Fuzzy Set Theory in Modeling Uncertainty Data via Interpolation Rational Bezier Surface Function

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Abstract

Lately, the research of rational spline function widely expanding because the designing of curve and surface, which is the additional parameter known as weight that it had. This weight use to changing the curve and surface shape to fit the real data form. Then, the interpolation method adapted in rational spline function as interpolation rational spline function, which can be constructed by using piecewise rational spline function. This method is being used to model the collection of data set to visualize the data for more the understanding the meaning of the data through the curve especially surface forms. However, the problem arises when the data become uncertain, which is unable to model them. Therefore, in this paper, we introduced fuzzy interpolation rational bicubic Bezier surface (later known as FIRBBS) which modeling the fuzzy data after had been defined by the fuzzy sets definition. The construction of FIRBBS is based on the definition of fuzzy set theory and interpolation rational bicubic Bezier surface. Then, the illustration given in figures for more understanding the method.

Keywords: Fuzzy set theory, rational bicubic Bezier surface, interpolation, fuzzy data, alpha-cut, defuzzification
1 Introduction

In Computer Aided Geometric Design (CAGD) field, rational spline function is a useful tool in modeling data which its advantage in weights which they had. The weights also named as additional parameters give full freedom to a user to control the shape of the design curve and surface which meaning that the curve changing when the weight is changing [5-7,13,15]. For rational spline curves, Sarfraz et. al [14] introduced a piecewise rational cubic spline curves where possesses parameters (weights) in each interval, which can be used to control the shape of the curves. This piecewise rational cubic spline also known as interpolation rational cubic spline. The extended of piecewise rational cubic function is the rational bicubic function which introduced by Hussain and Hussain [8].

The rational interpolation as the piecewise rational method, one of the methods in CAGD field fro designing. This interpolation method used in modeling data, especially in curves design and converted to surfaces form. The modeling data through the interpolation surface face the difficulty when the data become the uncertainty which is the data not precisely obtained. This uncertainty data cannot be modeled with the various method to construct a surface. Therefore, the fuzzy set theory which introduced by Zadeh [16] used as the solution of the uncertainty data. The extension of fuzzy set theory is fuzzy number concept [4,11,12,17] used to define the uncertainty data.

The structure of this paper begins with the basic definitions which these definitions are being used to constructed the proposed method through Section 2. Then, Section 3 discuss about the constructing model for modeling the uncertainty data by using interpolation rational Bezier surface function. Also, in this section, we will illustrate the fuzzification and defuzzification processes together with their illustration. Section 4 and 5 are both the discussion and conclusion of the constructed model respectively.

2 Preliminaries

In this Section 2, we give some definition which included the definition of fuzzy set theory, fuzzy number and fuzzy data point(defined from uncertainty data points using fuzzy number concepts). In addition, the fuzzification process(alpha-cut operation) and defuzzification process yet still defined in this section.

Definition 1. Let $X$ be a universal set and $A \subset X$ Set $A$ called fuzzy set denoted by $\tilde{A}$ if for every $x \in X$ there exists $\mu_A : X \to [0,1]$ a form of membership function that characterizing the membership grade for every element of $A$ in $X$ is defined by
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\[
\mu_A(x) = \begin{cases} 
1 & \text{if } x \in A \text{ (full membership)} \\
0 & \text{if } x \notin A \text{ (non-membership)} \\
c \in (0,1) & \text{if } x \notin A \text{ (non-full membership)}
\end{cases}
\]  

(1)

So, fuzzy set \( \tilde{A} \) can be written as \( \tilde{A} = \{(x, \mu_A(x))\} \) which is \( \tilde{A} \) in \( X \) is a set of order pair denoted generically by \( x \) in \( X \) with grade of membership \( \mu_A(x) \) in \( [0,1] \) [2,3].

