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Two Fixed Point Theorems for Maps on Incomplete G-Metric Spaces

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Abstract. In this paper, we prove two fixed point theorems on incomplete G-metric spaces. Examples are given to show that our results are proper generalizations of main results in [6].

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1. Introduction and preliminaries

In [7], Mustafa and Sims introduced the concept of G-metric spaces as follows.

Defenition 1.1 ([7], Definition 3). Let X be a nonempty set and $G : X \times X \times X \longrightarrow [0, \infty)$ satisfy the following

(G1) $G(x, y, z) = 0$ if $x = y = z$.
(G2) $0 < G(x, x, y)$ for all $x \neq y \in X$.
(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y \neq z \in X$.
(G4) The symmetry on three variables:

$$G(x, y, z) = G(x, z, y) = G(y, x, z) = G(y, z, x) = G(z, x, y) = G(z, y, x)$$

for all $x, y, z \in X$.

(G5) The rectangle inequality: $G(x, y, z) \leq G(x, a, a)+G(a, y, z)$ for all $x, y, z, a \in X$.

Then G is called a *G-metric* on X and the pair (X, G) is called a *G-metric space*.

An interesting work relating to G-metric spaces is to generalize fixed point theorems on metric spaces into this setting. In this way, many results on the fixed point problem of G-metric spaces have been obtained ([1]- [5]), ([7]- [15]). In [6], Mustafa et al have proved the existence of fixed points of maps defined on G-metric space where the completeness is replaced with weaker conditions as follows.

Theorem 1.2 ([6], Theorem 2.1). *Let (X, G) be a G-metric space and $T : X \longrightarrow X$ be a map such that*

1. *$G(Tx, Ty, Tz) \leq a.G(x, Tx, Tx) + b.G(y, Ty, Ty) + c.G(z, Tz, Tz)$ for all $x, y, z \in X$ and $a, b, c \geq 0$ with $0 \leq a + b + c < 1$;*
2. *T is G-continuous at a point $u \in X$;*
3. *There is $x \in X$; $\{T^n x\}$ has a subsequence $\{T^{n_i} x\}$ G-converges to u.*

Then u is the unique fixed point of T.

Theorem 1.3 ([6], Theorem 2.5). *Let (X, G) be a G-metric space and $T : X \longrightarrow X$ be a G-continuous map such that*

1. *$G(Tx, Ty, Tz) \leq k.\{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\}$ for all $x, y, z \in M$ where M is an everywhere dense subset of X with respect the G-metric topology and $0 \leq k < \frac{1}{6}$.*
2. *There exists $x \in X$ such that the sequence $\{T^n x\}$ is G-convergent to some $u \in X$.*

Then u is the unique fixed point of T.

Continuing these results, we prove two fixed point theorems on incomplete G-metric spaces. Examples are given to show that our results are proper generalizations of main results in [6].

First we recall some notions and lemmas.

Defenition 1.4 ([7]). Let (X, G) be a G-metric space and $x_0 \in X$, $r > 0$.

1. The set $B_G(x_0, r) = \{x \in X : G(x_0, x, x) < r\}$ is called a *G-ball with center x_0 and radius r.*
2. The family of all G-balls forms a base of a topology $\tau(G)$ on X, and $\tau(G)$ is called a *G-metric topology.*
3. The sequence $\{x_n\}$ is said to be *G-convergent* to x in X if $x_n \to x$ in the G-metric topology $\tau(G)$.
4. The sequence $\{x_n\}$ is said to be *G-Cauchy* in X if $G(x_n, x_m, x_l) \to 0$ as $m, n, l \to \infty$.
5. (X, G) is called a *complete G-metric* space if every G-Cauchy sequence is G-convergent.

