Asymptotic Classification of Distributed Linear Systems

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Abstract

This work is an extension to the asymptotic case of the concept of domination introduced for a class of controlled and observed systems. We give characterization results and the main properties of this notion for controlled systems, with respect to an output operator. We also examine the case of actuators and sensors. Various other situations are considered and applications are given. Then, we extend this study by comparing asymptotically observed systems with respect to a control operator. Finally, we study the relationship between the notion of asymptotic domination and the asymptotic compensation one, in the exact and weak cases.

Keywords: Distributed Systems, Asymptotic Domination, Actuators, Sensors, Compensation

1 Introduction

In this work, we present an extension of the notion of domination to the asymptotic case. The concept of domination was introduced in the finite time case
firstly for finite dimensional linear systems [1] and then for distributed parameter systems in [2, 3, 4, 5]. The initial problem consists to study the possibility of classification (comparison) of linear systems having the same dynamics. This is equivalent to study a possible comparison of input operators, and by duality output operators. A more general situation is examined in [5] where the considered systems do not necessarily have the same dynamics. The developed approach and the obtained results are more general to those established previously. In this paper, we give an extension of this general concept of domination to the asymptotic case. In such a case, the situation is different and as it is well known in systems theory, the approach and the developments requires more mathematical precautions. We consider without loss of generality, a class of linear distributed systems as follows

\[
\begin{aligned}
\dot{z}(t) &= Az(t) + Bu(t) \quad t > 0 \\
z(0) &= z_0
\end{aligned}
\]  

(1)

where \( A \) generates a strongly continuous semi-group (s.c.s.g.) \((S(t))_{t \geq 0}\) on the state \( Z \). \( B \in \mathcal{L}(U, Z) \), \( u \in L^2(0, T; +\infty) \); \( Z \) and \( U \) are respectively the state and the control spaces, assumed to be Hilbert spaces. The system (1) is augmented with the following output equation

\[
y(t) = Cz(t) \quad t > 0
\]  

(2)

with \( C \in \mathcal{L}(Z, Y) \), \( Y \) is the observation space, a Hilbert space. The operator \( A \) is the dynamics of the system, the operators \( B \) and \( C \) are respectively the input and output operators. The first problem consists to study a possible asymptotic comparison of controlled systems as system (1), with respect to an output operator \( C \). We give the main properties and characterization results. We also examine the case of sensors and actuators. Illustrative examples and applications are presented and various other situations are examined. Then by duality, we give an analogous analysis concerning the domination of observed systems, with respect to an input operator \( B \). Finally, we study the relationship between the notion of asymptotic domination and the asymptotic compensation problem [2].
2 Asymptotic domination for controlled systems

2.1 Problem statement and definitions

We consider the following linear distributed systems

\[(S_1) \begin{cases} \dot{z}_1(t) = A_1 z_1(t) + B_1 u_1(t) ; t > 0 \\ z_1(0) = z_{1,0} \in Z \end{cases}\]

\[(S_2) \begin{cases} \dot{z}_2(t) = A_2 z_2(t) + B_2 u_2(t) ; t > 0 \\ z_2(0) = z_{2,0} \in Z \end{cases}\]

where, for \(i = 1, 2\); \(A_i\) is a linear operator generating a s.c.s.g. \((S_i(t))_{t \geq 0}\) on the state space \(Z\). \(B_i \in L(U_i, Z)\), \(u_i \in L^2(0, +\infty; U_i)\); \(U_i\) is a control space. The systems \((S_1)\) and \((S_2)\) are respectively augmented with the output equations

\[(E_i) y_i(t) = C z_i(t) \text{ for } i = 1, 2 ; C \in \mathcal{L}(Z, Y)\]

The purpose is to study a possible comparison of systems \((S_1)\) and \((S_2)\) (or the input operators \(B_1\) and \(B_2\) if \(A_1 = A_2\)) with respect to the output operator \(C\). It is based on the dynamics \(A_1\) and \(A_2\), the control operators \(B_1\), \(B_2\) and the observation operator \(C\). We introduce hereafter the corresponding notion of asymptotic domination.

Definition 1 We say that

i. \((S_1)\) dominates asymptotically \((S_2)\) (or the pair \((A_1, B_1)\) dominates asymptotically \((A_2, B_2)\)) exactly, with respect to the operator \(C\), if

\[\text{Im}(P^{(2)}_\infty) \subset \text{Im}(P^{(1)}_\infty)\]

where, for \(i = 1, 2\)

\[P^{(i)}_\infty : L^2(0, +\infty; U_i) \rightarrow Z \quad u_i \rightarrow \int_0^{+\infty} C S_i(\tau) B_i u_i(\tau) d\tau\]

ii. \((S_1)\) dominates asymptotically \((S_2)\) (or the pair \((A_1, B_1)\) dominates asymptotically \((A_2, B_2)\)) weakly, with respect to the operator \(C\), if

\[\text{Im}(P^{(2)}_\infty) \subset \text{Im}(P^{(1)}_\infty)\]

In this situation, we note respectively

\[(A_2, B_2) \overset{\infty}{\sim} (A_1, B_1) \text{ and } (A_2, B_2) \overset{\infty}{\lesssim} (A_1, B_1)\]
Let us give following properties and remarks:

1. For \( i = 1, 2 \); if the s.c.s.g. \( (S_i(t))_{t \geq 0} \) is exponentially stable, i.e.
   \[
   \| S_i(t) \| \leq M_i e^{-\alpha_i t}; \quad t \geq 0
   \]
   with \( M_i \) and \( \alpha_i > 0 \), then the operator
   \[
   H^{(i)}_\infty : L^2(0, +\infty; U_i) \rightarrow Z \quad u_i \rightarrow \int_0^{+\infty} S_i(\tau) B_i u_i(\tau) d\tau
   \]
   is well defined, and hence \( P^{(i)}_\infty = C H^{(i)}_\infty \) is also well defined. But \( P^{(i)}_\infty \) may be well defined even if \( (S_i(t))_{t \geq 0} \) is not exponentially stable.

