Generalized Normal Type-2

Triangular Fuzzy Number

Abd. Fatah Wahab

Department of Mathematics, Faculty of Science and Technology, Universiti
Malaysia Terengganu, Malaysia.
fatah@umt.edu.my

Rozaimi Zakaria

Department of Mathematics, Faculty of Science and Technology, Universiti
Malaysia Terengganu, Malaysia.
rozaimi_z@yahoo.com

Copyright © 2013 Abd. Fatah Wahab and Rozaimi Zakaria. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract

Here, we present four theorems involving normal type-2 triangular fuzzy number (NT2TFN).

Keywords: Type-2 triangular fuzzy number, Alpha-cut, Type-reduction, Defuzzification

1 Introduction

Type-2 fuzzy number (T2FN) concept was introduced as the extension of type-1 fuzzy number (T1FN) [2,5] concept in dealing the problems of defining the complex uncertainty data in real data forms. This T2FN defined by the type-2 fuzzy set (T2FS) theory which was introduced by Zadeh [3] in order to solve the complex uncertainty problems of real data set.

Therefore, the definitions of T2FN, normal T2FN, alpha-cut operation, type-reduction and defuzzification process of normal T2FN are given as follows.

Definition 1. A T2FN is broadly defined as a T2FS that has a numerical domain. An interval T2FS is defined using the following four constraints, where
\(A_α = \left\{ [a^α, b^α], [c^α, d^α] \right\}, \quad ∀α \in [0,1], \quad ∀a^α, b^α, c^α, d^α \in \mathbb{R}\) \quad (Fig. 1) [1]:

1. \(a^α \leq b^α \leq c^α \leq d^α\)
2. \([a^α, d^α]\) and \([b^α, c^α]\) generate a function that is convex and \([a^α, d^α]\) generate a function is normal.
3. ∀\(α_1, α_2 \in [0,1]: (α_2 > α_1) \Rightarrow \left[ [a^{α_1}, c^{α_1}] \supset [a^{α_2}, c^{α_2}] \right], \right\}
   \left[ b^{α_1}, d^{α_1} \right] \supset \left[ b^{α_2}, d^{α_2} \right]\right)\), for \(c^{α_2} \geq b^{α_2}\).
4. If the maximum of the membership function generated by \([b^α, c^α]\) is the level \(α_m\), that is, \([b^{α_m}, c^{α_m}]\), then \([b^{α_m}, c^{α_m}] \subseteq [a^{α_m}, d^{α_m}]\).

**Figure 1. Definition of an interval T2FN.**

**Definition 2.** Given that T2FN, \(A\) which the height of lower membership function (LMF) and upper membership function (UMF) are \(h(A_{\alpha})\) and \(h(A_{\alpha})\) respectively, then T2FN called **Normal** of T2FN (NT2FN) if \(h(A_{\alpha}) < h(A_{\alpha}) = 1\) [4]. This Def. 2 can be illustrated through Fig. 2.

**Figure 2. The NT2FN.**
Definition 3. Based on Def. 2, let \( \overset{\text{\(\sim\)}}{A} \) be the set of T2FN in triangular form with \( \overset{\text{\(\sim\)}}{A}_i \in \overset{\text{\(\sim\)}}{A} \) where \( i = 0,1,...,n-1 \). Then \( \overset{\text{\(\sim\)}}{A}_{\alpha} \) is the alpha-cut operation of NT2TFN which is given as equation as follows [4].

\[
\overset{\text{\(\sim\)}}{A}_{\alpha} = \left\{ \overset{\text{\(\sim\)}}{A}_{\alpha}^{-}, A, \overset{\text{\(\sim\)}}{A}_{\alpha}^{+} \right\}
\]

\[
= \left\{ \left( \left[ \left( \left[ A; A_{\alpha}^{-}; A_{\alpha}^{+} \right) + \left( A_{\alpha}^{-}; A_{\alpha}^{+}; A_{\alpha}^{+} \right) \right] \right) \right) \right\}
\]

\[
\overset{\text{\(\sim\)}}{A}_{\alpha}^{-} = \left( A_{\alpha}^{-}; A_{\alpha}^{+}; A_{\alpha}^{+} \right), A_{\alpha}^{+}
\]

\[
\text{where} \ \alpha_{\text{LMF}} \ \text{and} \ \alpha_{\text{CLMF}} \ \text{are alpha values of lower membership function and crisp lower membership function of NT2TFN respectively. This definition can be illustrated through Fig. 3.}
\]

Figure 3. The alpha-cut operation towards NT2TFN.

