) = 2\left(\frac{\sum_{u,v\in V}(\mu(u, v))}{\sum_{u,v\in V}(\sigma(u) \wedge \sigma(v))}\right)$$

Consider the two IFGs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. An *Isomorphism* between two IFGs G_1 and G_2, denoted by $G_1 \cong G_2$, is a bijective map $h : V_1 \to V_2$ which satisfies

$$\mu_1(v_i) = \mu'_1(h(v_i)), \nu_1(v_i) = \nu'_1(h(v_i)) \text{ and}$$

$$\mu_2(v_i, v_j) = \mu'_2(h(v_i), h(v_j)),$$
$$\nu_2(v_i, v_j) = \nu'_2(h(v_i), h(v_j)) \text{ for every } \mathrm{v_i, v_j} \in \mathrm{V}.$$

3 Balanced Intuitionistic Fuzzy Graphs

Definition 1. The *density* of an intuitionistic fuzzy graph $G = (V, E)$ is

$$D(G) = (D_\mu(G), D_\nu(G)), \text{ where}$$

$D_\mu(G)$ is defined by

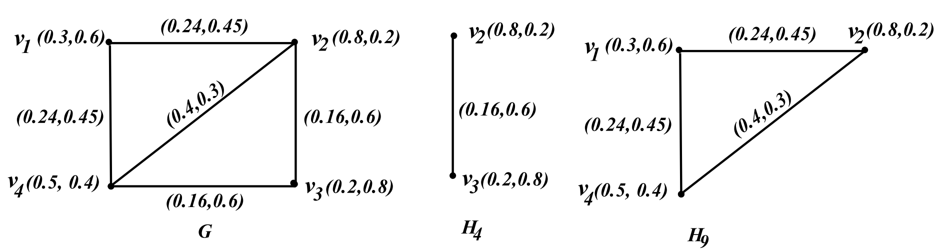
$$D_\mu(G) = \frac{2\sum_{u,v\in V}(\mu_2(u,v))}{\sum_{(u,v)\in E}(\mu_1(u)\wedge\mu_1(v))}, \text{ for u, v} \in \text{V}$$

and $D_\nu(G)$ is defined by

$$D_\nu(G) = \frac{2\sum_{u,v\in V}(\nu_2(u,v))}{\sum_{(u,v)\in E}(\nu_1(u)\vee\nu_1(v))}, \text{ for u, v} \in \text{V.}$$

Definition 2. *An intuitionistic fuzzy graph $G = (V, E)$ is balanced if $D(H) \leq D(G)$, that is, $D_\mu(H) \leq D_\mu(G), D_\nu(H) \leq D_\nu(G)$ for all subgraphs H of G.*

Example 3. *Consider a IFG, $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_2, v_4)\}$.*



Balanced Intuitionistic Fuzzy Graph

μ-density

$D_\mu(G) = 2\left(\frac{0.24+0.16+0.16+0.24+0.4}{0.3+0.2+0.2+0.3+0.5}\right) = 1.6$

ν-density

$D_\nu(G) = 2\left(\frac{0.45+0.6+0.6+0.45+0.3}{0.6+0.8+0.8+0.6+0.4}\right) = 1.5$

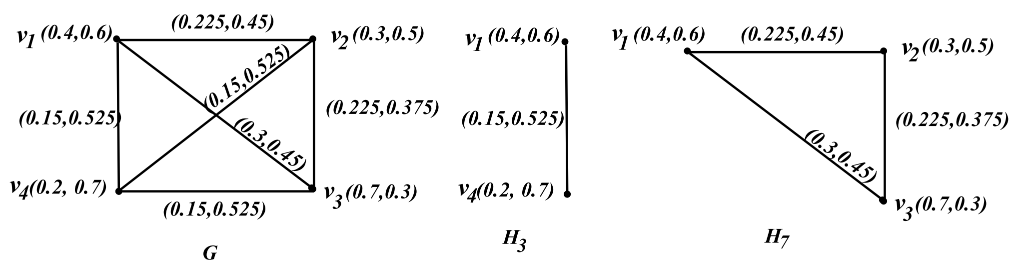
$D(G) = (D_\mu(G), D_\nu(G)) = (1.6, 1.5)$

Let $H_1 = \{v_1, v_2\}$, $H_2 = \{v_1, v_3\}$, $H_3 = \{v_1, v_4\}$, $H_4 = \{v_2, v_3\}$, $H_5 = \{v_2, v_4\}$, $H_6 = \{v_3, v_4\}$, $H_7 = \{v_1, v_2, v_3\}$, $H_8 = \{v_1, v_3, v_4\}$, $H_9 = \{v_1, v_2, v_4\}$, $H_{10} = \{v_2, v_3, v_4\}$, $H_{11} = \{v_1, v_2, v_3, v_4\}$ be a non empty subgraphs of G. Density $(D_\mu(H), D_\nu(H))$ is $D(H_1) = (1.6, 1.5)$, $D(H_2) = (0, 0)$, $D(H_3) = (1.6, 1.5)$,

$D(H_4) = (1.6, 1.5)$, $D(H_5) = (1.6, 1.5)$, $D(H_6) = (1.6, 1.5)$, $D(H_7) = (1.6, 1.5)$, $D(H_8) = (1.6, 1.5)$, $D(H_9) = (1.6, 1.5)$, $D(H_{10}) = (1.6, 1.5)$, $D(H_{11}) = (1.6, 1.5)$. *So* $D(H) \leq D(G)$ *for all subgraphs* H *of* G. *Hence* G *is balanced IFG.*

Definition 4. *An intuitionistic fuzzy graph* $G = (V, E)$ *is strictly balanced if for every* $u, v \in V$, $D(H) = D(G)$ *for all non empty subgraphs* H *of* G.