Lemma 1.5 ([7], Proposition 6). *Let (X, G) be a G-metric space. Then the following statements are equivalent.*

1. *x_n is G-convergent to x in X.*
2. *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*
3. *$G(x_n, x, x) \to 0$ as $n \to \infty$.*
4. *$G(x_n, x_m, x) \to 0$ as $n, m \to \infty$.*

Lemma 1.6 ([7], Proposition 7). *Let $T : X \longrightarrow X'$ be a map from a G-metric space (X, G) to a G-metric space (X', G'). Then T is G-continuous at $x \in X$ if and only if T is G-sequentially continuous at x, that is, whenever $\{x_n\}$ is G-convergent to x we have $\{f(x_n)\}$ is G-convergent to $f(x)$.*

Lemma 1.7 ([7], Proposition 8). *Let (X, G) be a G-metric space. Then G is jointly continuous in all three of its variables.*

Lemma 1.8 ([7], Proposition 9). *Let (X, G) be a G-metric space. Then the following statements are equivalent.*

1. *$\{x_n\}$ is a G-Cauchy sequence.*
2. *$G(x_n, x_m, x_m) \to 0$ as $m, n \to \infty$.*

2. Main results

Theorem 2.1. *Let (X, G) be a G-metric space and $T : X \longrightarrow X$ be a map such that*

(A_1) $G(Tx, Ty, Tz) \leq k. \max\big\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\big\}$ *for all $x, y, z \in X$ and some $k \in [0, 1)$;*

(A_2) T is G-continuous at $u \in X$;

(A_3) There exists $x \in X$ such that the sequence $\{T^n x\}$ has a subsequence $\{T^{n_i} x\}$ which is G-convergent to some $u \in X$.

Then u is the unique fixed point of T.

Proof. Since T is G-continuous at u and $\{T^{n_i}x\}$ is G-convergent to u, the sequence $\{T(T^{n_i}x)\}$ is G-convergent to Tu by Lemma 1.6, that is, $\{T^{n_i+1}x\}$ is G-convergent to Tu. We shall prove $Tu = u$. Suppose to the contrary that $Tu \neq u$. Then we get $G(u, Tu, Tu) > 0$. Since $\{T^{n_i}x\}$ is G-convergent to u and $\{T^{n_i+1}x\}$ is G-convergent to Tu, by choosing $\varepsilon = \dfrac{G(u, Tu, Tu)}{2} > 0$ and using Lemma 1.7 there exists $N_1 \in \mathbb{N}$ such that for all $i > N_1$ we have

$$G(T^{n_i}x, T^{n_i+1}x, T^{n_i+1}x) > \varepsilon. \tag{2.1}$$

Otherwise, by using the condition (A_1) we have

$$\begin{aligned} & G(T^{n_i+1}x, T^{n_i+2}x, T^{n_i+2}x) \\ \leq\ & k.\max\big\{G(T^{n_i}x, T^{n_i+1}x, T^{n_i+1}x), G(T^{n_i}x, T^{n_i+1}x, T^{n_i+1}x), \\ & G(T^{n_i+1}x, T^{n_i+2}x, T^{n_i+2}x), G(T^{n_i+1}x, T^{n_i+2}x, T^{n_i+2}x)\big\}. \end{aligned}$$

Since $0 \leq k < 1$, we get $G(T^{n_i+1}x, T^{n_i+2}x, T^{n_i+2}x) \leq k.G(T^{n_i}x, T^{n_i+1}x, T^{n_i+1}x)$.

Now for all $l > j > N_1$, by continuing the above process we obtain

$$\begin{aligned} G(T^{n_l}x, T^{n_l+1}x, T^{n_l+1}x) & \leq k.G(T^{n_l-1}x, T^{n_l}x, T^{n_l}x) \\ & \leq k^2.G(T^{n_l-2}x, T^{n_l-1}x, T^{n_l-1}x) \\ & \leq \ldots \\ & \leq k^{n_l-n_j}.G(T^{n_j}x, T^{n_j+1}x, T^{n_j+1}x). \end{aligned} \tag{2.2}$$

Taking the limit as $l \to \infty$ in (2.2) we get $\lim\limits_{l\to\infty} G(T^{n_l}x, T^{n_l+1}x, T^{n_l+1}x) = 0$. It is a contradiction to (2.1). This proves that $Tu = u$.