However, when \( \alpha_{\text{LMF}} < \alpha_i < \alpha_{\text{UMF}} \) for \( \alpha \)-cut operation of NT2TFN, then the Eq. 1 become

\[
\overset{\text{\(\sim\)}}{A}_{\alpha} = \left\{ \overset{\text{\(\sim\)}}{A}_{\alpha}^{-}, A, \overset{\text{\(\sim\)}}{A}_{\alpha}^{+} \right\}
\]

\[
= \left\{ \left( \left[ \left( \left[ A; A_{\alpha}^{-}; A_{\alpha}^{+} \right) + \left( A_{\alpha}^{-}; A_{\alpha}^{+}; A_{\alpha}^{+} \right) \right] \right) \right) \right\}
\]
\[
\begin{align*}
\alpha^{-} \rightarrow \alpha^{+} &= -\left(\alpha^{-} + \alpha^{+}\right) - A \langle 0;\alpha \rangle + \left\langle A_{\alpha}^{-}, A_{\alpha}^{+} \right\rangle \\
\text{which can be illustrated by given this following figure.}
\end{align*}
\]

Figure 4. The alpha-cut operation towards NT2TFN with $\alpha_{LMF} < \alpha < \alpha_{UMF}$.

**Definition 4.** Let $^N\overline{A_i}$ be a set of $(n + 1)$ NT2TFNs, then type-reduction method of $\alpha$-NT2TFNs(after fuzzification), $^N\overline{A_i}$ is defined [4] by

\[
^N\overline{A_i} = \left\{ ^N\overline{A_i} = \left\langle ^N\overline{A_{i_{\alpha}}^{\leftarrow}}, A_i, ^N\overline{A_{i_{\alpha}}^{\rightarrow}} \right\rangle; i = 0,1,...,n \right\}
\]

where $^N\overline{A_{i_{\alpha}}^{\leftarrow}}$ are left type-reduction of alpha-cut NT2TFNs, $^N\overline{A_{i_{\alpha}}^{\rightarrow}} = \frac{1}{3} \sum_{j=0}^{n} \left( A_{kj}^{-} + A_{kj}^{+} + A_{kj}^{\rightarrow} \right)$, $A_i$ is the crisp point of NT2TFN and $^N\overline{A_{i_{\alpha}}^{\rightarrow}}$ is right type-reduction of alpha-cut NT2TFNs, $^N\overline{A_{i_{\alpha}}^{\rightarrow}} = \frac{1}{3} \sum_{j=0}^{n} \left( \overline{P_{i_{\alpha}}^{-}} + \overline{P_{i_{\alpha}}^{+}} + \overline{P_{i_{\alpha}}^{\rightarrow}} \right)$.

**Definition 5.** Let $\alpha$-TR is the type-reduction method after $\alpha$-cut process had been applied for every NT2TFNs, $^N\overline{A_{i_{\alpha}}}$ then $^N\overline{A_{i_{\alpha}}}$ named as defuzzification NT2TFNs for $^N\overline{A_{i_{\alpha}}}$ if for every $^N\overline{A_{i_{\alpha}}} \in ^N\overline{A_{i_{\alpha}}} [4]$, $^N\overline{A_{i_{\alpha}}} = \left\{ ^N\overline{A_{i_{\alpha}}} \right\}$ for $i = 0,1,...,n$

where for every $^N\overline{A_{i_{\alpha}}} = \frac{1}{3} \sum_{j=0}^{n} < ^N\overline{A_{i_{\alpha}}^{\leftarrow}}, A_i, ^N\overline{A_{i_{\alpha}}^{\rightarrow}} >$. The process in defuzzifying the
NT2TFNs can be illustrated at Fig. 5.

Figure 5. Defuzzification process of NT2TFN.

2 Result

**Theorem 2.1.** Let $^N\tilde{A}$ be a NT2FN which centred at $c$ with $<\phi, \epsilon, \gamma>$ and $<\eta, \phi, \lambda>$ are left and right interval(intervals of footprint) of $^N\tilde{A}$ respectively. If $\alpha_i$ and $\alpha_j$ are the $\alpha$-cut of $^N\tilde{A}$ where $\alpha_i < \alpha_j$ with $\alpha_i < \alpha_j \leq \alpha' < \alpha''$ which $\alpha'$ and $\alpha''$ are lower and upper $\alpha$-cut of $^N\tilde{A}$, then $^N\tilde{A}_{\alpha_j(x)} \subseteq ^N\tilde{A}_{\alpha_i(x)}$.