Example 5. *Consider an IFG* $G = (V, E)$ *such that* $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3), (v_2, v_4)\}$.



Strictly Balanced IFG

$D(G) = (D_\mu(G), D_\nu(G)) = (1.5, 1.5)$. *Let* $H_1 = \{v_1, v_2\}$, $H_2 = \{v_1, v_3\}$, $H_3 = \{v_1, v_4\}$, $H_4 = \{v_2, v_3\}$, $H_5 = \{v_2, v_4\}$, $H_6 = \{v_3, v_4\}$, $H_7 = \{v_1, v_2, v_3\}$, $H_8 = \{v_1, v_3, v_4\}$, $H_9 = \{v_1, v_2, v_4\}$, $H_{10} = \{v_2, v_3, v_4\}$, $H_{11} = \{v_1, v_2, v_3, v_4\}$ *be a non empty subgraphs of* G. *Density* $(D_\mu(H), D_\nu(H))$ *is* $D(H_1) = (1.5, 1.5)$, $D(H_2) = (1.5, 1.5)$, $D(H_3) = (1.5, 1.5)$, $D(H_4) = (1.5, 1.5)$, $D(H_5) = (1.5, 1.5)$, $D(H_6) = (1.5, 1.5)$, $D(H_7) = (1.5, 1.5)$, $D(H_8) = (1.5, 1.5)$, $D(H_9) = (1.5, 1.5)$, $D(H_{10}) = (1.5, 1.5)$, $D(H_{11}) = (1.5, 1.5)$. *Hence* $D(H) = D(G)$ *for all non empty subgraphs* H *of* G. *Hence* G *is strictly balanced IFG.*

Theorem 6. *Every complete intuitionistic fuzzy graph is balanced.*

Proof. Let $G = (V, E)$ be a complete IFG, then by the definition of complete IFG, we have $\mu_2(u, v) = \mu_1(u) \wedge \mu_1(v)$ and $\nu_2(u, v) = \nu_1(u) \vee \nu_1(v)$ for every $u, v \in V$.

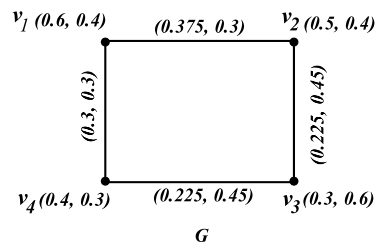
$$\therefore \sum_{u,v \in V} (\mu_2(u, v)) = \sum_{(u,v) \in E} (\mu_1(u) \wedge \mu_1(v)) \text{ and } \sum_{\mathrm{u,v} \in \mathrm{V}} (\nu_2(\mathrm{u,v})) = \sum_{(\mathrm{u,v}) \in \mathrm{E}} (\nu_1(\mathrm{u}) \vee \nu_1(\mathrm{v})).$$

Now $D(G) = \left(\left(\dfrac{2 \sum_{u,v \in V} (\mu_2(u, v))}{\sum_{(u,v) \in E} (\mu_1(u) \wedge \mu_1(v))} \right), \left(\dfrac{2 \sum_{u,v \in V} (\nu_2(u, v))}{\sum_{(u,v) \in E} (\nu_1(u) \vee \nu_1(v))} \right) \right)$ $D(G)$ $= \left(\left(\dfrac{2 \sum_{(u,v) \in E} (\mu_1(u) \wedge \mu_1(v))}{\sum_{(u,v) \in E} (\mu_1(u) \wedge \mu_1(v))} \right), \left(\dfrac{2 \sum_{(u,v) \in E} (\nu_1(u) \vee \nu_1(v))}{\sum_{(u,v) \in E} (\nu_1(u) \vee \nu_1(v))} \right) \right)$ $D(G) = (2,$

2). Let H be a non empty subgraph of G then, $D(H)$ =(2, 2) for every $H \subseteq G$. Thus G is Balanced. □

Note 7. *The converse of the above Theorem is need not be true. Every balanced IFG need not be complete.*

Example 8. *Consider an IFG $G = (V, E)$, such that $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4),$
$(v_4, v_1)\}$.*



Balanced IFG but not complete IFG

$D(G) = (D_\mu(G), D_\nu(G)) = (1.5, 1.5)$
Let $H_1 = \{v_1, v_2\}$, $H_2 = \{v_1, v_3\}$, $H_3 = \{v_1, v_4\}$, $H_4 = \{v_2, v_3\}$, $H_5 = \{v_2, v_4\}$, $H_6 = \{v_3, v_4\}$, $H_7 = \{v_1, v_2, v_3\}$, $H_8 = \{v_1, v_3, v_4\}$, $H_9 = \{v_1, v_2, v_4\}$, $H_{10} = \{v_2, v_3, v_4\}$, $H_{11} = \{v_1, v_2, v_3, v_4\}$ be a non empty subgraphs of G. Density $(D_\mu(H), D_\nu(H))$ is $D(H_1) = (1.5, 1.5)$, $D(H_2) = (0, 0)$, $D(H_3) = (1.5, 1.5)$, $D(H_4) = (1.5, 1.5)$, $D(H_5) = (0, 0)$, $D(H_6) = (1.5, 1.5)$, $D(H_7) = (1.5, 1.5)$, $D(H_8) = (1.5, 1.5)$, $D(H_9) = (1.5, 1.5)$, $D(H_{10}) = (1.5, 1.5)$, $D(H_{11}) = (1.5, 1.5)$. Hence $D(H) \leq D(G)$ for all subgraphs H of G. So G is balanced IFG. From the above graph easy to see that:
$\mu_2(u, v) \neq \mu_1(u) \wedge \mu_1(v)$ and $\nu_2(u, v) \neq \nu_1(u) \vee \nu_1(v))$. Hence G is balanced but not complete.