Next we prove the uniqueness of the fixed point of T. Let u, v be fixed points of T, that is, $Tu = u$ and $Tv = v$. It follows from the condition (A_1) we have

$$\begin{aligned} G(u, v, v) & = G(Tu, Tv, Tv) \\ & \leq k.\max\big\{G(u, v, v), G(u, Tu, Tu), G(v, Tv, Tv), G(v, Tv, Tv)\big\} \\ & = k.G(u, v, v). \end{aligned}$$

Since $0 \leq k < 1$, we get $G(u, v, v) = 0$. From the condition (G_2) we obtain $u = v$. This proves that the fixed point of T is unique. □

Remark 2.2. For all $x, y, z \in X$ and $a, b, c \geq 0$ with $0 \leq a + b + c < 1$ we have

$$\begin{aligned} & G(Tx, Ty, Tz) \\ \leq\ & aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz) \\ \leq\ & (a + b + c)\max\big\{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\big\} \\ \leq\ & (a + b + c)\max\big\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\big\} \\ =\ & k.\max\big\{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\big\} \end{aligned}$$

with $k = a + b + c \in [0, 1)$. This proves that Theorem 1.2 is a consequence of Theorem 2.1.

The following example shows that Theorem 2.1 is a proper generalization of Theorem 1.2.

Example 2.3. Let $X = [0, 1)$ and $G : X \times X \times X \to \mathbb{R}^+$ be given by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|$$

for all $x, y, z \in X$ and $T : X \to X$ be given by $T(x) = \frac{4}{5}x$ for all $x \in X$.

By [8, Example 1.2] we have (X, G) is a G-metric space. Next we will show that (X, G) is not complete. Indeed, we consider the sequence $\{x_n\}$ where $x_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$ to get

$$\begin{aligned} (2.3) \qquad & G(x_m, x_n, x_l) \\ = \; & |x_m - x_n| + |x_n - x_l| + |x_m - x_l| \\ = \; & |(1 - \frac{1}{m}) - (1 - \frac{1}{n})| + |(1 - \frac{1}{n}) - (1 - \frac{1}{l})| + |(1 - \frac{1}{m}) - (1 - \frac{1}{l})| \\ = \; & |\frac{1}{n} - \frac{1}{m}| + |\frac{1}{l} - \frac{1}{n}| + |\frac{1}{l} - \frac{1}{n}|. \end{aligned}$$

Taking the limit as $m, n, l \to \infty$ in (2.3) we obtain $G(x_m, x_n, x_l) \to 0$. Then $\{x_n\}$ is a G-Cauchy sequence. Suppose to the contrary that $x_n \to x$ in X. Then $G(x, x, x_n) = 2|x - 1 + \frac{1}{n}|$ which is convergent to 0 as $n \to \infty$, that implies $x = 1$. It is a contradiction since $1 \notin X$. Therefore, the sequence $\{x_n\}$ is not G-convergent in X. This proves that (X, G) is not complete.

Next we will show that Theorem 2.1 is applicable to T. For all $x, y, z \in X$ we have