2. Obviously, the exact asymptotic domination with respect to an output operator \( C \), implies the weak asymptotic one, with respect to \( C \). The converse is not true.

3. If the system \( (S_1) \) is controllable asymptotically exactly (respectively weakly), or equivalently
   \[
   \text{Im}(H^{(1)}_\infty) = Z \quad \text{(respectively Im}(H^{(1)}_\infty) = Z)
   \]
   then asymptotically, \( (S_1) \) dominates exactly (respectively weakly) any system \( (S_2) \), with respect to any output operator \( C \).

4. In the case where \( A_1 = A_2 \), \( (S_1) \) dominates asymptotically \( (S_2) \) exactly (respectively weakly), we say simply that \( B_1 \) dominates asymptotically \( B_2 \) exactly (respectively weakly). Then, we note
   \[
   B_1 \underset{C}{\leq} B_2 \quad \text{(respectively } B_1 \underset{C}{\leq} B_2)\).
   In such a situation, one can consider a single system with two inputs as follows
   \[
   (S) \quad \begin{cases} 
   \dot{z}(t) = Az(t) + B_1 u_1(t) + B_2 u_2(t); \quad t > 0 \\
   z(0) = z_0 \in Z
   \end{cases}
   \]
   augmented with an output equation
   \[
   (E) \quad y(t) = C z(t); \quad t > 0
   \]
   In this case, the asymptotic domination of control operators \( B_1 \) and \( B_2 \), with respect to the observation operator \( C \), is similar. The definitions and the obtained results remain practically the same.
5. The exact or weak asymptotic domination of systems (or operators) is a transitive and reflexive relation, but it is not antisymmetric. Thus, for example in the case where \( A_1 = A_2 \), for any non-zero operator \( B_1 \neq 0 \) and \( \alpha \neq 0 \), we have\(^1\) \( \text{Im}(P_\infty^{(1)}(B_1)) = \text{Im}(P_\infty^{(1)}(\alpha B_1)) \), even if \( B_1 \neq \alpha B_1 \) for \( \alpha \neq 1 \).

6. Concerning the relationship with the notion of asymptotic remediability [2], we consider without loss of generality, a class of linear distributed systems described by a state equation as follows

\[
\begin{align*}
\dot{z}(t) &= Az(t) + Bu(t) + d(t) ; \quad t > 0 \\
z(0) &= z_0
\end{align*}
\]

(6)

where \( d \in L^2(0, +\infty; Z) \) is a known or unknown disturbance. The system (6) is augmented with the following output equation

\[
y(t) = Cz(t) ; \quad t > 0
\]

(7)

If the system (6), augmented with (7), is exactly (respectively weakly) asymptotically remediable, or equivalently \( \text{Im}(R_\infty) \subset \text{Im}(P_\infty) \) (respectively \( \text{Im}(R_\infty) \subset \text{Im}(P_\infty) \)), then \( B_1 \) dominates asymptotically any operator \( B_2 \) exactly (respectively weakly) with respect to \( C \), where

\[
P_\infty u = \int_0^{+\infty} CS(\tau)Bu(\tau)d\tau
\]

and

\[
R_\infty d = \int_0^{+\infty} CS(\tau)d(\tau)d\tau
\]

We give hereafter characterization results concerning the exact and weak domination.

2.2 Characterizations

The following result gives a characterization of the asymptotic exact domination with respect to the output operator \( C \).

**Proposition 2** The following properties are equivalent

\(^1\)\( P_\infty^{(1)}(B_1) \) denotes the operator \( P_\infty^{(1)} \) corresponding to \( B_1 \), i.e. defined by

\[
P_\infty^{(1)}(B_1)u_1 = \int_0^{+\infty} CS_1(\tau)B_1u_1(\tau)d\tau
\]
i. The system \((S_1)\) dominates asymptotically \((S_2)\) exactly with respect to the operator \(C\).

ii. For any \(u_2 \in L^2(0, +\infty; U_2)\), there exists \(u_1 \in L^2(0, +\infty; U_1)\) such that
\[
P^{(1)}_\infty u_1 + P^{(2)}_\infty u_2 = 0
\]  
(8)

iii. There exists \(\gamma > 0\) such that for any \(\theta \in Y'\), we have
\[
||B^*_2 S^*_2(\cdot) C^* \theta||_{L^2(0, +\infty; U_2)} \leq \gamma ||B^*_1 S^*_1(\cdot) C^* \theta||_{L^2(0, +\infty; U_1)}
\]  
(9)

Concerning the weak case, we have the following characterization result.