Corollary 9. *Every strong IFG is balanced.*

Theorem 10. *Let $G = (V, E)$ be a self complementary IFG. Then $D(G) = (1, 1)$.*

Theorem 11. *et $G = (V, E)$ be a strictly balanced IFG and $\overline{G} = (\overline{V}, \overline{E})$ be its complement then $D(G) + D(\overline{G}) = (2, 2)$.*

Proof. Let $G = (V, E)$ be a strictly balanced IFG and $\overline{G} = (\overline{V}, \overline{E})$ be its complement. Let H be a non empty subgraph of G. Since G is strictly balanced

$D(G) = D(H)$ for every $H \subseteq G$ and $u, v \in V$,

$$\text{In } \overline{\mathrm{G}},\ \overline{\mu_2(\mathrm{u},\mathrm{v})} = \mu_1(\mathrm{u}) \wedge \mu_1(\mathrm{v}) - \mu_2(\mathrm{u},\mathrm{v}) \longrightarrow (1)$$
$$\text{and } \overline{\nu_2(\mathrm{u},\mathrm{v})} = \nu_1(\mathrm{u}) \vee \nu_1(\mathrm{v}) - \nu_2(\mathrm{u},\mathrm{v}) \longrightarrow (2)$$

for every $u, v \in V$. Dividing (1) by $\mu_1(u) \wedge \mu_1(v)$ gives

$$\frac{\overline{\mu_2(u,v)}}{\mu_1(u) \wedge \mu_1(v)} = 1 - \frac{\mu_2(u,v)}{\mu_1(u) \wedge \mu_1(v)},$$

for every $u, v \in V$ and dividing (2) by $\nu_1(u) \vee \nu_1(v)$,

$$\frac{\overline{\nu_2(u,v)}}{\nu_1(u) \vee \nu_1(v)} = 1 - \frac{\nu_2(u,v)}{\nu_1(u) \vee \nu_1(v)},$$

for every $u, v \in V$.

$$\text{then } \sum_{\mathrm{u},\mathrm{v} \in \mathrm{V}} \frac{\overline{\mu_2(\mathrm{u},\mathrm{v})}}{\mu_1(\mathrm{u}) \wedge \mu_1(\mathrm{v})} = 1 - \sum_{\mathrm{u},\mathrm{v} \in \mathrm{V}} \frac{\mu_2(\mathrm{u},\mathrm{v})}{\mu_1(\mathrm{u}) \wedge \mu_1(\mathrm{v})},$$

where $u, v \in V$.

$$\text{and } \sum_{\mathrm{u},\mathrm{v} \in \mathrm{V}} \frac{\overline{\nu_2(\mathrm{u},\mathrm{v})}}{\nu_1(\mathrm{u}) \vee \nu_1(\mathrm{v})} = 1 - \sum_{\mathrm{u},\mathrm{v} \in \mathrm{V}} \frac{\nu_2(\mathrm{u},\mathrm{v})}{\nu_1(\mathrm{u}) \vee \nu_1(\mathrm{v})},$$

where $u, v \in V$.

$$2 \sum_{u,v \in V} \frac{\overline{\mu_2(u,v)}}{\overline{\overline{\mu_1(u)} \wedge \overline{\mu_1(v)}}} = 2 - 2 \sum_{u,v \in V} \frac{\mu_2(u,v)}{\mu_1(u) \wedge \mu_1(v)},$$

where $u, v \in V$

$$\text{and } 2 \sum_{\mathrm{u},\mathrm{v} \in \mathrm{V}} \frac{\overline{\nu_2(\mathrm{u},\mathrm{v})}}{\overline{\overline{\nu_1(\mathrm{u})} \vee \overline{\nu_1(\mathrm{v})}}} = 2 - 2 \sum_{\mathrm{u},\mathrm{v} \in \mathrm{V}} \frac{\nu_2(\mathrm{u},\mathrm{v})}{\nu_1(\mathrm{u}) \vee \nu_1(\mathrm{v})},$$

where $u, v \in V$

$$D_\mu(\overline{G}) = 2 - D_\mu(G) \text{ and } \mathrm{D}_\nu(\overline{\mathrm{G}}) = 2 - \mathrm{D}_\nu(\mathrm{G})$$

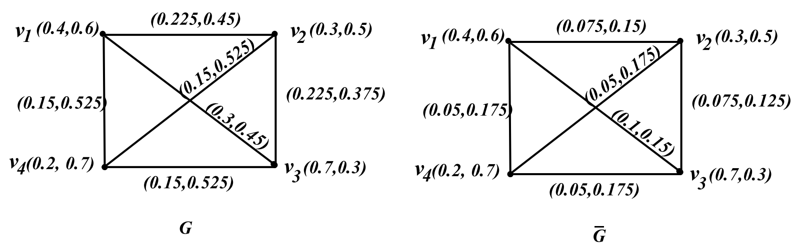
Now,

$$\begin{aligned} D(G) + D(\overline{G}) &= (D_\mu(G), D_\nu(G)) + (D_\mu(\overline{G}), D_\nu(\overline{G})) \\ &= (D_\mu(G) + D_\mu(\overline{G})), (D_\nu(G) + D_\nu(\overline{G})) \end{aligned}$$

Hence $D(G) + D(\overline{G})$= (2,2). □

Theorem 12. *The complement of strictly balanced IFG is strictly balanced.*

Example 13. *Consider an IFG,* $G = (V, E)$*, such that* $V = \{v_1, v_2, v_3, v_4\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3), (v_2, v_4)\}$ *and its complement* $\overline{G} = (\overline{V}, \overline{E})$*, such that* $\overline{V} = \{v_1, v_2, v_3, v_4\}$,
$\overline{E} = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3), (v_2, v_4)\}$.