$$G(x, Tx, Tx) = | x - Tx | + | Tx - Tx | + | Tx - x | = \frac{2}{5}x$$

and

$$G(y, Ty, Ty) = \frac{2}{5}y, \quad G(z, Tz, Tz) = \frac{2}{5}z.$$

Therefore,

$$\begin{aligned}G(Tx,Ty,Tz) &= \mid Tx-Ty\mid + \mid Ty-Tz\mid + \mid Tz-Tx\mid \\ &= \mid \frac{4}{5}x-\frac{4}{5}y\mid + \mid \frac{4}{5}y-\frac{4}{5}z\mid + \mid \frac{4}{5}z-\frac{4}{5}x\mid \\ &= \frac{4}{5}(\mid x-y\mid + \mid y-z\mid + \mid z-x\mid) \\ &\leq \frac{9}{10}(\mid x-y\mid + \mid y-z\mid + \mid z-x\mid) \\ &\leq \frac{9}{10}.\max\Big\{\mid x-y\mid + \mid y-z\mid + \mid z-x\mid, \frac{2}{5}x, \frac{2}{5}y, \frac{2}{5}z\Big\} \\ &= k.\max\big\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\big\}\end{aligned}$$

where $k=\dfrac{9}{10}$. This proves that T satisfies the condition (A_1) in Theorem 2.1.

For each $\varepsilon>0$, $\delta=\varepsilon$ and $G(0,y,z)=|0-y|+|y-z|+|0-z|<\delta$ we have

$$\begin{aligned}G(T0,Ty,Tz) &= |0-\frac{4y}{5}|+|\frac{4y}{5}-\frac{4z}{5}|+|0-\frac{4z}{5}| \\ &= \frac{4}{5}(|0-y|+|y-z|+|0-z|)=\frac{4}{5}G(0,y,z) \\ &< \varepsilon.\end{aligned}$$

This proves that T is G-continuous at $0\in X$, that is, T satisfies the condition (A_2) of Theorem 2.1.

By choosing $x=\dfrac{1}{4}\in X$ we have $T^n(\dfrac{1}{4})=\dfrac{1}{4}\Big(\dfrac{4}{5}\Big)^n$ that is G-convergent to $0\in X$. Then T satisfies the condition (A_3) in Theorem 2.1.

Therefore, all assumptions of Theorem 2.1 are satisfied. Then Theorem 2.1 is applicable to T. We see that $x=0$ is the unique fixed point of T.

Now we show that T does not satisfy the condition (1) in Theorem 1.2. For all $x,y,z\in X$ we may assume that $x\geq y\geq z$, then

$$\begin{aligned}(2.4)\qquad G(Tx,Ty,Tz) &= \mid Tx-Ty\mid + \mid Ty-Tz\mid + \mid Tz-Tx\mid \\ &= \mid \frac{4}{5}x-\frac{4}{5}y\mid + \mid \frac{4}{5}y-\frac{4}{5}z\mid + \mid \frac{4}{5}z-\frac{4}{5}x\mid \\ &= \frac{4}{5}(x-y+y-z-z+x)=\frac{8}{5}(x-z)\end{aligned}$$

and

(2.5)

$$\max\big\{G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\big\}=\max\Big\{\frac{2}{5}x, \frac{2}{5}y, \frac{2}{5}z\Big\}=\frac{2}{5}x.$$

If T satisfies the condition (1) in Theorem 1.2, then

$$\begin{aligned} (2.6) \qquad & G(Tx,Ty,Tz) \\ \leq\ & a.G(x,Tx,Tx) + b.G(y,Ty,Ty) + c.G(z,Tz,Tz) \\ \leq\ & (a+b+c)\max\left\{G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\right\} \end{aligned}$$

for all $x, y, z \in X$ and $a, b, c \geq 0$ with $0 \leq a+b+c < 1$. By combining (2.4), (2.5) and (2.6) we have

$$\frac{8}{5}(x-z) \leq (a+b+c)\frac{2}{5}x$$

which is equivalent to

$$4x - 4z \leq (a+b+c)x.$$

This inequality does not hold if $x > y > z = \frac{3}{4}x$ because of $0 \leq (a+b+c) < 1$. That is, the condition (1) in Theorem 1.2 does not hold.

