Curvilinear Interpolation:

Hyperbola with a Linear Term

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Abstract

This paper illustrates a method for interpolating curvilinear data by means of the hyperbola plus a linear term. The method is based on the least squares principle. It requires a minimum of four curvilinear data and it is easy to apply.

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1. Introduction

A recent paper in This Journal illustrated curvilinear interpolation by means of Eq. (1) [1]. The equation can be expanded by the addition of a linear term as in Eq. (2).

\[ R = A + \frac{B}{x + C} \]  \hspace{1cm} (1)

\[ R = A + (B)x + \frac{C}{x + D} \]  \hspace{1cm} (2)
2. Four-point method based on Equation (2)

The approach based on Eq. (2) requires four curvilinear data. In one respect, it is more versatile than the method based on Eq. (1). The latter method is not suitable for data containing an interior extremum [1]. At least some of these cases can be treated by Eq. (2). Let four trial (x,y) data be (1,2), (2,4), (3,8), (4,16), respectively. Using Eq. (2), the sum of the squared deviations is Eq. (3). See also [1].

\[
\sum = (2 - A - B - C/(D+1))^2 + (4 - A - 2B - C/(D+2))^2 + (8 - A - 3B - C/(D+3))^2 + (16 - A - 4B - C/(D+4))^2 = 0
\]  
(3)

The derivative of \(\sum\) with respect to A is denoted dA, the derivative with respect to B is denoted dB. They appear as Eqs. (4) and (5), respectively. Expressions for dC and dD are developed similarly. The cited derivatives are partial derivatives.

\[
dA = -60 + 8A + 20B + 2C/(D+1) + 2C/(D+2) + 2C/(D+3) + 2C/(D+4) = 0 \quad (4)
\]

\[
dB = -196 + 20A + 60B + 2C/(D+1) + 4C/(D+2) + 6C/(D+3) + 8C/(D+4) = 0 \quad (5)
\]

The equations \{dA, dB, dC\} form a simultaneous set. This set can be solved for symbolic expressions for \{A, B, C\} in terms of D. Substitute \{A, B, C\} into dD. The simplified result is Eq. (6).

\[
(-36)(5D + 12)(D + 7) = 0
\]  
(6)

Solve Eq. (6) for the numerical value of D. In the typical case, Eq. (6) will render several numerical solutions for D. The solution that lies outside the domain of \((-x)\) is the chosen value of D. In this example D = –7. The four-point method, like the three-point method [1], does not apply if all numerical estimates of D lie inside the domain of \((-x)\). The present choice of D also renders A = –16, B = –2, C = –120.

The interpolating equation for the trial data is Eq. (7). It reproduces the four trial data. Evenly-spaced data are not required by Eq. (1) or by Eq. (2). Both equations are useful for more data than their minimum requirements [1].

\[
R = -16 - 2x - 120 / (x - 7)
\]  
(7)

Let the generating function \(u(x)\) be \(\cosh(x)\). Five curvilinear \([x,y]\) points are: 

\([-0.7, u(-0.7)], [-0.4, u(-0.4)], [-0.1, u(-0.1)], [0.2, u(0.2)], [0.5, u(0.5)]\). The interpolating equation is Eq. (8). In 10-digit precision, the sum of the squares of
deviations of Eq. (8) from \( u(x) \), over the range \( x = -0.7 \ldots 0.5 \), is about \( (5)(10^{-7}) \). Plotting Eq. (8) reveals its U-shape like \( \cosh(x) \) when centered at \( x = 0 \).

\[
y = -470.0181 + 15.5635x + 14257.9141 / (30.2705 + x)
\]

Interpolation of an extremum is a desirable feature of Eq. (2). It is not possessed by the three-point hyperbola method in Eq. (1) [1]. Equation (2) applies in the present case but it may not apply to all data containing an interior extremum.

The illustrated four-point method, Eq. (2), is an expansion of the three-point method described in [1]. Equation (2) is invariant under translation of the data within the precision of the calculations.

Let four equidistant data \([x,y]\) be: \([1, u(x)], [2, u(x)], [3, u(x)], [4, u(x)]\). Let \( u(x) = \sinh(x/4) \). A polynomial equation, a double-exponential equation, and Eq. (2) interpolate the data over the range \( x = 1 \ldots 4 \). They are Eqs. (9)-(11), respectively. All numerical coefficients have been rounded.

\[
y = 0.0031541x^3 - 0.0025553x^2 + 0.25407x - 0.0020567
\]

\[
y = (0.64201)(1.28403)^{(x-1)} - (0.38940)(0.77880)^{(x-1)}
\]

\[
y = -0.64304 + 0.16716x - 5.96622 / (x - 9.18982)
\]

Let four trial generating functions be: (1) \( y = \sinh(x/4) \), (2) \( y = \ln(x^{1/2}+1) \), (3) \( y = \exp(1.1x)+100/x \), (4) \( y = 1024/(2^x)+111x \), respectively. Table 1 lists the sums of squared deviations of three interpolating equations from their generating functions over the range \( x = 1 \ldots 4 \). The table illustrates that the cubic polynomial, the double-exponential, and Eq. (2) can render similar or dissimilar results. In the laboratory, the choice of the interpolating equation is typically empirical. The double-exponential equation lacks invariance under data translation.

As the number of data increases the choices for \( D \) proliferate rapidly. This effect complicates the applications of Eq. (2). Some curvilinear data fail to yield an estimate of \( D \) that lies outside the domain of \( (-x) \). In that case, Eq. (2) does not apply. The author has not found a case where more than one real value of \( D \) lies outside the domain of \( (-x) \) provided the data are strictly monotonic-increasing or monotonic-decreasing. However, the possibility of more than one numerical candidate for \( D \) is not excluded. Equation (2) is then ambiguous. The user makes the best choice for his purpose or selects another approach.
The hyperbola-based approaches to curvilinear interpolation apply primarily to small sets of (x,y) data. The limit of eight curvilinear data is suggested in [1]. The merits of the illustrated methods are their simplicity and ease of application [1].

Table 2. Approximate sums of squares of deviations of three interpolating equations from generating functions over the range $x = 1 \ldots 4$. See the text.

<table>
<thead>
<tr>
<th>Function</th>
<th>Cubic polynomial</th>
<th>Double-exponential</th>
<th>Equation (2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sinh(x/4)$</td>
<td>$(1)(10^{-8})$</td>
<td>$(6)(10^{-19})$</td>
<td>$(2)(10^{-7})$</td>
</tr>
<tr>
<td>$\ln(x^{(3/2)} + 1)$</td>
<td>$(6)(10^{-7})$</td>
<td>$(5)(10^{-7})$</td>
<td>$(1)(10^{-8})$</td>
</tr>
<tr>
<td>$\exp(1.1x) + 100/x$</td>
<td>11.7</td>
<td>4.5</td>
<td>11.7</td>
</tr>
<tr>
<td>$1024/2^x + 111x$</td>
<td>4.0</td>
<td>0.35</td>
<td>0.62</td>
</tr>
</tbody>
</table>

References


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