Strictly Balanced and its complement

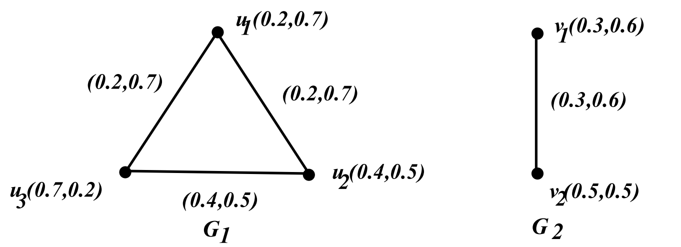
$D(G) = (D_\mu(G), D_\nu(G)) = (1.5, 1.5)$ $D(\overline{G}) = (D_\mu(\overline{G}), D_\nu(\overline{G})) = (0.5, 0.5)$. *Hence* $D(G) + D(\overline{G}) = (1.5 + 0.5, 1.5 + 0.5) = (2, 2)$.

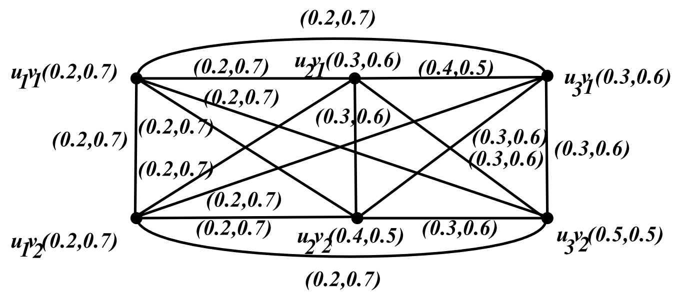
Definition 14. *Let* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$ *be an intuitionistic fuzzy graphs, where* $V = V_1 \times V_2$ *and* $E = \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$*. Then the direct product of* G_1 *and* G_2 *is an IFG denoted by* $G_1 \sqcap G_2 = (V, E)$*, where*

- $(\mu_1 \sqcap \mu_1^{'})(u_1, v_1) = \min(\mu_1(u_1), \mu_1^{'}(v_1))$ *for every* $(u_1, v_1) \in V_1 \times V_2$ *and* $(\nu_1 \sqcap \nu_1^{'})(u_1, v_1) = \max(\nu_1(u_1), \nu_1^{'}(v_1))$ *for every* $(u_1, v_1) \in V_1 \times V_2$.

- $(\mu_2 \sqcap \mu_2^{'})(u_1, v_1)(u_2, v_2) = min(\mu_2(u_1, u_2), \mu_2^{'}(v_1, v_2)$ *for every* $(u_1, u_2) \in E_1$, $(v_1, v_2) \in E_2$ *and* $(\nu_2 \sqcap \nu_2^{'})(u_1, v_1)(u_2, v_2) = \max(\nu_2(u_1, u_2), \nu_2^{'}(v_1, v_2))$ *for every* $(u_1, u_2) \in E_1, (v_1, v_2) \in E_2$.

Example 15. Consider an IFG, such that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $V_1 = \{u_1, u_2, u_3\}$, $E_1 = \{(u_1, u_2), (u_2, u_3), (u_3, u_1)\}$, $V_2 = \{v_1, v_2\}$ and $E_1 = \{(v_1, v_2)\}$.