Theorem 2.4. *Let (X,G) be a G-metric space and $T : X \longrightarrow X$ be a G-continuous map such that*

(B_1) $G(Tx,Ty,Tz) \leq k.\max\left\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\right\}$ *for all $x, y, z \in M$ where M is an everywhere dense subset of X with respect the G-metric topology and $0 \leq k < \frac{1}{2}$.*

(B_2) *There exists $x \in X$ such that the sequence $\{T^n x\}$ is G-convergent to some $u \in X$.*

Then u is the unique fixed point of T.

Proof. For all $x, y, z \in X$, since $\overline{M} = X$, there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in M such that $x_n \to x$, $y_n \to y$ and $z_n \to z$. From the condition (G_5) we have

$$\begin{aligned} (2.7) \qquad & G(Tz,Ty,Ty) \\ \leq\ & G(Tz,Tz_n,Tz_n) + G(Tz_n,Ty,Ty) \\ \leq\ & G(Tz,Tz_n,Tz_n) + G(Tz_n,Ty_n,Ty_n) + G(Ty_n,Ty,Ty). \end{aligned}$$

Since $y_n, z_n \in M$ for all $n \in \mathbb{N}$, from the condition (B_1) we have

(2.8)
$$G(Tz_n,Ty_n,Ty_n) \leq k.\max\left\{G(z_n,y_n,y_n), G(z_n,Tz_n,Tz_n), G(y_n,Ty_n,Ty_n)\right\}$$

Using the condition (G_5) again we get

$$\begin{aligned} (2.9) \qquad G(z_n,y_n,y_n) &\leq G(z_n,z,z) + G(z,y_n,y_n) \\ &= G(z_n,z,z) + G(y_n,y_n,z) \\ &\leq G(z_n,z,z) + G(y_n,y_n,y) + G(y,y,z) \\ &\leq G(z_n,z,z) + G(y_n,y_n,y) + G(x,y,z) \end{aligned}$$

and

$$(2.10)\quad \begin{aligned} G(z_n, Tz_n, Tz_n) &\leq G(z_n, z, z) + G(z, Tz_n, Tz_n) \\ &\leq G(z_n, z, z) + G(z, Tz, Tz) + G(Tz, Tz_n, Tz_n) \end{aligned}$$

and

$$(2.11)\quad \begin{aligned} G(y_n, Ty_n, Ty_n) &\leq G(y_n, y, y) + G(y, Ty_n, Ty_n) \\ &\leq G(y_n, y, y) + G(y, Ty, Ty) + G(Ty, Ty_n, Ty_n). \end{aligned}$$

From (2.8), (2.9), (2.10) and (2.11) we have

$$(2.12)\quad \begin{aligned} & G(Tz_n, Ty_n, Ty_n) \\ \leq\ & k. \max\left\{G(z_n, y_n, y_n), G(z_n, Tz_n, Tz_n), G(y_n, Ty_n, Ty_n)\right\} \\ \leq\ & k. \max\left\{G(z_n, z, z) + G(y_n, y_n, y) + G(x, y, z),\right. \\ & G(z_n, z, z) + G(z, Tz, Tz) + G(Tz, Tz_n, Tz_n), \\ & \left. G(y_n, y, y) + G(y, Ty, Ty) + G(Ty, Ty_n, Ty_n)\right\}. \end{aligned}$$

From (2.7) and (2.12) we get

$$(2.13)\quad \begin{aligned} & G(Tz, Ty, Ty) \\ \leq\ & G(Tz, Tz_n, Tz_n) + G(Tz_n, Ty, Ty) \\ \leq\ & G(Tz, Tz_n, Tz_n) + G(Ty_n, Ty, Ty) + G(Tz_n, Ty_n, Ty_n) \\ \leq\ & G(Tz, Tz_n, Tz_n) + G(Ty_n, Ty, Ty) + \\ & +k. \max\left\{G(z_n, z, z) + G(y_n, y_n, y) + G(x, y, z),\right. \\ & G(z_n, z, z) + G(z, Tz, Tz) + G(Ty, Ty_n, Ty_n), \\ & \left. G(y_n, y, y) + G(y, Ty, Ty) + G(Ty, Ty_n, Ty_n)\right\}. \end{aligned}$$