**Proof.** Let the membership function of $^N\tilde{A}$ given as

$$^N\tilde{A}(x) = \begin{cases} 
1 - \frac{c-x}{(\phi, \epsilon, \gamma)}, & \text{if } (\phi, \epsilon, \gamma) \leq x \leq c \\
1 - \frac{x-c}{(\eta, \phi, \lambda)}, & \text{if } c \leq x \leq (\eta, \phi, \lambda) \\
0, & \text{otherwise}
\end{cases}$$

where $\phi \leq \epsilon \leq \gamma$ and $\eta \leq \phi \leq \lambda$. Then, the interval of $^N\tilde{A}$ is $<c-(\phi, \epsilon, \gamma), c, c+(\eta, \phi, \lambda)>$. From Eq. 4, if the type-2 fuzzy interval was obtained by $\alpha$-cut operation, then the interval of $^N\tilde{A}_{\alpha_i}$ achieved which given as $^N\tilde{A}_{\alpha_i} = <A_{\tilde{a_i}, \tilde{a_i}}, A_{\tilde{a_i}, \tilde{a_i}}>$ and $^N\tilde{A}_{\alpha_j} = <A_{\tilde{a_j}, \tilde{a_j}}, A_{\tilde{a_j}, \tilde{a_j}}>$ with $A_{\tilde{a_i}, \tilde{a_i}}$ and $A_{\tilde{a_i}, \tilde{a_i}}$ and $A_{\tilde{a_j}, \tilde{a_j}}$ and $A_{\tilde{a_j}, \tilde{a_j}}$ are left and right footprint of
\( \alpha \)-cut of \( A \) where \( \alpha_i < \alpha_j \) for all \( \alpha_i, \alpha_j \in (0, 1] \). For \( \alpha_i \),
\[
\left\{ \begin{aligned}
&\frac{(c-(\varphi, \epsilon, \gamma))(\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l}) - (c - (\varphi, \epsilon, \gamma))}{c - (c-(\varphi, \epsilon, \gamma))} = (\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}), c, \\
&\frac{(c + (\eta, \phi, \lambda)) - (c + (\eta, \phi, \lambda))}{(c + (\eta, \phi, \lambda)) - c} = (\tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}, \tilde{\alpha}_i, \tilde{\alpha}_i^{u})
\end{aligned} \right.
\]
\[
\left\{ \begin{aligned}
&\frac{(c-(\varphi, \epsilon, \gamma))(\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l}) - (c - (\varphi, \epsilon, \gamma))}{c - (c-(\varphi, \epsilon, \gamma))} = (\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}), c, \\
&\frac{(c + (\eta, \phi, \lambda)) - (c + (\eta, \phi, \lambda))}{(c + (\eta, \phi, \lambda)) - c} = (\tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}, \tilde{\alpha}_i, \tilde{\alpha}_i^{u})
\end{aligned} \right.
\]
\[
\left\{ \begin{aligned}
&\frac{(c-(\varphi, \epsilon, \gamma))(\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l}) - (c - (\varphi, \epsilon, \gamma))}{c - (c-(\varphi, \epsilon, \gamma))} = (\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}), c, \\
&\frac{(c + (\eta, \phi, \lambda)) - (c + (\eta, \phi, \lambda))}{(c + (\eta, \phi, \lambda)) - c} = (\tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}, \tilde{\alpha}_i, \tilde{\alpha}_i^{u})
\end{aligned} \right.
\]
\[
\left\{ \begin{aligned}
&\frac{(c-(\varphi, \epsilon, \gamma))(\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l}) - (c - (\varphi, \epsilon, \gamma))}{c - (c-(\varphi, \epsilon, \gamma))} = (\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}), c, \\
&\frac{(c + (\eta, \phi, \lambda)) - (c + (\eta, \phi, \lambda))}{(c + (\eta, \phi, \lambda)) - c} = (\tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}, \tilde{\alpha}_i, \tilde{\alpha}_i^{u})
\end{aligned} \right.
\]
Since \( \alpha_i < \alpha_j \), then
\[
\left\{ \begin{aligned}
&\frac{(c-(\varphi, \epsilon, \gamma))(\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l}) - (c - (\varphi, \epsilon, \gamma))}{c - (c-(\varphi, \epsilon, \gamma))} = (\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}), c, \\
&\frac{(c + (\eta, \phi, \lambda)) - (c + (\eta, \phi, \lambda))}{(c + (\eta, \phi, \lambda)) - c} = (\tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}, \tilde{\alpha}_i, \tilde{\alpha}_i^{u})
\end{aligned} \right.
\]
\[
\left\{ \begin{aligned}
&\frac{(c-(\varphi, \epsilon, \gamma))(\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l}) - (c - (\varphi, \epsilon, \gamma))}{c - (c-(\varphi, \epsilon, \gamma))} = (\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}), c, \\
&\frac{(c + (\eta, \phi, \lambda)) - (c + (\eta, \phi, \lambda))}{(c + (\eta, \phi, \lambda)) - c} = (\tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}, \tilde{\alpha}_i, \tilde{\alpha}_i^{u})
\end{aligned} \right.
\]
\[
\left\{ \begin{aligned}
&\frac{(c-(\varphi, \epsilon, \gamma))(\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l}) - (c - (\varphi, \epsilon, \gamma))}{c - (c-(\varphi, \epsilon, \gamma))} = (\tilde{\alpha}_i^{u}, \tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}), c, \\
&\frac{(c + (\eta, \phi, \lambda)) - (c + (\eta, \phi, \lambda))}{(c + (\eta, \phi, \lambda)) - c} = (\tilde{\alpha}_i^{l} = \frac{\tilde{\alpha}_i}{\alpha_i}, \tilde{\alpha}_i, \tilde{\alpha}_i^{u})
\end{aligned} \right.
\]
Generalized normal type-2 triangular fuzzy number