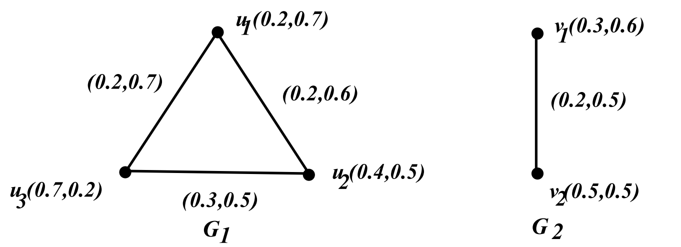
Direct Product $(G_1 \sqcap G_2)$

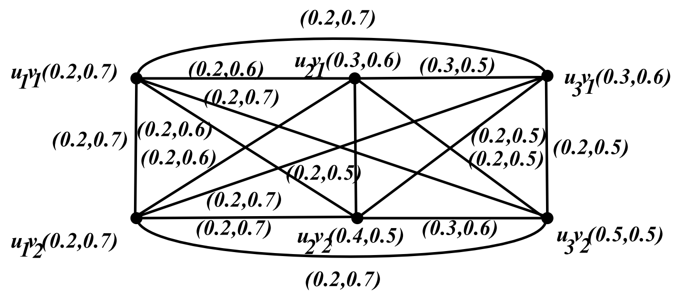
By computations, it is easy to see that:
$(\mu_1 \sqcap \mu_1')(u_1, v_1) = 0.2$ and $(\nu_1 \sqcap \nu_1')(u_1, v_1) = 0.7$, $(\mu_1 \sqcap \mu_1')(u_1, v_2) = 0.2$ and $(\nu_1 \sqcap \nu_1')(u_1, v_2) = 0.7$, $(\mu_1 \sqcap \mu_1')(u_2, v_1) = 0.3$ and $(\nu_1 \sqcap \nu_1')(u_2, v_1) = 0.6$, $(\mu_1 \sqcap \mu_1')(u_2, v_2) = 0.4$ and $(\nu_1 \sqcap \nu_1')(u_2, v_2) = 0.5$, $(\mu_1 \sqcap \mu_1')(u_3, v_1) = 0.3$ and $(\nu_1 \sqcap \nu_1')(u_3, v_1) = 0.6$, $(\mu_1 \sqcap \mu_1')(u_3, v_2) = 0.5$ and $(\nu_1 \sqcap \nu_1')(u_3, v_2) = 0.5$. $(\mu_2 \sqcap \mu_2')(u_1, v_1)(u_2, v_2) = 0.2$ and $(\nu_2 \sqcap \nu_2')(u_1, v_1)(u_2, v_2) = 0.7$, $(\mu_2 \sqcap \mu_2')(u_2, v_1)(u_3, v_2) = 0.3$ and $(\nu_2 \sqcap \nu_2')(u_2, v_1)(u_3, v_2) = 0.6$, $(\mu_2 \sqcap \mu_2')(u_1, v_1)(u_3, v_2) = 0.2$ and $(\nu_2 \sqcap \nu_2')(u_1, v_1)(u_3, v_2) = 0.7$.

Definition 16. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be an intuitionistic fuzzy graphs, where $V = V_1 \times V_2$ and $E = \{(u, u_2)(u, v_2) : u \in V_1, (u_2, v_2) \in E_2\} \bigcup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$. Then the semi strong product of G_1 and G_2 is an IFG denoted by $G_1 \odot G_2 = (V, E)$, where*

- $(\mu_1 \odot \mu_1')(u_1, u_2) = min(\mu_1(u_1), \mu_1'(u_2))$ *for every* $(u_1, u_2) \in V_1 \times V_2$ *and* $(\nu_1 \odot \nu_1')(u_1, u_2) = max(\nu_1(u_1), \nu_1'(u_2))$ *for every* $(u_1, u_2) \in V_1 \times V_2$.
- $(\mu_2 \odot \mu_2')(u, u_2)(u, v_2) = min(\mu_1(u), \mu_2'(u_2, v_2)$ *for every* $u \in V_1, (u_2, v_2) \in E_2$ *and* $(\nu_2 \odot \nu_2')(u, u_2)(u, v_2) = max(\nu_1(u), \nu_2'(u_2, v_2)$ *for every* $u \in V_1, (u_2, v_2) \in E_2$ *and* $(\mu_2 \odot \mu_2')(u_1, u_2)(v_1, v_2) = min(\mu_2(u_1, u_2), \mu_2'(v_1, v_2))$ *for every* $(u_1, u_2) \in E_1, (v_1, v_2) \in E_2$ *and* $(\nu_2 \odot \nu_2')(u_1, u_2)(v_1, v_2) = max(\nu_2(u_1, u_2), \nu_2'(v_1, v_2))$ *for every* $(u_1, u_2) \in E_1, (v_1, v_2) \in E_2$.

Example 17. *Consider an IFG, such that $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, $V_1 = \{u_1, u_2, u_3\}$, $E_1 = \{(u_1, u_2), (u_2, u_3), (u_3, u_1)\}$, $V_2 = \{v_1, v_2\}$ and $E_1 = \{(v_1, v_2)\}$.*





Semi Strong Product $(G_1 \odot G_2)$

By computations, it is easy to see that:
$(\mu_1 \odot \mu_1^{'})(u_1, v_1) = 0.2$ and $(\nu_1 \odot \nu_1^{'})(u_1, v_1) = 0.7$, $(\mu_1 \odot \mu_1^{'})(u_1, v_2) = 0.2$ and $(\nu_1 \odot \nu_1^{'})(u_1, v_2) = 0.7$, $(\mu_1 \odot \mu_1^{'})(u_2, v_1) = 0.3$ and $(\nu_1 \odot \nu_1^{'})(u_2, v_1) = 0.6$, $(\mu_1 \odot \mu_1^{'})(u_2, v_2) = 0.4$ and $(\nu_1 \odot \nu_1^{'})(u_2, v_2) = 0.5$, $(\mu_1 \odot \mu_1^{'})(u_3, v_1) = 0.3$ and $(\nu_1 \odot \nu_1^{'})(u_3, v_1) = 0.6$, $(\mu_1 \odot \mu_1^{'})(u_3, v_2) = 0.5$ and $(\nu_1 \odot \nu_1^{'})(u_3, v_2) = 0.5$.
$(\mu_2 \odot \mu_2^{'})(u_1, v_1)(u_2, v_2) = 0.2$ and
$(\nu_2 \odot \nu_2^{'})(u_1, v_1)(u_2, v_2) = 0.6$, $(\mu_2 \odot \mu_2^{'})(u_2, v_1)(u_3, v_2) = 0.2$ and $(\nu_2 \odot \nu_2^{'})(u_2, v_1)(u_3, v_2) = 0.5$,
$(\mu_2 \odot \mu_2^{'})(u_1, v_1)(u_3, v_2) = 0.2$ and $(\nu_2 \odot \nu_2^{'})(u_1, v_1)(u_3, v_2) = 0.7$ $(\mu_2 \odot \mu_2^{'})(u_1, v_1)(u_1, v_2) = 0.2$ and $(\nu_2 \odot \nu_2^{'})(u_1, v_1)(u_1, v_2) = 0.7$ $(\mu_2 \odot \mu_2^{'})(u_2, v_1)(u_2, v_2) = 0.2$ and $(\nu_2 \odot \nu_2^{'})(u_2, v_1)(u_2, v_2) = 0.5$ $(\mu_2 \odot \mu_2^{'})(u_3, v_1)(u_3, v_2) = 0.2$ and $(\nu_2 \odot \nu_2^{'})(u_3, v_1)(u_3, v_2) = 0.5$.