Since T is G-continuous and $y_n \to y, z_n \to z$, by using Lemma 1.6 we have

$$Ty_n \to Ty, Tz_n \to Tz.$$

On the other hand, by using Lemma 1.5 we get

$$y_n \to y,\ z_n \to z,\ Ty_n \to Ty,\ Tz_n \to Tz.$$

This implies that

$$G(y_n, y, y) \to 0,\ G(z_n, z, z) \to 0,\ G(Ty, Ty_n, Ty_n) \to 0,\ G(Tz, Tz_n, Tz_n) \to 0$$

as $n \to \infty$. Therefore, by taking the limit as $n \to \infty$ in (2.13) we get

$$(2.14)\quad G(Tz, Ty, Ty) \leq k. \max\left\{G(x, y, z), G(y, Ty, Ty), G(z, Tz, Tz)\right\}$$

Since $y_n, x_n \in M$ for all $n \in \mathbb{N}$, by a similar argument we obtain

$$(2.15)\quad G(Tx, Ty, Ty) \leq k. \max\left\{G(x, y, z), G(y, Ty, Ty), G(x, Tx, Tx)\right\}.$$

It follows from (G_5), (2.14) and (2.15) we have

$$\begin{aligned} G(Tx,Ty,Tz) &\leq G(Tx,Ty,Ty)+G(Tz,Ty,Ty) \\ &\leq k.\max\big\{G(x,y,z),G(y,Ty,Ty),G(z,Tz,Tz)\big\} \\ &\quad +k.\max\big\{G(x,y,z),G(y,Ty,Ty),G(x,Tx,Tx)\big\} \\ &\leq 2k.\max\big\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\big\}. \end{aligned}$$

Therefore,

$$G(Tx,Ty,Tz)\leq 2k.\max\big\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\big\}$$

for all $x,y,z\in X$ and $0\leq 2k<1$. It follows from Theorem 2.1 that u is a unique fixed point of T. □

Remark 2.5. For all $x,y,z\in M$ and $0\leq k<\dfrac{1}{6}$ we have

$$\begin{aligned} &G(Tx,Ty,Tz) \\ \leq\ & k.\{G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)\} \\ \leq\ & 3k.\max\big\{G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\big\} \\ \leq\ & 3k.\max\big\{G(x,y,z),G(x,Tx,Tx),G(y,Ty,Ty),G(z,Tz,Tz)\big\}. \end{aligned}$$

Since $0\leq k<\dfrac{1}{6}$, we get $0\leq 3k<\dfrac{1}{2}$. Therefore, Theorem 1.3 is a consequence of Theorem 2.4.

The following example shows that Theorem 2.4 is a proper generalization of Theorem 1.3.

Example 2.6. Let $X=[0;1]$, $M=(0;1)$ be an everywhere dense subset of X and $G:X\times X\times X\to\mathbb{R}^+$ be a map given by

$$G(x,y,z)=|x-y|+|y-z|+|z-x|$$

for all $x,y,z\in X$; $T:X\to X$ be a map given by $T(x)=\dfrac{2}{5}x$ for all $x\in X$.

As in Example 2.3 we have (X,G) is a G-metric space. For each $a\in X$ and $\varepsilon>0$ we put $\delta=\varepsilon$. Then for all $x,y\in X$ with

$$G(a,y,z)=|a-y|+|y-z|+|a-z|<\delta$$

we have

$$G(T(a),T(y),T(z))=|\frac{2a}{5}-\frac{2y}{5}|+|\frac{2y}{5}-\frac{2z}{5}|+|\frac{2a}{5}-\frac{2z}{5}|=\frac{2}{5}G(a,y,z)<\varepsilon.$$

This proves that T is G-continuous at a.