\[ \alpha \left( \tilde{\alpha}_j = \frac{\tilde{\alpha}_l}{\alpha}, \tilde{\alpha}_j, \tilde{\alpha}_j \right) + c \left( \eta, \phi, \lambda \right) \]

\[ \alpha \left( (\tilde{\alpha}_j, \tilde{\alpha}_j, \tilde{\alpha}_j = \frac{\tilde{\alpha}_l}{\alpha} \right) (\tilde{\alpha}_j, \tilde{\alpha}_j, \tilde{\alpha}_j) < (\tilde{\alpha}_j = \frac{\tilde{\alpha}_l}{\alpha}, \tilde{\alpha}_j, \tilde{\alpha}_j) \]

Then, \( \left( A_{(\tilde{\alpha}_l, \tilde{\alpha}_l, \tilde{\alpha}_l)^e} A_{(\tilde{\alpha}_l, \tilde{\alpha}_l, \tilde{\alpha}_l)^e} \right) < \left( A_{(\tilde{\alpha}_l, \tilde{\alpha}_l, \tilde{\alpha}_l)^e} A_{(\tilde{\alpha}_l, \tilde{\alpha}_l, \tilde{\alpha}_l)^e} \right) \). Therefore,

\[ A_{(\tilde{\alpha}_l, \tilde{\alpha}_l, \tilde{\alpha}_l)^e} A_{(\tilde{\alpha}_l, \tilde{\alpha}_l, \tilde{\alpha}_l)^e} \] and

\[ N_{A_{(\tilde{\alpha}_l, \tilde{\alpha}_l, \tilde{\alpha}_l)^e}} \subseteq N_{A_{(\tilde{\alpha}_l, \tilde{\alpha}_l, \tilde{\alpha}_l)^e}} \]

Figure 6. \( \alpha_i < \alpha_j \Leftrightarrow N_{A_{\alpha_i}} N_{A_{\alpha_j}} \).

**Theorem 2.1.1.** If \( \alpha_i \) and \( \alpha_j \) are the \( \alpha \)-cut of \( N_{\tilde{A}} \) where \( \alpha_i < \alpha_j \) with \( \alpha^i < \alpha_i < \alpha_j \leq \alpha^u \) which \( \alpha^i \) and \( \alpha^u \) are lower and upper \( \alpha \)-cut of \( N_{\tilde{A}} \), then

\[ N_{A_{\alpha_i}} \subseteq N_{A_{\alpha_j}} \]

**Proof.** For \( \alpha_i \),
\[
\begin{align*}
\left\{ \frac{(c - (\varphi, \varepsilon, 0))_{(\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu)}}{c - (c - (\varphi, \varepsilon, 0))} - \frac{(c - (\varphi, \varepsilon, 0))}{c - (c - (\varphi, \varepsilon, 0))} \right\} &= (\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu, 0), c, \\
\left\{ \frac{(c + (0, \phi, \lambda)) - (c + (0, \phi, \lambda))}{(c + (0, \phi, \lambda)) - c} \right\} &= (0, \tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu, 0), c,
\end{align*}
\]

For \( \alpha_j \),
\[
\left\{ \frac{(c - (\varphi, \varepsilon, 0))_{(\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu)}}{c - (c - (\varphi, \varepsilon, 0))} - \frac{(c - (\varphi, \varepsilon, 0))}{c - (c - (\varphi, \varepsilon, 0))} \right\} &= (\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu, 0), c, \\
\left\{ \frac{(c + (0, \phi, \lambda)) - (c + (0, \phi, \lambda))_{(0, \tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu)}}{(c + (0, \phi, \lambda)) - c} \right\} &= (0, \tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu, 0), c,
\]

Since \( \alpha_i < \alpha_j \), then
\[
\left\{ \frac{(c - (\varphi, \varepsilon, 0))_{(\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu)}}{c - (c - (\varphi, \varepsilon, 0))} < \frac{(c - (\varphi, \varepsilon, 0))_{(\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu)}}{c - (c - (\varphi, \varepsilon, 0))} \right\}, c,
\]

\[
\begin{align*}
\left\{ \frac{(c + (0, \phi, \lambda))}{(c + (0, \phi, \lambda))_{(0, \tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu)}} < \frac{(c + (0, \phi, \lambda))}{(c + (0, \phi, \lambda))_{(0, \tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu)}} \right\} &= \left\{ ((c - (c - (\varphi, \varepsilon, 0)))_{(\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu)} \right\}, c, ((c - (c - (\varphi, \varepsilon, 0)))_{(\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu)} \right\}, c,
\end{align*}
\]