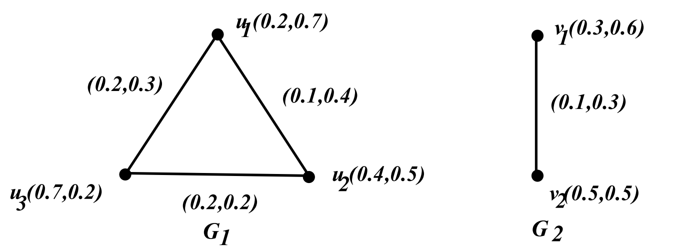
Definition 18. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be an intuitionistic fuzzy graphs, where $V = V_1 \times V_2$ and $E = \{(u, v_1)(u, v_2) : u \in V_1, (v_1, v_2) \in E_2\} \bigcup \{(u_1, w)(u_2, w) : w \in V_2, (u_1, u_2) \in E_1\} \bigcup \{(u_1, v_1)(u_2, v_2) : (u_1, u_2) \in E_1, (v_1, v_2) \in E_2\}$. Then the strong product of G_1 and G_2 is an IFG denoted by $G_1 * G_2 = (V, E)$, where*

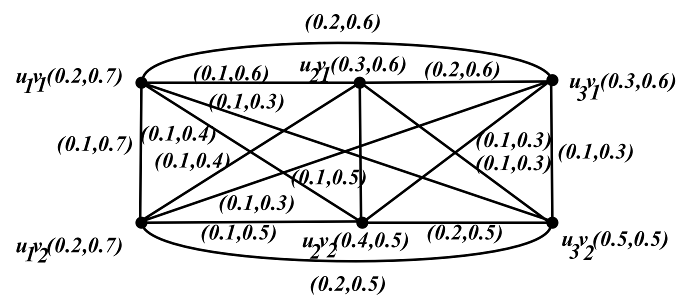
1. *$(\mu_1 * \mu_1^{'})(u_1, u_2) = \min(\mu_1(u_1), \mu_1^{'}(u_2))$ for every $(u_1, u_2) \in V_1 \times V_2$ and $(\nu_1 * \nu_1^{'})(u_1, u_2) = \max(\nu_1(u_1), \nu_1^{'}(u_2))$ for every $(u_1, u_2) \in V_1 \times V_2$.*

2. *$(\mu_2 * \mu_2^{'})(u, u_2)(u, v_2) = \min(\mu_1(u), \mu_2^{'}(u_2, v_2)$ for every $u \in V_1, (u_2, v_2) \in E_2$ and $(\mu_2 * \nu_2^{'})(u, u_2)(u, v_2) = \max(\nu_1(u), \nu_2^{'}(u_2, v_2)$ for every $u \in V_1, (u_2, v_2) \in$*

E_2 *and* $(\mu_2 * \mu_2^{'})(u_1, w)(u_2, w) = \min(\mu_2(u_1, u_2), \mu_2^{'}(w)$ *for every* $(u_1, u_2) \in E_1, w \in V_2$ *and* $(\nu_2 * \nu_2^{'})(u_1, w)(u_2, w) = \max(\nu_2(u_1, u_2), \nu_2^{'}(w)$ *for every* $(u_1, u_2) \in E_1, w \in V_2$. $(\mu_2 * \mu_2^{'})(u_1, u_2)(v_1, v_2) = \min(\mu_2(u_1, u_2), \mu_2^{'}(v_1, v_2)$ *for every* $(u_1, u_2) \in E_1, (v_1, v_2) \in E_2$ *and* $(\nu_2 * \nu_2^{'})(u_1, u_2)(v_1, v_2) = \max(\nu_2(u_1, u_2), \nu_2^{'}(v_1, v_2)$ *for every* $(u_1, u_2) \in E_1, (v_1, v_2) \in E_2$.

Example 19. *Consider an IFG, such that* $G_1 = (V_1, E_1)$ *and* $G_2 = (V_2, E_2)$, $V_1 = \{u_1, u_2, u_3\}$, $E_1 = \{(u_1, u_2), (u_2, u_3), (u_3, u_1)\}$, $V_2 = \{v_1, v_2\}$ *and* $E_1 = \{(v_1, v_2)\}$.