**Proposition 3** The system \((S_1)\) dominates asymptotically \((S_2)\) weakly, with respect to \(C\), if and only if
\[
\ker[B^*_1 S^*_1(\cdot) C^*] \subset \ker[B^*_2 S^*_2(\cdot) C^*]
\]  
(10)

It is well known in the finite time and in the asymptotic cases, the dynamics of the considered systems, as well as the choice of the input and output operators play an important role in control systems theory [8, 9, 10, 11, 12, 13]. Here also, the asymptotic domination for controlled systems, with respect to an output operator \(C\), depends on the dynamics \(A_i\) and particularly on the choice of the control operators \(B_i\). Even if \(B_1 = B_2 = B\), i.e. with the same actuators, the pair \((A_1, B)\) may dominates asymptotically \((A_2, B)\) (weakly or exactly). This is illustrated in the following example.

**Example 4** We consider the system described by the one dimension equation
\[
\begin{align*}
\frac{\partial z(x,t)}{\partial t} &= \alpha \frac{\partial^2 z(x,t)}{\partial x^2} + \beta z(x,t) + g(x)u(t) \text{ in } [0, 1] \times [0, +\infty[
\quad \frac{z(0,t)}{z(1,t)} = 0 \text{ in } [0, +\infty[
\quad z(x,0) = z_0(x) \text{ in } [0, 1]
\end{align*}
\]

The operator \(M(\alpha, \beta) = \alpha \frac{\partial^2}{\partial x^2} + \beta I\) generates the s.c.s.g. \((S_{(\alpha, \beta)}(t))_{t \geq 0}\) defined by
\[
S_{(\alpha, \beta)}(t)z = \sum_{n=1}^{+\infty} e^{(\beta - \alpha n^2 \pi^2) t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj}
\]
where \((\varphi_n)_n\), with \(\varphi_n(x) = \sqrt{2} \sin(n\pi x)\), is a complete system of eigenfunctions of \(M(\alpha, \beta)\) associated to the eigenvalues \(\lambda_n = \beta - \alpha n^2 \pi^2\) \((\alpha, \beta \in \mathbb{R})\). We
Hence, for example if \( \alpha \) and \( \beta \) are such that \( \beta - \alpha n^2 \pi^2 < 0 \), \( \forall n \geq 1 \); i.e. \( \beta < \alpha \pi^2 \). Hence, for example if \( \alpha = 1 \), \( \beta - \alpha n^2 \pi^2 \leq \beta - \pi^2 \). It is sufficient to consider \( \beta < \pi^2 \). For \( z^* \in \mathbb{Z}' = L^2(0,1) \), we have

\[
\|B^*S^{*}_{(\alpha,\beta)}(t)z^*\|^2_{L^2(0,+,\mathbb{R})} = \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{2(\beta-\alpha n^2 \pi^2)t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle^2 (g, \varphi_{nj})^2 dt (11)
\]

Hence, if \( g = \varphi_{n_0} \) \((n_0 \geq 1)\), equation (11) becomes

\[
\|B^*S^{*}_{(\alpha,\beta)}(t)z^*\|^2_{L^2(0,+,\mathbb{R})} = \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{2(\beta-\alpha n^2 \pi^2)t} \langle z, \varphi_{n_0} \rangle^2 dt
\]

Let \( A_1 = M_{1,0} = \frac{\partial^2}{\partial x^2} \) and \( A_2 = M_{1,\beta} = \frac{\partial^2}{\partial x^2} + \beta I \); \( \beta \neq 0 \). The corresponding semi-groups, noted \((S_1(t))_{t \geq 0}\) and \((S_2(t))_{t \geq 0}\), are respectively defined by

\[
S_1(t)z = \sum_{n=1}^{+\infty} e^{-n^2 \pi^2 t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj}
\]

and

\[
S_2(t)z = \sum_{n=1}^{+\infty} e^{(\beta-n^2 \pi^2) t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj}
\]

Then for \( B_1 = B_2 = B \), with \( Bu = \varphi_{n_0}u \)

1) If \( 0 < \beta < \alpha \pi^2 \), then for any \( z^* \in \mathbb{Z}' \), we have

\[
\|B^*S^*_1(t)z^*\|^2_{L^2(0,+,\mathbb{R})} = \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{-2n^2 \pi^2 t} \langle z^*, \varphi_n \rangle^2 dt
\]

\[
\leq \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{2(\beta-n^2 \pi^2) t} \langle z^*, \varphi_n \rangle^2 dt
\]

\[
= \|B^*S^*_2(t)z^*\|^2_{L^2(0,+,\mathbb{R})}
\]

consequently, the pair \((A_2,B)\) dominates asymptotically the pair \((A_1,B)\) exactly, and hence weakly.

2) If \( \beta < 0 \), then for any \( z^* \in \mathbb{Z}' \),

\[
\|B^*S^*_2(t)z^*\|^2_{L^2(0,+,\mathbb{R})} = \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{2(\beta-n^2 \pi^2) t} \langle z^*, \varphi_n \rangle^2 dt
\]

\[
\leq \sum_{n=1}^{+\infty} \int_0^{+\infty} e^{-2n^2 \pi^2 t} \langle z^*, \varphi_n \rangle^2 dt
\]

\[
= \|B^*S^*_1(t)z^*\|^2_{L^2(0,+,\mathbb{R})}
\]

Hence, the pair \((A_1,B)\) dominates the pair \((A_2,B)\) exactly (and weakly).
Remark 5  In the case of a Neumann boundary condition, the eigenfunctions and the eigenvalues of \( M(\alpha, \beta) \) are as follows

\[
\begin{align*}
\varphi_0 & \equiv 1 \quad ; \quad \sqrt{2}\varphi_n(x) = \cos(n\pi x) \quad \text{for} \quad n \geq 1 \\
\lambda_0 & = \beta \quad ; \quad \lambda_n = \beta - \alpha n^2 \pi^2 \quad \text{for} \quad n \geq 1 
\end{align*}
\]

The corresponding semi-group is defined by

\[
S_{(\alpha, \beta)}(t)z = \sum_{n=0}^{+\infty} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj}
\]

\((S_{(\alpha, \beta)})_{t \geq 0}\) is not necessarily exponentially stable. Consequently, the operators \( H_{(i)}^\infty \) are not necessarily well defined.