Then, \( A_{(\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu, 0)} < A_{(\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu, 0)} \), \( A_{(\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu, 0)} = A_{(\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu, 0)} \), \( A_{(\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu, 0)} > A_{(\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu, 0)} \), \( A_{(\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu, 0)} > A_{(\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu, 0)} \). Therefore,

\[
\left\{ A_{(\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu, 0)} \right\} \subseteq \left\{ A_{(\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu, 0)} \right\}, \quad \left\{ A_{(\tilde{\alpha}_i^\mu, \tilde{\alpha}_i^\nu, 0)} \right\} \subseteq \left\{ A_{(\tilde{\alpha}_j^\mu, \tilde{\alpha}_j^\nu, 0)} \right\},
\]

where \( \alpha_i, \alpha_j \in (\alpha^\prime, 1) \).

Theorem 2.1.2. If $\alpha_i$ and $\alpha_j$ are the $\alpha$-cut of $^N \tilde{A}$ where $\alpha_i < \alpha_j$ with $\alpha_i < \alpha' < \alpha_j \leq \alpha''$ which $\alpha'$ and $\alpha''$ are lower and upper $\alpha$-cut of $^N \tilde{A}$, then $^N \tilde{A}_{\alpha_j} \subseteq ^N \tilde{A}_{\alpha_i}$.

Proof. For $\alpha_i$,

\[
\begin{aligned}
\frac{(c - (\varphi, \varepsilon, \gamma))_{(\tilde{a}_i, \tilde{a}_i, \tilde{a}_i)} - (c - (\varphi, \varepsilon, \gamma))}{c - (c - (\varphi, \varepsilon, \gamma))} &= \frac{\tilde{a}^u_i, \tilde{a}_i, \tilde{a}_i^b}{\alpha_i}, c,

\frac{(c + (\eta, \phi, \lambda)) - (c + (\eta, \phi, \lambda))_{(\tilde{a}_i, \tilde{a}_i, \tilde{a}_i)}}{(c + (\eta, \phi, \lambda)) - c} &= \frac{\tilde{a}^b_i = \frac{\tilde{a}_i}{\alpha_i}, \tilde{a}_i, \tilde{a}_i^u}{\alpha_i}.
\end{aligned}
\]

\[
\begin{aligned}
\left\{ (c - (\varphi, \varepsilon, \gamma))_{(\tilde{a}_i, \tilde{a}_i, \tilde{a}_i)} \right\} = [c - (c - (\varphi, \varepsilon, \gamma))](\tilde{a}^u_i, \tilde{a}_i, \tilde{a}_i) \left( \frac{\tilde{a}_i}{\alpha_i} \right) + (c - (\varphi, \varepsilon, \gamma))c,

(c + (\eta, \phi, \lambda))_{(\tilde{a}_i, \tilde{a}_i, \tilde{a}_i)} = -[(c + (\eta, \phi, \lambda)) - c](\tilde{a}^b_i = \frac{\tilde{a}_i}{\alpha_i}, \tilde{a}_i, \tilde{a}_i^u) + (c + (\eta, \phi, \lambda))c.
\end{aligned}
\]

For $\alpha_j$,
\[
\left\{ \left( c - (\varphi, \varepsilon, \gamma) \right)_{(\bar{a}_j^p, \bar{a}_j, 0)} - \left( c - (\varphi, \varepsilon, \gamma) \right) \right\} = (\tilde{\alpha}_j^u, \tilde{\alpha}_j, 0), c,
\]
\[
\left( c + (\eta, \phi, \lambda) \right)_{(0, \tilde{\alpha}_j^u, \tilde{\alpha}_j)} = \left( 0, \tilde{\alpha}_j^u, \tilde{\alpha}_j \right)
\]
\[
\left\{ \left( c - (\varphi, \varepsilon, \gamma) \right)_{(\bar{a}_j^p, \bar{a}_j, 0)} = \left[ c - (c - (\varphi, \varepsilon, \gamma)) \right] (\tilde{\alpha}_j^u, \tilde{\alpha}_j, 0) + (c - (\varphi, \varepsilon, \gamma)), c,
\]
\[
\left( c + (\eta, \phi, \lambda) \right)_{(0, \tilde{\alpha}_j^u, \tilde{\alpha}_j)} = \left[ -\left( (c + (\eta, \phi, \lambda)) - c \right) \right] (0, \tilde{\alpha}_j, \tilde{\alpha}_j) + (c + (\eta, \phi, \lambda)).
\]