**Definition 2.** Let \( R \) be a universal set which \( R \) is a real number and \( A \) is subset to \( R \). Fuzzy set \( \tilde{A} \) in \( R \) (number around \( A \) in \( R \)) called fuzzy number which explained through the \( \alpha \)-level set (strong \( \alpha \)-cut) that is if for every \( \alpha \in (0,1] \), there exist set \( A_\alpha \) in \( R \) until \( \tilde{A}_\alpha = \{ x \in R : \mu_A(x) > \alpha \} \). If alpha value are \( \alpha_i > \alpha_j > \cdots > \alpha_k \), then \( \tilde{A}_{\alpha_i} \subset \tilde{A}_{\alpha_j} \subset \cdots \subset \tilde{A}_{\alpha_k} \) [1-3].

**Definition 3.** Fuzzy set \( \tilde{P} \) in a space of \( S \) said set of fuzzy control points if for every \( \alpha \) – level set was chosen, there exists pointed interval that is \( \tilde{P} = \langle P^+_i, P^-_i, P^+_i \rangle \) in \( S \) with every \( P_i \) is crisp point and membership function \( \mu_p : S \rightarrow [0,1] \) which is defined as \( \mu_p(P_i) = 1, \)

\[
\mu_p(P^+_i) = \begin{cases} 
0 & \text{if } P^+_i \notin S \\
c \in (0,1) & \text{if } P^+_i \in S \text{ and } \mu_p(P^+_i) = 1 \\
1 & \text{if } P^-_i \in S
\end{cases}
\]

where \( \mu_p(P^+_i) \) and \( \mu_p(P^-_i) \) are left membership grade value and right membership grade value respectively and generally written as \( \tilde{P} = \{ \tilde{P}_i : i = 0,1,2,\ldots,n \} \)

(2)

for every \( i \), \( \tilde{P}_i = \langle \tilde{P}^+_i, P_i, \tilde{P}^-_i \rangle \) with \( \tilde{P}^+_i, P_i \) and \( \tilde{P}^-_i \) are left fuzzy control point, crisp control point and right fuzzy control point respectively. This is also same as applied to two and three dimension space [2,3].

**Definition 4.** A fuzzy number \( \tilde{A} \) is a triangular fuzzy number which is centred at \( p \) with left width, \( \delta > 0 \) and right width, \( \beta > 0 \) if its membership function forms as

\[
A(x) = \begin{cases} 
1 - \frac{x-a}{\delta} & \text{if } a - \delta \leq x \leq a \\
1 - \frac{x-a}{\beta} & \text{if } a \leq x \leq a + \beta \\
0 & \text{otherwise}
\end{cases}
\]

(3)

which can be written as \( \tilde{A} = (a, \delta, \beta) \). Therefore, if the crisp interval is obtained
by using the alpha cut operation, then the interval of $\tilde{A}_\alpha$ surely can be obtained [17]. $\forall \alpha \in (0,1]$, from

$$\frac{(a-\delta)^{(\alpha)}}{a-(a-\delta)} = \alpha, \quad \frac{(a+\beta)-(a+\beta)^{(\alpha)}}{(a+\beta)-a} = \alpha$$

$$(a-\delta)^{(\alpha)} = (a-(a-\delta))\alpha + (a-\delta)$$

$$(a+\beta)^{(\alpha)} = -(a+\beta-a)\alpha + (a+\beta)$$

Then,

$$\tilde{A}_\alpha = [(a-\delta)^{(\alpha)}, (a+\beta)^{(\alpha)}]$$

$$= [(a-(a-\delta))\alpha + (a-\delta), -(a+\beta-a)\alpha + (a+\beta)].$$

**Definition 5.** Let the $\alpha$-cut of fuzzy data points, $\tilde{P}_i$ has been applied by using Def. 4. Then $\tilde{P}$ named as defuzzification of $\tilde{P}_i$ if for every $\tilde{P}_i \in \tilde{P}$,

$$\tilde{P} = \{\tilde{P}_i\} \quad \text{for} \quad i = 0,1,...,n$$

(5)

where for every $\tilde{P}_i = \frac{1}{3} \left( \tilde{p}_{i\alpha\leftarrow} + \tilde{p}_i + \tilde{p}_{i\alpha\rightarrow} \right)$ where $\tilde{p}_{i\alpha\leftarrow}$ and $\tilde{p}_{i\alpha\rightarrow}$ are the $i$th left and right values of fuzzy data points after the alpha-cut process respectively.