1) If \( \beta < 0 \), \((S_{(\alpha, \beta)})_{t \geq 0}\) is exponentially stable, and hence the operators \( H_{(i)}^\infty \) are well defined. The results are practically the same with respect to the case of the Dirichlet case.

2) In the general case, even if \((S_{(\alpha, \beta)})_{t \geq 0}\) is not necessarily exponentially stable, the operators \( P_{(i)}^\infty \) depending on the choice of \( C \), may be well defined. Indeed if \( C \) is as follows \( C\varphi_0 = 0 \), or equivalently \( \varphi_0 \in \ker C \), then \( P_{(i)}^\infty \) are well defined. The results are similar to those obtained in the case of a Dirichlet boundary condition.

In the next section, we examine the case of a finite number of actuators, and then the case where the observation is given by sensors.

2.3 Case of actuators and sensors

This section is focused on the notions of actuators and sensors [7, 9, 12], i.e. on input and output operators. In what follows, we assume that \( Z = L^2(\Omega) \) and, without loss of generality, we consider the analytic case where \( A_1 \) and \( A_2 \) generate respectively the s.c.s.g. \((S_1(t))_{t \geq 0}\) and \((S_2(t))_{t \geq 0}\) defined by\(^2\)

\[
S_1(t)z = \sum_{n=1}^{+\infty} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z, \varphi_{nj} \rangle \varphi_{nj}
\]

and

\[
S_2(t)z = \sum_{n=1}^{+\infty} e^{\gamma_n t} \sum_{j=1}^{s_n} \langle z, \psi_{nj} \rangle \psi_{nj}
\]

where \( \{\varphi_{nj}, j = 1, \ldots, r_n; n \geq 1\} \) is a complete orthonormal basis of eigenfunctions of \( A_1 \), associated to the real eigenvalues \( (\lambda_n)_{n \geq 1} \) such that \( \lambda_1 > \lambda_2 > \lambda_3 > \ldots; r_n \) is the multiplicity of \( \lambda_n \). \( \{\psi_{nj}, j = 1, \ldots, s_n; n \geq 1\} \) is a complete orthonormal basis of eigenfunctions of \( A_2 \), associated to the real eigenvalues \( (\gamma_n)_{n \geq 1} \) such that \( \gamma_1 > \gamma_2 > \gamma_3 > \ldots; s_n \) is the multiplicity of \( \gamma_n \).

\(^2\)Such a situation is considered in example 4
2.3.1 Case of actuators

In the case where \((S_1)\) is excited by \(p\) zone actuators \((\Omega_i, g_i)_{1 \leq i \leq p}\), we have \(U_1 = \mathbb{R}^p\) and \(B_1 u(t) = \sum_{i=1}^p g_i u_i(t)\), where \(u = (u_1, \ldots, u_p)^{tr} \in L^2(0, T; \mathbb{R}^p)\) and \(g_i \in L^2(\Omega) ; \Omega_i = \text{supp}(g_i) \subset \Omega\). We have \(B_1^* z = (\langle g_1, z \rangle, \ldots, \langle g_p, z \rangle)^{tr}\).

By the same, if \((S_2)\) is excited by \(q\) zone actuators \((D_i, h_i)_{1 \leq i \leq q}\), we have \(U_2 = \mathbb{R}^q\) and \(B_2 v(t) = \sum_{i=1}^q h_i v_i(t)\) with \(v = (v_1, \ldots, v_q)^{tr} \in L^2(0, T; \mathbb{R}^q)\), \(h_i \in L^2(\Omega), D_i = \text{supp}(h_i) \subset \Omega\) and \(B_2^* z = (\langle h_1, z \rangle, \ldots, \langle h_q, z \rangle)^{tr}\).

The following result deriving from proposition 2, gives characterizations of exact and weak domination in the case of actuators.

**Proposition 6** The system \((S_1)\) dominates asymptotically \((S_2)\) exactly, if and only if, there exists \(\gamma > 0\) such that for any \(\theta \in Y\), we have

\[
\begin{align*}
\| \sum_{n=1}^{+\infty} \sum_{j=1}^{s_n} \langle C^* \theta, \psi_{nj} \rangle_{L^2(\Omega)} \langle h_i, \psi_{nj} \rangle_{L^2(\Omega)} & \|_{L^2(0, T; \mathbb{R}^q)} \\
& \leq \gamma \| \sum_{n=1}^{+\infty} \sum_{j=1}^{r_n} \langle C^* \theta, \varphi_{nj} \rangle_{L^2(\Omega)} \langle g_i, \varphi_{nj} \rangle_{L^2(\Omega)} & \|_{L^2(0, T; \mathbb{R}^p)}
\end{align*}
\]

\begin{equation}
(14)
\end{equation}

for \(\Omega\) and \(\supp(\Omega) \subset \Omega\) and \(B_2^* z = (\langle h_1, z \rangle, \ldots, \langle h_q, z \rangle)^{tr}\).