Since \( \alpha_i < \alpha_j \), then
\[
\left\{ \left( c - (\varphi, \varepsilon, \gamma) \right)_{(\bar{a}_j^p, \bar{a}_j, 0)} < \left( c - (\varphi, \varepsilon, 0) \right)_{(\bar{a}_j^p, \bar{a}_j, 0)}, c,
\]
\[
\left( c + (\eta, \phi, \lambda) \right)_{(\bar{a}_j^p, \bar{a}_j, 0)} < \left( c + (0, \phi, \lambda) \right)_{(\bar{a}_j^p, \bar{a}_j, 0)}
\]
\[
= \left\{ \left[ c - (c - (\varphi, \varepsilon, \gamma)) \right] (\tilde{\alpha}_j^u, \tilde{\alpha}_j, \tilde{\alpha}_j^u) + (c - (\varphi, \varepsilon, \gamma)) \right\} < \left\{ \left[ c - (c - (\varphi, \varepsilon, \gamma)) \right] (\tilde{\alpha}_j^u, \tilde{\alpha}_j, 0) + (c - (\varphi, \varepsilon, \gamma)) \right\}
\]
\[
+ (c - (\varphi, \varepsilon, \gamma)), c, \left[ -\left( (c + (\eta, \phi, \lambda)) - c \right) \right] (\tilde{\alpha}_j^u, \tilde{\alpha}_j, \tilde{\alpha}_j^u) + (c + (\eta, \phi, \lambda)) \right\} < \left\{ -\left[ (c + (\eta, \phi, \lambda)) - c \right] (0, \tilde{\alpha}_j, \tilde{\alpha}_j) + (c + (\eta, \phi, \lambda)) \right\}
\]
\[
= \left\{ (\tilde{\alpha}_j^u, \tilde{\alpha}_j, \tilde{\alpha}_j^u) < \left( \tilde{\alpha}_j^u, \tilde{\alpha}_j, 0 \right), c, (\tilde{\alpha}_j^u, \tilde{\alpha}_j, \tilde{\alpha}_j^u) < (0, \tilde{\alpha}_j, \tilde{\alpha}_j) \right\}
\]

Then, \( \left\{ A_{(\tilde{\alpha}_j^u, \tilde{\alpha}_j, 0)} < A_{(\tilde{\alpha}_j^u, \tilde{\alpha}_j, 0)}, A_{(\tilde{\alpha}_j^u, \tilde{\alpha}_j, 0)} < A_{(0, \tilde{\alpha}_j^u, \tilde{\alpha}_j)} \right\}. \)

Therefore,
\[
A_{(\tilde{\alpha}_j^u, \tilde{\alpha}_j, 0)}, A_{(0, \tilde{\alpha}_j^u, \tilde{\alpha}_j)} \subseteq A_{(\tilde{\alpha}_j^u, \tilde{\alpha}_j, 0)}, A_{(0, \tilde{\alpha}_j^u, \tilde{\alpha}_j)} >
\]
\[
\exists \tilde{\alpha}_j^u \subseteq \tilde{\alpha}_j^u, \text{ where } \alpha_i \in (0, 1) \text{ and } \alpha_j \in (\alpha_i', 1) \]
Theorem 2.2. Based on the definition of defuzzification for NT2TFN, let $\alpha$ be a representation of $\alpha$-cut operation of NT2TFN, $^N\bar{A}_{\alpha}$ and $^N\bar{A}_{\alpha(TR)}$ be type-reduction of $^N\bar{A}$, which gives

$$^N\bar{A}_{\alpha} = \left\{ ^N\bar{A}_{\alpha}^c = (A - (\phi, \epsilon, \gamma))_{\alpha}, A, ^N\bar{A}_{\alpha}^+ = (A + (\eta, \phi, \lambda))_{\alpha} \right\}$$ and

$$^N\bar{A}_{\alpha(TR)} = \left\{ ^N\bar{A}_{\alpha(TR)}^c = (A - (\phi, \epsilon, \gamma))_{\alpha(TR)}, A, ^N\bar{A}_{\alpha(TR)}^+ = (A + (\eta, \phi, \lambda))_{\alpha(TR)} \right\}$$

where $^N\bar{A}_{\alpha}^c$ and $^N\bar{A}_{\alpha}^+$ are left and right footprint of NT2TFN after $\alpha$-cut operation was applied and $^N\bar{A}_{\alpha(TR)}^c$ and $^N\bar{A}_{\alpha(TR)}^+$ are left and right footprint $\alpha$-NT2TFN after type-reduction has been applied, $A$ is a crisp point and $(\phi, \epsilon, \gamma)$ are left-left, left and right-left lengths and $(\eta, \phi, \lambda)$ are left-right, right, right-right lengths from $A$ respectively with $\gamma < \epsilon < \phi$ and $\lambda < \phi < \eta$. If $(\eta, \phi, \lambda) < (\phi, \epsilon, \gamma)$ or $(\phi, \epsilon, \gamma) < (\eta, \phi, \lambda)$, then crisp type-2 fuzzy solution is on left or right of $A$ respectively.