Strong Product $(G_1 * G_2)$

By computations, it is easy to see that:
$(\mu_1 * \mu_1^{'})(u_1, v_1) = 0.2$ and $(\nu_1 * \nu_1^{'})(u_1, v_1) = 0.7$, $(\mu_1 * \mu_1^{'})(u_1, v_2) = 0.2$ and $(\nu_1 * \nu_1^{'})(u_1, v_2) = 0.7$, $(\mu_1 * \mu_1^{'})(u_2, v_1) = 0.3$ and $(\nu_1 * \nu_1^{'})(u_2, v_1) = 0.6$, $(\mu_1 * \mu_1^{'})(u_2, v_2) = 0.4$ and $(\nu_1 * \nu_1^{'})(u_2, v_2) = 0.5$, $(\mu_1 * \mu_1^{'})(u_3, v_1) = 0.3$ and $(\nu_1 * \nu_1^{'})(u_3, v_1) = 0.6$, $(\mu_1 * \mu_1^{'})(u_3, v_2) = 0.5$ and $(\nu_1 * \nu_1^{'})(u_3, v_2) = 0.5$. $(\mu_2 * \mu_2^{'})(u_1, v_1)(u_2, v_2) = 0.1$ and $(\nu_2 * \nu_2^{'})(u_1, v_1)(u_2, v_2) = 0.4$, $(\mu_2 * \mu_2^{'})(u_2, v_1)(u_3, v_2) = 0.1$ and $(\nu_2 * \nu_2^{'})(u_2, v_1)(u_3, v_2) = 0.3$, $(\mu_2 * \mu_2^{'})(u_1, v_1)(u_3, v_2) = 0.1$ and $(\nu_2 * \nu_2^{'})(u_1, v_1)(u_3, v_2) = 0.3$ $(\mu_2 * \mu_2^{'})(u_1, v_1)(u_1, v_2) = 0.1$ and $(\nu_2 * \nu_2^{'})(u_1, v_1)(u_1, v_2) = 0.7$ $(\mu_2 * \mu_2^{'})(u_2, v_1)(u_2, v_2) = 0.1$ and $(\nu_2 * \nu_2^{'})(u_2, v_1)(u_2, v_2) = 0.5$ $(\mu_2 * \mu_2^{'})(u_3, v_1)(u_3, v_2) = 0.1$ and $(\nu_2 * \nu_2^{'})(u_3, v_1)(u_3, v_2) = 0.3$ $(\mu_2 * \mu_2^{'})(u_1, v_1)(u_2, v_1) = 0.1$ and $(\nu_2 * \nu_2^{'})(u_1, v_1)(u_2, v_1) = 0.6$ $(\mu_2 * \mu_2^{'})(u_2, v_1)(u_3, v_1) = 0.2$ and $(\nu_2 * \nu_2^{'})(u_2, v_1)(u_3, v_1) = 0.6$ $(\mu_2 * \mu_2^{'})(u_1, v_2)(u_3, v_2) = 0.2$ and $(\nu_2 * \nu_2^{'})(u_1, v_2)(u_3, v_2) = 0.5$ $(\mu_2 * \mu_2^{'})(u_2, v_2)(u_3, v_2) = 0.2$ and $(\nu_2 * \nu_2^{'})(u_2, v_2)(u_3, v_2) = 0.5$.

Theorem 20. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are complete intuitionistic fuzzy graphs, then $G_1 \sqcap G_2$, $G_1 \odot G_2$ and $G_1 * G_2$ are complete.*

Theorem 21. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two intuitionistic fuzzy graphs such that $G_1 \sqcap G_2$ is complete, then either G_1 or G_2 must be complete.*

Theorem 22. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two intuitionistic fuzzy graphs such that ($G_1 \odot G_2$) or ($G_1 * G_2$) is complete, then either G_1 or G_2 must be complete.*

Theorem 23. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two intuitionistic fuzzy graphs. Then $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$ if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.*

Proof. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two intuitionistic fuzzy graphs. If $D(G_i) \leq D(G_1 \sqcap G_1)$ for $i = 1, 2$, then $D(G_1) = \left(\left(\frac{2\sum_{u_1,u_2\in V_1} \mu_2(u_1,u_2)}{\sum_{(u_1,u_2)\in E_1} \mu_1(u_1)\wedge\mu_1(u_2)}\right), \left(\frac{2\sum_{u_1,u_2\in V_1} \nu_2(u_1,u_2)}{\sum_{(u_1,u_2)\in E_1} \nu_1(u_1)\vee\nu_1(u_2)}\right)\right) \geq \left(\left(\frac{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2} (\mu_2\sqcap\mu_2^{'})((u_1,u_2)(v_1,v_2))}{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2} (\mu_1\sqcap\mu_1^{'})((u_1,u_2)(v_1,v_2))}\right), \left(\frac{2\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2} (\nu_2\sqcap\nu_2^{'})((u_1,u_2)(v_1,v_2))}{\sum_{u_1,u_2\in V_1,v_1,v_2\in V_2} (\nu_1\sqcap\nu_1^{'})((u_1,u_2)(v_1,v_2))}\right)\right)$. $D(G_1) = D(G_1 \sqcap G_2) D(G_1) \geq D(G_1 \sqcap G_2)$. Thus, $D(G_1) = D(G_1 \sqcap G_2)$. Similarly, $D(G_2) = D(G_1 \sqcap G_2)$. Hence $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.