**ii)** weakly, if and only if, for any \(n \geq 1\), we have

\[
\text{Ker}[M_n P_n] \subset \text{Ker}[G_n Q_n]
\]

where, for \(n \geq 1\), \(M_n = (\langle g_i, \psi_{nj} \rangle)_{1 \leq i \leq p; 1 \leq j \leq r_n}\) and \(G_n = (\langle h_i, \psi_{nj} \rangle)_{1 \leq i \leq q; 1 \leq j \leq s_n}\)

are the controllability matrices corresponding to the actuators \((D_i, g_i)_{1 \leq i \leq p}\) and \((\Omega_j, h_j)_{1 \leq j \leq q}\) respectively, \(P_n \) and \(Q_n \) are linear maps defined on \(Y\) by

\[
P_n(\theta) = \begin{pmatrix} \langle C^* \theta, \varphi_{n,1} \rangle_{L^2(\Omega)} \\ \vdots \\ \langle C^* \theta, \varphi_{n,r_n} \rangle_{L^2(\Omega)} \end{pmatrix} \in \mathbb{R}^{r_n}, Q_n(\theta) = \begin{pmatrix} \langle C^* \theta, \psi_{n,1} \rangle_{L^2(\Omega)} \\ \vdots \\ \langle C^* \theta, \psi_{n,s_n} \rangle_{L^2(\Omega)} \end{pmatrix} \in \mathbb{R}^{s_n}
\]

We examine hereafter the case of sensors.

2.3.2 Case of sensors

Now, if the output is given by \(m\) sensors \((E_i, f_i)_{1 \leq i \leq m}\), we have

\[
C z = \begin{pmatrix} \langle z, f_1 \rangle \\ \vdots \\ \langle z, f_m \rangle \end{pmatrix} \in \mathbb{R}^m \quad \text{and} \quad C^* \theta = \sum_{i=1}^m \theta_i f_i \quad \text{for} \quad \theta \in \mathbb{R}^m
\]

In this case, the weak and exact asymptotic domination are equivalent. We have the following proposition.
Proposition 7 \((S_1)\) dominates asymptotically \((S_2)\), with respect to the sensors \((E_i, f_i)_{1\leq i \leq m}\), if and only if
\[
\cap_{n \geq 1} \ker(M_n G_n^{tr}) \subset \cap_{n \geq 1} \ker(Q_n R_n^{tr})
\]
where \(G_n\) and \(R_n\) are the corresponding observability matrices defined by
\[
G_n = ((f_i, \varphi_n))_{1 \leq i \leq m; 1 \leq j \leq r_n} \quad \text{and} \quad R_n = ((f_i, \psi_n))_{1 \leq i \leq m; 1 \leq j \leq s_n}
\]

Proof. \((S_1)\) dominates asymptotically \((S_2)\), with respect to the sensors \((E_i, f_i)_{1\leq i \leq m}\), if and only if, for any \(\theta = (\theta_k)_{1 \leq k \leq m} \in \mathbb{R}^m\),
\[
\forall n \in \mathbb{N}^*, (\sum_{i=1}^m \theta_k \langle f_k, \varphi_n \rangle)_{1 \leq j \leq r_n} \in \ker(M_n)
\]
implies that
\[
\forall n \in \mathbb{N}^*, (\sum_{i=1}^m \theta_k \langle f_k, \psi_n \rangle)_{1 \leq j \leq s_n} \in \ker(Q_n)
\]
or equivalently, for any \(\theta \in \mathbb{R}^m\),
\[
[\forall n \geq 1, \theta \in \ker(M_n G_n^{tr})] \implies [\forall n \geq 1, \theta \in \ker(Q_n G_n^{tr})]
\]
we then have the result.

Let us give the following remarks.

1. If \(A_1 = A_2\), we have \(G_n = R_n\), for \(n \geq 1\).

2. One actuator may dominates \(p\) actuators \((p > 1)\), with respect to an output operator \(C\) (sensors).

3. In the case of one actuator and one sensor, i.e. for \(p = q = 1\) and \(m = 1\), we have
\[
M_n = ((g, \varphi_n), \ldots, (g, \varphi_{nr_n})) \quad \text{and} \quad Q_n = ((h, \psi_n), \ldots, (h, \psi_{ns_n}))
\]
and
\[
G_n^{tr} = \begin{pmatrix}
\langle f, \varphi_n \rangle \\
. \\
. \\
\langle f, \varphi_{nr_n} \rangle
\end{pmatrix}, \quad R_n^{tr} = \begin{pmatrix}
\langle f, \psi_n \rangle \\
. \\
. \\
\langle f, \psi_{ns_n} \rangle
\end{pmatrix}
\]
Then
\[
M_n G_n^{tr} = (\sum_{j=1}^{r_n} g, \varphi_n) \langle f, \varphi_n \rangle) \quad \text{and} \quad Q_n R_n^{tr} = (\sum_{j=1}^{s_n} h, \psi_n) \langle f, \psi_n \rangle)
\]

4. In the case of a finite number of sensors, the exact and weak domination are equivalent.
3 Applications to diffusion systems

To illustrate previous results and other specific situations, we consider without loss of generality, a class of diffusion systems described by the following parabolic equation.

\[
\begin{cases}
\frac{\partial z(x,t)}{\partial t} = \Delta z(x,t) + g(x)u(t) & \Omega \times ]0, +\infty[ \\
z(x,0) = 0 & \Omega \\
z(\xi,t) = 0 & \partial \Omega \times ]0, +\infty[ 
\end{cases}
\]

where $\Omega$ is a bounded subset of $\mathbb{R}^n$ with a sufficiently regular boundary $\partial \Omega = \Gamma$; $Z = L^2(\Omega)$ and $Az = \Delta z$ for $z \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. (S) is augmented with the output equation

\[
(E) \ y(t) = Cz(t), \ t > 0
\]

We examine respectively, hereafter the case of one and two space dimension.