Proof. Given that $^N\bar{A}_{\alpha} = \left\{ ^N\bar{A}_{\alpha}^c = (A - (\phi, \epsilon, \gamma))_{\alpha}, A, ^N\bar{A}_{\alpha}^+ = (A + (\eta, \phi, \lambda))_{\alpha} \right\}$. Then, we obtained

Case 1: For $\alpha < \alpha' < \alpha''$ where $\alpha'$ and $\alpha''$ are lower and upper membership function of $^N\bar{A}$, then
\[
\left\{ \left( A - \left( \varphi, \varepsilon, \gamma \right) \right)_{\alpha} - \left( A - \left( \varphi, \varepsilon, \gamma \right) \right) \right\}_{\alpha} A, \frac{(A + (\eta, \phi, \lambda)) - (A + (\eta, \phi, \lambda))_{\alpha}}{A - \left( \varphi, \varepsilon, \gamma \right)} A = -\left( (A + (\eta, \phi, \lambda)) - A \right)(\alpha, \alpha, \frac{\alpha}{\alpha'}) + (A + (\eta, \phi, \lambda)) < \left( (A - \left( \varphi, \varepsilon, \gamma \right) \right)_{\alpha} A, (A + (\eta, \phi, \lambda))_{\alpha}
\]

For \((\eta, \phi, \lambda) < (\varphi, \varepsilon, \gamma)\), then
\[
= -\left( (A + (\eta, \phi, \lambda)) - A \right)(\alpha, \alpha, \alpha) + (A + (\eta, \phi, \lambda)) < \left( (A - \left( \varphi, \varepsilon, \gamma \right) \right)(\alpha, \alpha, \frac{\alpha}{\alpha'})
\]
\[
= -\left( (A - \left( \varphi, \varepsilon, \gamma \right) \right)(\alpha, \alpha, \alpha) + (\alpha, \alpha, \frac{\alpha}{\alpha'}) A + (\eta, \phi, \lambda) < (\alpha, \alpha, \frac{\alpha}{\alpha'}) A - \left( (A - \left( \varphi, \varepsilon, \gamma \right) \right)_{\alpha} A
\]
\[
= -\left( (\eta, \phi, \lambda) (\alpha, \alpha, \frac{\alpha}{\alpha'}) - 1 \right) < (\alpha, \alpha, \frac{\alpha}{\alpha'}) - 1
\]
\[
= -\left( (\eta, \phi, \lambda) < (\varphi, \varepsilon, \gamma) \right)(\alpha, \alpha, \frac{\alpha}{\alpha'}) - 1
\]

For \((\varphi, \varepsilon, \gamma) < (\eta, \phi, \lambda)\), then
\[
= -\left( (A - \left( \varphi, \varepsilon, \gamma \right) \right)(\alpha, \alpha, \frac{\alpha}{\alpha'}) + (A + (\eta, \phi, \lambda)) < -(\alpha, \alpha, \frac{\alpha}{\alpha'}) A
\]
\[
= -\left( (\alpha, \alpha, \frac{\alpha}{\alpha'}) A - (\alpha, \alpha, \frac{\alpha}{\alpha'}) A + (\alpha, \alpha, \frac{\alpha}{\alpha'}) A + (\eta, \phi, \lambda) < (\alpha, \alpha, \frac{\alpha}{\alpha'}) A - (\alpha, \alpha, \frac{\alpha}{\alpha'}) A
\]
\[
= (\varphi, \varepsilon, \gamma)(\alpha, \alpha, \frac{\alpha}{\alpha'}) - (\alpha, \alpha, \frac{\alpha}{\alpha'}) < -(\alpha, \alpha, \frac{\alpha}{\alpha'}) A + (\eta, \phi, \lambda)
\]
\[
= -(\alpha, \alpha, \frac{\alpha}{\alpha'}) A + (\alpha, \alpha, \frac{\alpha}{\alpha'}) A + (\eta, \phi, \lambda) < (\alpha, \alpha, \frac{\alpha}{\alpha'}) A - (\alpha, \alpha, \frac{\alpha}{\alpha'}) A
\]
\[
= -\left( (\varphi, \varepsilon, \gamma) < (\eta, \phi, \lambda) \right)(\alpha, \alpha, \frac{\alpha}{\alpha'}) - 1
\]

The type-reduction process of \(\alpha A_{\alpha}, \alpha A_{\alpha(\eta)}\) given as
Generalized normal type-2 triangular fuzzy number

\[ \overline{N} \overline{A}_{a(\text{TR})} = \left\{ \frac{(A-\varphi)_{\alpha} + (A-\varepsilon)_{\alpha} + (A-\gamma)_{\alpha} \alpha}{3}, \frac{(A+\eta)_{\alpha} + (A+\phi)_{\alpha} + (A+\lambda)_{\alpha} \alpha}{3} \right\} \]