Next, for all $x,y,z\in M$ we have

$$G(x,y,z)=\mid x-y\mid+\mid y-z\mid+\mid z-x\mid$$

and

$$G(x,Tx,Tx) = \mid x - Tx \mid + \mid Tx - Tx \mid + \mid Tx - x \mid = \frac{6}{5}x.$$

Similarly we get

$$G(y,Ty,Ty) = \frac{6}{5}y, \text{ and } G(z,Tz,Tz) = \frac{6}{5}z.$$

Therefore, for all $x, y, z \in M$ we have

$$\begin{aligned} G(Tx,Ty,Tz) &= \mid Tx - Ty \mid + \mid Ty - Tz \mid + \mid Tz - Tx \mid \\ &= \mid \frac{2}{5}x - \frac{2}{5}y \mid + \mid \frac{2}{5}y - \frac{2}{5}z \mid + \mid \frac{2}{5}z - \frac{2}{5}x \mid \\ &= \frac{2}{5}(\mid x-y \mid + \mid y-z \mid + \mid z-x \mid) \\ &\leq \frac{7}{15}(\mid x-y \mid + \mid y-z \mid + \mid z-x \mid) \\ &\leq \frac{7}{15}\max\big\{ \mid x-y \mid + \mid y-z \mid + \mid z-x \mid, \frac{6}{5}\mid x \mid, \frac{6}{5}\mid y \mid, \frac{6}{5}\mid z \mid \big\} \\ &= k.\max\big\{G(x,y,z), G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\big\} \end{aligned}$$

where $0 \leq k = \dfrac{7}{15} < \dfrac{1}{2}$. This proves that the condition (B_1) in Theorem 2.4 is satisfied.

By choosing $x = \frac{1}{2} \in X$ we have the sequence $\Big\{T^n\big(\frac{1}{2}\big)\Big\} = \Big\{\frac{1}{2}\big(\frac{2}{5}\big)^n\Big\}$ is G-convergent to $0 \in X$. This proves that T satisfies the condition (B_2) in Theorem 2.4.

Therefore, all assumptions of Theorem 2.4 are satisfied. Then Theorem 2.4 is applicable to T. We see that $x = 0$ is the unique fixed point of T.

Now we prove that T does not satisfy the condition (1) in Theorem 1.3. For all $x, y, z \in M$, we may assume that $x \geq y \geq z$. Then we have

$$\begin{aligned} G(Tx,Ty,Tz) &= \mid Tx - Ty \mid + \mid Ty - Tz \mid + \mid Tz - Tx \mid \\ &= \mid \frac{2}{5}x - \frac{2}{5}y \mid + \mid \frac{2}{5}y - \frac{2}{5}z \mid + \mid \frac{2}{5}z - \frac{2}{5}x \mid \\ &= \frac{2}{5}(x - y + y - z - z + x) \\ &= \frac{4}{5}(x - z). \end{aligned} \tag{2.16}$$

and

(2.17)

$$\max\big\{G(x,Tx,Tx), G(y,Ty,Ty), G(z,Tz,Tz)\big\} = \max\Big\{\frac{6}{5}x, \frac{6}{5}y, \frac{6}{5}z\Big\} = \frac{6}{5}x.$$

Suppose to the contrary that T satisfies the condition (1) in Theorem 1.3. Then we have

$$G(Tx,Ty,Tz) \leq k\{G(x,Tx,Tx)+G(y,Ty,Ty)+G(z,Tz,Tz)\}$$

for all $x,y,z \in M$ and $0 < k < \frac{1}{6}$. Combining (2.16) and (2.17) to give $\frac{4}{5}(x-z) \leq 3k.\frac{6}{5}x$ which is equivalent to $2x-2z \leq 9k.x$. By choosing $x > y > z = \frac{1}{4}x$ we get $k \geq \frac{1}{6}$. It is a contradiction. Therefore, Theorem 1.3 is not applicable to T.

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