□

Theorem 24. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two balanced intuitionistic fuzzy graphs. Then $G_1 \sqcap G_2$ is balanced if and only if $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.*

Proof. Let $G_1 \sqcap G_2$ is balanced intuitionistic fuzzy graphs. Then by definition, $D(G_i) \leq D(G_1 \sqcap G_2)$ for $i = 1, 2$. So by Theorem 4.6 $D(G_1) = D(G_2) = D(G_1 \sqcap G_2)$.
Conversely,

$$\text{suppose that } \mathrm{D(G_1) = D(G_2) = D(G_1 \sqcap G_2)}.$$

We have to prove that $G_1 \sqcap G_2$ is balanced. Let $(\frac{n_1}{r_1}, \frac{n_2}{r_2})$ be the density of an IFG G_1. Let $(\frac{a_1}{b_1}, \frac{a_2}{b_2})$ and $(\frac{a_3}{b_3}, \frac{a_4}{b_4})$ be the density of intuitionistic fuzzy subgraph H_1 and H_2 of G_1 and G_2 respectively. Since G_1 and G_2 are balanced and

$$D(G_1) = D(G_2) = (\frac{n_1}{r_1}, \frac{n_2}{r_2}),$$

$$\text{where } 0 \leq (\frac{\mathrm{n_1}}{\mathrm{r_1}}, \frac{\mathrm{n_2}}{\mathrm{r_2}}) \leq (2, 2),$$

$$D(H_1) = (\frac{a_1}{b_1}, \frac{a_2}{b_2}) \leq (\frac{n_1}{r_1}, \frac{n_2}{r_2}), D(H_2) = (\frac{a_3}{b_3}, \frac{a_4}{b_4}) \leq (\frac{n_1}{r_1}, \frac{n_2}{r_2}).$$

$$\text{Thus } a_1r_1 + a_3r_1 \leq b_1n_1 + b_3n_1 \text{and } a_2r_2 + a_4r_2 \leq b_2n_2 + b_4n_2.$$

$$\begin{aligned}\text{Hence } D(H_1) + D(H_2) &\leq (\frac{a_1 + a_3}{b_1 + b_3}, \frac{a_2 + a_4}{b_2 + b_4}) \\ &\leq (\frac{n_1}{r_1}, \frac{n_2}{r_2}) = D(G_1 \sqcap G_2).\end{aligned}$$

$\Rightarrow D(H) \leq D(G_1 \sqcap G_2)$. Therefore, $D(G_1 \sqcap G_2)$ is balanced. □

Theorem 25. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be isomorphic intuitionistic fuzzy graphs. If G_2 is balanced, then G_1 is balanced.*

Proof. Let $h : V_1 \to V_2$ be a bijection such that $\mu_1(u) = \mu_1'(h(u))$, $\nu_1(u) = \nu_1'(h(u))$ and $\mu_2(u, v) = \mu_2'(h(u), h(v))$, $\nu_2(u, v) = \nu_2'(h(u), h(v))$ for every $u, v \in V_1$

$$\text{then} \sum_{u \in V_1} \mu_1(u) = \sum_{u \in V_2} \mu_1'(u),$$

$$\sum_{u \in V_1} \nu_1(u) = \sum_{u \in V_2} \nu_1'(u),$$

$$\sum_{u,v \in V_1} \mu_2(u, v) = \sum_{u,v \in V_2} \mu_2'(u, v) \text{and} \sum_{u,v \in V_1} \nu_2(u, v) = \sum_{u,v \in V_2} \nu_2'(u, v).$$

If $H_1 = G(V(H_1), E(H_1))$ is an intuitionistic fuzzy subgraph of G_1 and $H_2 = G(V(H_2), E(H_2))$ is an intuitionistic fuzzy subgraph of G_2, where $\mu_1'(h(u)) = \mu_1(u)$, $\nu_1'(h(u)) = \nu_1(u)$ and $\mu_2'(h(u), h(v)) = \mu_2(u, v)$, $\nu_2'(h(u), h(v)) = \nu_2(u, v)$ for all $u, v \in V(H_1)$ Since G_2 is balanced, $D(H_2) \leq D(G_2)$ ie., $D_\mu(H_2) \leq D_\mu(G_2)$ and $D_\nu(H_2) \leq D_\nu(G_2)$

$$\frac{2 \sum_{u,v \in V(H_1)} \mu_2'(u, v)}{\sum_{(u,v) \in E(H_1)} \mu_1'(u) \wedge \mu_1'(v)} \leq \frac{2 \sum_{u,v \in V_2} \mu_2'(u, v)}{\sum_{(u,v) \in E_2} \mu_1'(u) \wedge \mu_1'(v)}$$

and

$$\frac{2 \sum_{u,v \in V(H_1)} \nu_2'(u, v)}{\sum_{(u,v) \in E(H_1)} \nu_1'(u) \wedge \nu_1'(v)} \leq \frac{2 \sum_{u,v \in V_2} \nu_2'(u, v)}{\sum_{(u,v) \in E_2} \nu_1'(u) \wedge \nu_1'(v)},$$

$$\frac{2 \sum_{u,v \in V(H_1)} \mu_2(u, v)}{\sum_{(u,v) \in E(H_1)} \mu_1'(u) \wedge \mu_1'(v)} \leq \frac{2 \sum_{u,v \in V_1} \mu_1(u, v)}{\sum_{(u,v) \in E_1} \mu_1'(u) \wedge \mu_1'(v)}$$

and

$$\frac{2 \sum_{u,v \in V(H_1)} \nu_2(u, v)}{\sum_{(u,v) \in E(H_1)} \nu_1'(u) \wedge \nu_1'(v)} \leq \frac{2 \sum_{u,v \in V_1} \nu_1(u, v)}{\sum_{(u,v) \in E_1} \nu_1'(u) \wedge \nu_1'(v)}$$

ie., $D_\mu(H_1) \leq D_\mu(G_1)$ and $D_\nu(H_1) \leq D_\nu(G_1)$. $\therefore D(H_1) \leq D(G_1)$. Hence G_1 is balanced. □

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