3.1 One dimension case

In this section, we consider the systems (S_1) and (S_2) described by the following one dimension equations, with $\Omega = ]0, a[$ and $A_1 = A_2 = \Delta$.

\[
\begin{cases}
\frac{\partial z_1(x,t)}{\partial t} = \frac{\partial^2 z_1(x,t)}{\partial x^2} + g(x)u_1(t) \text{ in } ]0, a[ \times ]0, +\infty[ \\
z_1(0,t) = z_1(a,t) = 0 \text{ in } ]0, +\infty[ \\
z_1(x,0) = 0 \text{ in } ]0, a[ 
\end{cases}
\]

\[
\begin{cases}
\frac{\partial z_2(x,t)}{\partial t} = \frac{\partial^2 z_2(x,t)}{\partial x^2} + h(x)u_2(t) \text{ in } ]0, a[ \times ]0, +\infty[ \\
z_2(0,t) = z_2(a,t) = 0 \text{ in } ]0, +\infty[ \\
z_2(x,0) = 0 \text{ in } ]0, a[ 
\end{cases}
\]

$A = \Delta$ admits a complete orthonormal system of eigenfunctions $(\varphi_n)_{n \in \mathbb{N}^*}$ associated to the eigenvalues $\lambda_n = -\frac{n^2 \pi^2}{a^2}$ with $\varphi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$. Each system (S_i) is augmented with the output equation corresponding to a sensor $(D, f)$,

\[
(E_i) \ y_i(t) = \langle f, z_i(t) \rangle_{L^2(D)} ; \ t > 0
\]
According to proposition 7, \((\Omega, g)\) dominates asymptotically \((\Omega, h)\) with respect to the sensor \((D, f)\), if and only if,

\[
\forall n \in \mathbb{N}^*, \langle g, \varphi_n \rangle \langle f, \varphi_n \rangle = 0 \implies \forall n \in \mathbb{N}^*, \langle h, \varphi_n \rangle \langle f, \varphi_n \rangle = 0
\] (21)

Let \(m, n \in \mathbb{N}^*\) such that \(m \neq n\). We suppose that \((S_1)\) and \((S_2)\) are respectively excited by the actuators \((\Omega, \varphi_n)\) and \((\Omega, \varphi_m)\), i.e. \(g = \varphi_n\) and \(h = \varphi_m\).

Then - \((\Omega, g)\) dominates asymptotically \((\Omega, h)\) with respect to the sensor \((\Omega, \varphi_n)\) and - \((\Omega, h)\) dominates asymptotically \((\Omega, g)\) with respect to the sensor \((\Omega, \varphi_m)\).

Let us also note that in the one dimension case, any operators \(B_1\) and \(B_2\) are asymptotically comparable. This is not always possible in the two-dimension case which will be examined in the next section.

### 3.2. Two dimension case

Now, we consider the case where \(\Omega = ]0, 1[ \times ]0, 1[\) and the systems described by the following equations

\[
(S_1) \begin{cases}
\frac{\partial z_1(x, y, t)}{\partial t} = \Delta z_1(x, y, t) + g_1(x, y)u_1(t) + g_2(x, y)u_2(t) \text{ in } \Omega \times ]0, +\infty[ \\
z_1(x, y, t) = 0 \text{ in } \Gamma \times ]0, +\infty[ \\
z_1(x, y, 0) = 0 \text{ in } \Omega 
\end{cases}
\]

\[
(S_2) \begin{cases}
\frac{\partial z_2(x, y, t)}{\partial t} = \Delta z_2(x, y, t) + h_1(x, y)v_1(t) + h_2(x, y)v_2(t) \text{ in } \Omega \times ]0, +\infty[ \\
z_2(x, y, t) = 0 \text{ in } \Gamma \times ]0, +\infty[ \\
z_2(x, y, 0) = 0 \text{ in } \Omega 
\end{cases}
\]

Here, we have \(Z = L^2(\Omega)\) and \(Az = \Delta z = \frac{\partial z}{\partial x^2} + \frac{\partial z}{\partial y^2}\), for \(z \in D(A) = H^2(\Omega) \cap H_0^1(\Omega)\). \(A\) admits a complete orthonormal system of eigenfunctions \((\varphi_{m,n})_{m,n \in \mathbb{N}^*}\) associated to the eigenvalues \((\lambda_{m,n})_{m,n \in \mathbb{N}^*}\) defined by

\[
\begin{align*}
\lambda_{m,n} &= -(m^2 + n^2)\pi^2 \\
\varphi_{m,n}(x, y) &= 2\sin(m\pi x)\sin(n\pi y)
\end{align*}
\] (22)
(S_1) and (S_2) are respectively augmented with the output equations

\[(E_1) \ y_1(t) = (\langle f_1, z_1(t) \rangle_{L^2(D_1)}, \langle f_2, z_2(t) \rangle_{L^2(D_2)}), \ t > 0 \]

and

\[(E_2) \ y_2(t) = (\langle f_1, z_2(t) \rangle_{L^2(D_1)}, \langle f_2, z_2(t) \rangle_{L^2(D_2)}), \ t > 0 \]

Let us first note that: \(200 = 14^2 + 2^2 = 10^2 + 10^2\), then \(-200\) is a double eigenvalue, corresponding to the eigenfunctions \(\varphi_{10,10}, \varphi_{10,10}\) and \(\varphi_{2,14}\).