Then, the defuzzification process of \( \overline{N} \overline{A}_{a(\text{TR})} \) can be given as follow

\[
\overline{N} \overline{A}_{a(\text{TR})} = \frac{\left( \frac{(A-\varphi)_{\alpha} + (A-\varepsilon)_{\alpha} + (A-\gamma)_{\alpha} \alpha}{3} \right) + \left( \frac{(A+\eta)_{\alpha} + (A+\phi)_{\alpha} + (A+\lambda)_{\alpha} \alpha}{3} \right)}{3}
\]

Therefore, if crisp type-2 fuzzy solution was obtained at the left side of \( A \) then \( N \overline{A}_{a(\text{TR})} < A \) which \( \left\{ \left( A + (\eta, \phi, \lambda) \right)_{\alpha} \right\} < \left\{ (A - (\varphi, \varepsilon, \gamma))_{\alpha} \right\} \) with \( (\eta, \phi, \lambda) < (\varphi, \varepsilon, \gamma) \) and if crisp type-2 fuzzy solution at the right side of \( A \), then \( A < N \overline{A}_{a(\text{TR})} \) which \( \left\{ (A - (\varphi, \varepsilon, \gamma))_{\alpha} \right\} < \left\{ (A + (\eta, \phi, \lambda)) \right\} \) with \( (\varphi, \varepsilon, \gamma) < (\eta, \phi, \lambda) \).

**Case 2:** For \( \alpha' < \alpha < \alpha^o \), then

\[
\left\{ \frac{(A-\left(0, \varphi, \varepsilon, 0\right))_{\alpha} - (A-\left(0, \varphi, \varepsilon, 0\right))_{\alpha}}{A-(A-(\varphi, \varepsilon, 0))}, \frac{(A+\left(0, \phi, \lambda\right))-(A+(0, \phi, \lambda))_{\alpha}}{(A+\left(0, \phi, \lambda\right))-(A+(0, \phi, \lambda))_{\alpha}} \right\} = \left\{ (A-(\varphi, \varepsilon, 0))_{\alpha} = (A-(\varphi, \varepsilon, 0))(\alpha, \alpha, 0) + (\varphi, \varepsilon, 0), A, \right\}
\]

\[
(A + (0, \phi, \lambda))_{\alpha} = -(A + (0, \phi, \lambda))_{\alpha} = (A - (0, \phi, \lambda))_{\alpha} = A \right\}
\]

For \( (\eta, \phi, \lambda) < (\varphi, \varepsilon, \gamma) \), then

\[
= -((A + (0, \phi, \lambda)) - A)(0, \alpha, \alpha) + (A + (0, \phi, \lambda)) - (A - (\varphi, \varepsilon, 0))(\alpha, \alpha, 0) + (A - (\varphi, \varepsilon, 0))
\]

The type-reduction process of \( \overline{N} \overline{A}_{a} \), \( \overline{N} \overline{A}_{a(\text{TR})} \) given as
Then, the defuzzification process of $N\overrightarrow{A}(TR)$ can be given as follow:

$$N\overrightarrow{A}(TR) = \left\{ N\overrightarrow{A}^{-}(TR) = (A - (\varphi, \varepsilon, 0))_{a(TR)}, A, N\overrightarrow{A}^{+}(TR) = (A + (\eta, \phi, 0))_{a(TR)} \right\}$$

$$= \left\{ \frac{(A - \varphi)_{a} + (A - \varepsilon)_{a} + 0}{2}, A, \frac{0 + (A + \phi)_{a} + (A + \lambda)_{a}}{2} \right\}.$$

Therefore, if crisp type-2 fuzzy solution was obtained at the left side of $A$, then $\frac{N\overrightarrow{A}(TR)}{A} < A$ which $\langle A + (0, \phi, \lambda) \rangle < \langle A - (\varphi, \varepsilon, 0) \rangle$ with $(0, \varphi, \lambda) < (\varphi, \varepsilon, 0)$ and if crisp type-2 fuzzy solution at the right side of $A$, then $A < \frac{N\overrightarrow{A}(TR)}{A}$ which $\langle A - (\varphi, \varepsilon, 0) \rangle < \langle A + (0, \phi, \lambda) \rangle$ with $(\varphi, \varepsilon, 0) < (0, \phi, \lambda)$.

Acknowledgement

The authors would like to thank Research Management and Innovation Centre (RMIC) of Universiti Malaysia Terengganu and Ministry of Higher Education (MOHE) Malaysia for funding(FRGS, vot59244) and providing the facilities to carry out this research.

References


Received: February 1, 2013