### 3 Fuzzy Data Modeling via Interpolation Rational Bezier Surface Function

In this Section 3, we will discuss of how the fuzzy data points had been defined by using fuzzy number concepts and being modeled in surface form by interpolate them by using Bezier surface function. This modeling of fuzzy data points, then called as fuzzy interpolation rational bicubic Bezier surface (FIRBBS). Here, we choose bicubic surface form as we want to illustrate the surface as the rectangular patch.

**Definition 6.** Let $\pi: a = x_0 < x_1 < ... < x_m = b$ be partition of $[a,b]$ and $\lambda: c = y_0 < y_1 < ... < y_n = d$ be partition of $[c,d]$. The fuzzy rational bicubic Bezier function is defined over each rectangular patch $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$, where $i = 0,1,2,...,m-1$; $j = 0,1,2,...,n-1$ as:

$$\tilde{C}(x,y) = \tilde{C}_{i,j}(x,y) = A_i(u)\tilde{F}(i,j)A_j(v),$$

(6)

Where
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\[ \tilde{F}(i, j) = \begin{pmatrix}
\tilde{F}_x(i, j) & \tilde{F}_y(i, j) & \tilde{F}_y(i, j+1)
\end{pmatrix}, \]

\[ A_i(u) = [a_0(u) \quad a_1(u) \quad a_2(u) \quad a_3(u)], \quad A_j(v) = [a_0(v) \quad a_1(v) \quad a_2(v) \quad a_3(v)], \]

with

\[ a_0(u) = \frac{(1-u)^3 + 3u\alpha(1-u)^2}{q(u)}, \quad a_0(v) = \frac{(1-v)^3 + 3v\alpha_j(1-v)^2}{q_j(v)}, \]

\[ a_1(u) = \frac{u^3 + 3u^2\beta(1-u)}{q(u)}, \quad a_1(v) = \frac{v^3 + 3v^2\beta_j(1-v)}{q_j(v)}, \]

\[ a_2(u) = \frac{u(1-u)^2}{q(u)}, \quad a_2(v) = \frac{v(1-v)^2}{q_j(v)}, \]

\[ a_3(u) = \frac{-u^2(1-u)}{q(u)}, \quad a_3(v) = \frac{-v^2(1-v)}{q_j(v)}, \]

\[ \tilde{q}_i(u) = (1-u)^3 + 3u\alpha(1-u)^2 + 3u^2\tilde{\beta}(1-u) + u^3 \]

\[ \tilde{q}_j(v) = (1-v)^3 + 3v\alpha_j(1-v)^2 + 3v^2\tilde{\beta}_j(1-v) + v^3. \]

Substituting the values \( A, \tilde{F}, \) and \( \tilde{A} \) in Eq. 6, then the fuzzy rational bicubic Bezier function \( \tilde{C}(x, y) \) can be expressed as:

\[ \tilde{C}(x, y) = \frac{(1-u)^3 \tilde{\sigma}_{i,j} + 3u(1-u)^2 \tilde{\tau}_{i,j} + 3u^2(1-u)\tilde{\varepsilon}_{i,j} + u^3 \tilde{\kappa}_{i,j}}{(1-u)^3 + 3u\alpha(1-u)^2 + 3u^2\tilde{\beta}(1-u) + u^3} \]

(7)

where
Unfortunately, these fuzzy rational functions are not very useful for fuzzy surface design as any one of the free fuzzy parameter $\tilde{\alpha}_i$, $\tilde{\beta}_i$, $\tilde{\gamma}_i$ and $\tilde{\delta}_i$ applies to the entire network of fuzzy curves. Thus, there is no local control on the fuzzy surface. This ambiguity is overcome by introducing variable fuzzy weights and desired local control has been achieved. For this purpose new free fuzzy parameters $\tilde{\alpha}_{i,j}$, $\tilde{\beta}_{i,j}$, $\tilde{\gamma}_{i,j}$ and $\tilde{\delta}_{i,j}$ are introduced such that:

\[
\tilde{\alpha}_{i,j} = \left(1-v\right)^3 \tilde{F}_{i,j} + 3v\left(1-v\right)^2 \left(\tilde{\alpha}_i F_{i,j} + \tilde{F}_{i,j}^y\right) + 3v^2\left(1-v\right)\left(\tilde{\beta}_i F_{i,j} - \tilde{F}_{i,j}^y\right) + v^3 \tilde{F}_{i,j},
\]

\[
\tilde{\beta}_{i,j} = \left(1-v\right)^3 \left(\tilde{\alpha}_i F_{i,j} + \tilde{F}_{i,j}^y\right) + 3v\left(1-v\right)^2 \left(\tilde{\gamma}_i \left(F_{i,j} - \tilde{F}_{i,j}^y\right) + \tilde{\alpha}_i F_{i,j} + \tilde{F}_{i,j}^y\right) + 3v^2\left(1-v\right)\left(\tilde{\beta}_i \left(F_{i,j} - \tilde{F}_{i,j}^y\right) + \tilde{\gamma}_i \left(F_{i,j} - \tilde{F}_{i,j}^y\right)\right) + v^3 \tilde{F}_{i,j},
\]

\[
\tilde{\gamma}_{i,j} = \left(1-v\right)^3 \left(F_{i,j} - \tilde{F}_{i,j}^y\right) + 3v\left(1-v\right)^2 \left(\tilde{\gamma}_i \left(F_{i,j} - \tilde{F}_{i,j}^y\right) + \tilde{\gamma}_i \left(F_{i,j} - \tilde{F}_{i,j}^y\right)\right) + 3v^2\left(1-v\right)\left(\tilde{\gamma}_i \left(F_{i,j} - \tilde{F}_{i,j}^y\right) + \tilde{\gamma}_i \left(F_{i,j} - \tilde{F}_{i,j}^y\right)\right) + v^3 \tilde{F}_{i,j},
\]

The shape of the fuzzy surface can be modified by assigning different values to these fuzzy parameters. This property of free fuzzy parameters will impose different constraint on $\tilde{\alpha}_{i,j}$, $\tilde{\beta}_{i,j}$, $\tilde{\gamma}_{i,j}$ and $\tilde{\delta}_{i,j}$.

### 3.1 Choices of Derivatives

In most applications, the derivative parameters $d_i$, $F_{i,j}^x$, $F_{i,j}^y$ and $F_{i,j}^z$ are not given and hence must be determined either from given fuzzy data or by some other means. These method are the approximation based on various mathematical theories. An obvious is mentioned here [8-10]:

#### 3.1.1 Fuzzy Arithmetic Mean Method

Fuzzy arithmetic mean method is the three-fuzzy point difference approximation based on arithmetic manipulation. This method is defined as:
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\[
\tilde{F}_{x,0,j} = \tilde{A}_{0,j} + (\tilde{A}_{0,j} - \tilde{A}_{1,j}), \\
\tilde{F}_{x,i,j} = \frac{\tilde{A}_{i,j} + \tilde{A}_{i-1,j}}{2}, \quad i = 1, 2, 3, \ldots, m - 1; \quad j = 0, 1, 2, \ldots, n, \\
\tilde{F}_{x,m,j} = \tilde{A}_{m-1,j} + (\tilde{A}_{m-1,j} - \tilde{A}_{m-2,j}), \\
\tilde{F}_{y,0,j} = \tilde{A}_{i,0} + (\tilde{A}_{i,0} - \tilde{A}_{i,1}), \\
\tilde{F}_{y,i,0} = \frac{\tilde{A}_{i,j} + \tilde{A}_{i,j-1}}{2}, \quad i = 0, 1, 2, \ldots, m; \quad j = 1, 2, 3, \ldots, n - 1, \\
\tilde{F}_{y,i,n} = \tilde{A}_{i,n-1} + (\tilde{A}_{i,n-1} - \tilde{A}_{i,n-2}), \\
\tilde{F}_{y,i,n} = \frac{\tilde{A}_{i,j} + \tilde{A}_{i,j+1}}{2}(\tilde{F}_{i+1,j} - \tilde{F}_{i-1,j}), \quad i = 1, 2, \ldots, m - 1; \quad j = 1, 2, \ldots, n - 1,
\]

where \( \tilde{A}_{i,j} = F_{i+1,j} - F_{i,j} \) and \( \tilde{A}_{i,j} = F_{i,j+1} - F_{i,j} \). Thus, for illustrated of FIRBBS through Eq. 7 which the illustrations are used fuzzy data points is given in Fig. 1.