By the same, \(250 = 15^2 + 5^2 = 13^2 + 9^2\), then \(-250\) is also a double eigenvalue, corresponding to the eigenfunctions \(\varphi_{5,15}, \varphi_{9,13}\).

The examples given hereafter show the following situations:
- An actuator may dominates another one with respect to a sensor.
- None of the systems does not dominates the other.

**Example 8** In the case where \(g_1 = \varphi_{10,10}, g_2 = 0, \ h_1 = 0, h_2 = \varphi_{5,15}, f_1 = \varphi_{10,10} \) and \(f_2 = \varphi_{2,14}\), we have

\[\bigcap_{n \geq 1} \ker(M_n G_n^{tr}) = \mathbb{R}(0,1) \quad \text{and} \quad \bigcap_{n \geq 1} \ker(Q_n G_n^{tr}) = \{0\} \quad (23)\]

where \(\mathbb{R}(0,1)\) denotes the y-axis. Therefore (S_2) dominates (S_1) with respect to the corresponding output operator \(C\). On the other hand, for \(g_1 = \varphi_{10,10}, g_2 = 0, \ h_1 = 0, h_2 = \varphi_{5,15} \) and \(f_1 = \varphi_{5,15}, f_2 = \varphi_{9,13}\), we have

\[\bigcap_{n \geq 1} \ker(M_n G_n^{tr}) = \{0\} \quad \text{and} \quad \bigcap_{n \geq 1} \ker(Q_n G_n^{tr}) = \mathbb{R}(1,0) \quad (24)\]

where \(\mathbb{R}(1,0)\) denotes the x-axis. Then (S_1) dominates (S_2) with respect to the corresponding output operator \(C\).

**Example 9** Now, for \(g_1 = \varphi_{10,10}, g_2 = 0, \ h_1 = 0, h_2 = \varphi_{2,14}, f_1 = \varphi_{10,10} \) and \(f_2 = \varphi_{2,14}\), we have

\[\bigcap_{n \geq 1} \ker(M_n G_n^{tr}) = \mathbb{R}(0,1) \quad \text{and} \quad \bigcap_{n \geq 1} \ker(Q_n G_n^{tr}) = \mathbb{R}(1,0) \quad (25)\]

Then none of the operators \(B_1\) and \(B_2\) does not dominates the other.

### 4 Asymptotic domination of output operators

In this section, we introduce and we study the notion of asymptotic domination for a general controlled system and then relationship between the notions of domination and compensation.
4.1 Domination of output operators

We consider the following linear distributed system

\[
(S) \begin{cases}
\dot{z}(t) = Az(t) + Bu(t) ; t > 0 \\
 z(0) = z_0 = 0
\end{cases}
\]  

where \( A \) generates a s.c.s.g. \((S(t))_{t \geq 0}\) on the state space \( Z \); \( B \in \mathcal{L}(U, Z) \) and \( u \in L^2(0, +\infty; U) \); \( U \) is the control space and the system \((S)\) is augmented with the output equations

\[ y_i(t) = C_i z_i(t) , \quad t > 0; \quad i = 1, 2. \]

where \( C_i \in L(Z, Y); \quad i = 1, 2; \quad Y \) is an Hilbert space. We introduce hereafter the appropriate notion of domination for the considered case.

Definition 10 We say that

i. \( C_1 \) dominates asymptotically \( C_2 \) exactly with respect to the system \((S)\) (or the pair \((A, B)\)) on \([0, T]\), if \( \text{Im}(\Theta_2) \subset \text{Im}(\Theta_1) \).

ii. \( C_1 \) dominates asymptotically \( C_2 \) weakly with respect to the system \((S)\) (or the pair \((A, B)\)) on \([0, T]\), if \( \text{Im}(\Theta_2(\infty)) \subset \text{Im}(\Theta_1(\infty)) \).

where, for \( i = 1, 2 \)

\[
\Theta^{(i)}_\infty : L^2(0, +\infty; U_i) \rightarrow Z \\
u_i \rightarrow \int_0^{+\infty} C_i S_i(\tau) B u_i(\tau) d\tau
\]

Here also, we can deduce similar characterization results in the weak and exact cases. On the other hand, one can consider a natural question on a possible transitivity of such a domination. As it will be seen, this may be possible under convenient hypothesis. In order to examine this question, we consider without loss of generality, the linear distributed systems with the same dynamics \( A \) \((A_1 = A_2 = A)\)

\[
(S_i) \begin{cases}
\dot{z}_i(t) = A z_i(t) + B_i u_i(t) ; t > 0 \\
z_i(0) = z_{i,0}
\end{cases}
\]

where \( A \) generates a s.c.s.g. \( (S(t))_{t \geq 0}\) on the state space \( Z \); and for \( i = 1, 2; \) \( B_i \in \mathcal{L}(U_i, Z), \) \( u_i \in L^2(0, +\infty; U_i); \) \( U_i \) is a control space. The systems \((S_1)\) and \((S_2)\) are augmented with the output equations

\[
(E_{1,i}) : \quad y_{i,1}(t) = C_1 z_i(t) ; \quad i = 1, 2 ; \quad t > 0 \\
(E_{2,j}) : \quad y_{j,2}(t) = C_2 z_j(t) ; \quad j = 1, 2 ; \quad t > 0
\]
where \( C_i \in \mathcal{L}(Z,Y) \), for \( i = 1, 2 \); \( Y \) is a Hilbert space. We have the following result deriving from the definitions.

**Proposition 11** If the following conditions are satisfied

i. \( B_1 \) dominates asymptotically \( B_2 \) exactly (respectively weakly) with respect to operator \( C_1 \),

ii. \( C_1 \) dominates asymptotically \( C_2 \) exactly (respectively weakly) with respect to operator \( B_2 \),

iii. \( C_2 \) dominates asymptotically \( C_1 \) exactly (respectively weakly) with respect to operator \( B_1 \),

then \( B_1 \) dominates asymptotically \( B_2 \) exactly (respectively weakly) with respect to operator \( C_2 \).

We examine hereafter, the relationship between the notions of domination and compensation.

### 4.2 Asymptotic domination and asymptotic compensation

In this section, we study the relationship between the notions of domination and compensation \([1, 2, 7]\). We consider without loss of generality, the following systems

\[
\begin{cases}
\dot{z}_i(t) = A z_i(t) + d_i(t) + B_i u_i(t), & t > 0 \\
z_i(0) = z_{i,0}
\end{cases}
\]  

(27)

where \( A \) generates a s.c.s.g. \( (S(t))_{t \geq 0} \) on the state space \( Z \); for \( i = 1, 2 \), \( B_i \in \mathcal{L}(U_i, Z) \), \( u_i \in L^2(0, +\infty; U_i) \), \( d_i \in L^2(0, +\infty; Z) \); \( U_i \) is a control space. Each system \( (S_i) \) is augmented with the output equation

\[
(E_i) \quad y_i(t) = C_i z_i(t)
\]

First let us recall the notion of asymptotic compensation.

**Definition 12** The system \( (S_i) \) augmented with output equation \( (E_i) \) (or \( (S_i)+ (E_i) \)) is

i. exactly remediable asymptotically if for any \( d_i \in L^2(0, +\infty; Z) \), there exists \( u_i \in L^2(0, +\infty; U_i) \) such that \( \Xi^{(i)} u_i + R^{(i)} d_i = 0 \), or equivalently

\[
\text{Im} (R^{(i)}_{\infty}) \subset \text{Im} (\Xi^{(i)}_{\infty})
\]

(28)
ii. weakly remediable asymptotically if for any \( d_i \in L^2(0, +\infty; Z) \) and any \( \epsilon > 0 \), there exists \( u_i \in L^2(0, +\infty; U_i) \) such that \( ||\Xi_{\infty}^{(i)} u_i + R_{\infty}^{(i)} d_i|| < \epsilon \), or equivalently

\[
Im(R_{\infty}^{(i)}) \subset Im(\Xi_{\infty})
\]  

(29)

where, for \( i = 1, 2 \) \( \Xi_{\infty}^{(i)} \) and \( R_{\infty}^{(i)} \) are defined by

\[
\Xi_{\infty}^{(i)} : L^2(0, +\infty; U_i) \rightarrow Z
\]

\[
u_i \rightarrow \int_0^{+\infty} C_i S_i(\tau) B_i u_i(\tau) d\tau
\]

\[
R_{\infty}^{(i)} : L^2(0, +\infty; Z) \rightarrow Z
\]

\[
d_i \rightarrow \int_0^{+\infty} C_i S_i(s) d_i(s) ds
\]

Here, the question is not to examine if a system is (or not) asymptotically remediable (for this one can see [2, 7]), but to study the nature of the relation between the notions of domination and compensation, respectively in the exact and weak cases. We have the following result.

**Proposition 13** If the following conditions are verified

i. \( (S_1) + (E_1) \) is exactly (respectively weakly) remediable asymptotically.

ii. \( C_2 \) dominates asymptotically \( C_1 \) exactly (respectively weakly) with respect to the operator \( B_1 \).

iii. \( Im(R_{\infty}^{2}) \subset Im(R_{\infty}^{1}) \) (respectively \( \overline{Im(C_2 R_{\infty}^{2})} \subset \overline{Im(C_1 R_{\infty}^{1})} \)).

then \( (S_1) + (S_2) \) is exactly (respectively weakly) remediable asymptotically.

We have the similar result concerning the output domination and the remediability notion.

**Proposition 14** If the following conditions are satisfied

i. \( (S_1) + (E_1) \) is exactly (respectively weakly) remediable asymptotically.

ii. \( B_2 \) dominates asymptotically \( B_1 \) exactly (respectively weakly) with respect to the operator \( C_1 \).

then \( (S_2) + (E_1) \) is exactly (respectively weakly) remediable asymptotically.

Let us note that this section is a generalization of the previous one where \( d(t) \) has the form \( B_2 u_2(t) \). The results can be applied easily to a diffusion systems and to other systems and situations.
References


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