![Figure 1](a) Uniform FIRBBS (a) with meshes (b) without meshes which interpolate all fuzzy points data. ![Figure 1](b)

In Fig. 1a and 1b, FIRBBS was constructed through Eq. 7, which interpolated all the 16 uniform fuzzy data points. The uniform fuzzy weight values of the surface can change the fuzzy surface shape as shows in Fig. 2.

![Figure 2](a) Uniform fuzzy weight changing the uniform FIRBBS due to fuzzy weight values changing with (a) all fuzzy weights values equal to 6.5 and (b) all fuzzy weight values equal to 0.5. ![Figure 2](b)
Fig. 2 shows that how the uniform fuzzy weight values of uniform FIRBBS can change the uniform fuzzy surface shape. These uniform fuzzy weight values, $\tilde{a}_{i,j}$, $\tilde{b}_{i,j}$, $\tilde{c}_{i,j}$, and $\tilde{d}_{i,j}$, where $\tilde{a}_{i,j} = \tilde{b}_{i,j} = \tilde{c}_{i,j} = \tilde{d}_{i,j} = 6.5$ shows uniform fuzzy surface shape as in Fig. 2a and $\tilde{a}_{i,j} = \tilde{b}_{i,j} = \tilde{c}_{i,j} = \tilde{d}_{i,j} = 0.5$ shows surface shape as in Fig. 2b. In Fig. 1, uniform FIRBBS was constructed through Eq. 7 interpolate all 16 uniform fuzzy points data which also describe the uniform fuzzy data at $z$-axis (fuzzy height). For the operation of alpha-cut toward FIRBBS based on Def. 4, the visualization can be shown through Fig. 3.

**Figure 3.** The uniform alpha-cut operation against uniform FIRBBS.

Fig. 3 illustrated the operation of uniform alpha-cut was applied on uniform fuzzy data which at $z$-axis. The alpha value is 0.5, which means that if the alpha value is tended to 1, then the FIRBBS approached to single surface (crisp surface). Next is the defuzzification process of uniform FIRBBS after fuzzification process has being applied. This defuzzification process is applied to obtain crisp fuzzy solution of FIRBBS. Based on Def. 5, then we obtained the defuzzification surface of FIRBBS which illustrated via Fig. 4.

**Figure 4.** The crisp fuzzy solution surface of FIRBBS after defuzzification process.

Fig. 4 shows that the modeling of crisp fuzzy solution of FIRBBS after the defuzzification process has been applied. The defuzzification surface is exactly the same with the crisp surface because the fuzzy interval of fuzzy data points are symmetry, meaning that the length between crisp data point and left fuzzy data
point is same with the length between crisp data point and right fuzzy data point.

4. Discussion

The construction of FIRBBS became one of the methods in modeling fuzzy data in surface form. This modeling surface used to interpolate the data to give the illustration of that surface where this method is extended from fuzzy interpolation bicubic Bezier surface. The advantage of this method compared with the other method such as fuzzy interpolation bicubic Bezier surface is the additional fuzzy parameter known as fuzzy weights, which can change the shape of the fuzzy surface due to fuzzy weight values.

5. Conclusion

As the conclusion, the FIRBBS was introduced, which can be used in modeling fuzzy data (after defining the uncertainty data by using the fuzzy number) which represent the fuzzy surface. This method also can be used for modeling the non-fuzzy data or exact data. Furthermore, the fuzzy weights as the additional fuzzy parameters which FIRBBS had, bring the advantage to change the fuzzy surface with changing the values of fuzzy weights. Therefore, with this method, the visualizations of these fuzzy data are suitable and this method also can be reduced to fuzzy interpolation bicubic Bezier surface when all fuzzy weight's values are set to 1.

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