Two Fixed Point Theorems for Maps on Incomplete $G$-Metric Spaces

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Abstract. In this paper, we prove two fixed point theorems on incomplete $G$-metric spaces. Examples are given to show that our results are proper generalizations of main results in [6].

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1. Introduction and preliminaries

In [7], Mustafa and Sims introduced the concept of $G$-metric spaces as follows.

**Definition 1.1** ([7], Definition 3). Let $X$ be a nonempty set and $G : X \times X \times X \rightarrow [0, \infty)$ satisfy the following

1. $G(x, y, z) = 0$ if $x = y = z$.
2. $0 < G(x, x, y)$ for all $x \neq y \in X$.
3. $G(x, x, y) \leq G(x, y, z)$ for all $x, y \neq z \in X$.
4. The symmetry on three variables: $G(x, y, z) = G(x, z, y) = G(y, x, z) = G(y, z, x) = G(z, x, y) = G(z, y, x)$ for all $x, y, z \in X$.
5. The rectangle inequality: $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

Then $G$ is called a $G$-metric on $X$ and the pair $(X, G)$ is called a $G$-metric space.

An interesting work relating to $G$-metric spaces is to generalize fixed point theorems on metric spaces into this setting. In this way, many results on the fixed point problem of $G$-metric spaces have been obtained ([1]-[5]), ([7]-[15]). In [6], Mustafa et al have proved the existence of fixed points of maps defined on $G$-metric space where the completeness is replaced with weaker conditions as follows.

**Theorem 1.2** ([6], Theorem 2.1). Let $(X, G)$ be a $G$-metric space and $T : X \rightarrow X$ be a map such that

1. $G(Tx, Ty, Tz) \leq a.G(x, Tx, Tx) + b.G(y, Ty, Ty) + c.G(z, Tz, Tz)$ for all $x, y, z \in X$ and $a, b, c \geq 0$ with $0 \leq a + b + c < 1$;
2. $T$ is $G$-continuous at a point $u \in X$;
3. There is $x \in X$; $\{T^n x\}$ is $G$-convergent to some $u \in X$.

Then $u$ is the unique fixed point of $T$.

**Theorem 1.3** ([6], Theorem 2.5). Let $(X, G)$ be a $G$-continuous map such that

1. $G(Tx, Ty, Tz) \leq k.\{G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz)\}$ for all $x, y, z \in M$ where $M$ is an everywhere dense subset of $X$ with respect the $G$-metric topology and $0 \leq k < \frac{1}{6}$.
2. There exists $x \in X$ such that the sequence $\{T^n x\}$ is $G$-convergent to some $u \in X$.

Then $u$ is the unique fixed point of $T$. 
Continuing these results, we prove two fixed point theorems on incomplete \( G \)-metric spaces. Examples are given to show that our results are proper generalizations of main results in [6].

First we recall some notions and lemmas.

**Definition 1.4** ([7]). Let \((X, G)\) be a \( G \)-metric space and \( x_0 \in X, r > 0 \).

1. The set \( B_G(x_0, r) = \{ x \in X : G(x_0, x, x) < r \} \) is called a \( G \)-ball with center \( x_0 \) and radius \( r \).
2. The family of all \( G \)-balls forms a base of a topology \( \tau(G) \) on \( X \), and \( \tau(G) \) is called a \( G \)-metric topology.
3. The sequence \( \{x_n\} \) is said to be \( G \)-convergent to \( x \) in \( X \) if \( x_n \to x \) in the \( G \)-metric topology \( \tau(G) \).
4. The sequence \( \{x_n\} \) is said to be \( G \)-Cauchy in \( X \) if \( G(x_n, x_m, x_l) \to 0 \) as \( m, n, l \to \infty \).
5. \((X, G)\) is called a complete \( G \)-metric space if every \( G \)-Cauchy sequence is \( G \)-convergent.

**Lemma 1.5** ([7], Proposition 6). Let \((X, G)\) be a \( G \)-metric space. Then the following statements are equivalent.

1. \( x_n \) is \( G \)-convergent to \( x \) in \( X \).
2. \( G(x_n, x_n, x) \to 0 \) as \( n \to \infty \).
3. \( G(x_n, x, x) \to 0 \) as \( n \to \infty \).
4. \( G(x_n, x_m, x) \to 0 \) as \( n, m \to \infty \).

**Lemma 1.6** ([7], Proposition 7). Let \( T : X \to X' \) be a map from a \( G \)-metric space \((X, G)\) to a \( G \)-metric space \((X', G')\). Then \( T \) is \( G \)-continuous at \( x \in X \) if and only if \( T \) is \( G \)-sequentially continuous at \( x \), that is, whenever \( \{x_n\} \) is \( G \)-convergent to \( x \) we have \( \{f(x_n)\} \) is \( G \)-convergent to \( f(x) \).

**Lemma 1.7** ([7], Proposition 8). Let \((X, G)\) be a \( G \)-metric space. Then \( G \) is jointly continuous in all three of its variables.

**Lemma 1.8** ([7], Proposition 9). Let \((X, G)\) be a \( G \)-metric space. Then the following statements are equivalent.

1. \( \{x_n\} \) is a \( G \)-Cauchy sequence.
2. \( G(x_n, x_m, x_m) \to 0 \) as \( m, n \to \infty \).

2. **Main results**

**Theorem 2.1.** Let \((X, G)\) be a \( G \)-metric space and \( T : X \to X \) be a map such that

- \((A_1)\) \( G(Tx, Ty, Tz) \leq k \cdot \max \{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\} \) for all \( x, y, z \in X \) and some \( k \in [0, 1) \);
- \((A_2)\) \( T \) is \( G \)-continuous at \( u \in X \);
- \((A_3)\) There exists \( x \in X \) such that the sequence \( \{T^nx\} \) has a subsequence \( \{T^{n_i}x\} \) which is \( G \)-convergent to some \( u \in X \).
Then $u$ is the unique fixed point of $T$.

Proof. Since $T$ is $G$-continuous at $u$ and $\{T^n x\}$ is $G$-convergent to $u$, the sequence $\{T(T^n x)\}$ is $G$-convergent to $Tu$ by Lemma 1.6, that is, $\{T^{n+1} x\}$ is $G$-convergent to $Tu$. We shall prove $Tu = u$. Suppose to the contrary that $Tu \neq u$. Then we get $G(u, Tu, Tu) > 0$. Since $\{T^n x\}$ is $G$-convergent to $u$ and $\{T^{n+1} x\}$ is $G$-convergent to $Tu$, by choosing $\varepsilon = \frac{G(u, Tu, Tu)}{2} > 0$ and using Lemma 1.7 there exists $N_1 \in \mathbb{N}$ such that for all $i > N_1$ we have

$$G(T^{n_i} x, T^{n_i+1} x, T^{n_i+1} x) > \varepsilon. \quad (2.1)$$

Otherwise, by using the condition $(A_1)$ we have

$$G(T^{n_i+1} x, T^{n_i+2} x, T^{n_i+2} x) \leq k \cdot \max \left\{ G(T^{n_i} x, T^{n_i+1} x, T^{n_i+1} x), G(T^{n_i} x, T^{n_i+1} x, T^{n_i+1} x), G(T^{n_i+1} x, T^{n_i+2} x, T^{n_i+2} x), G(T^{n_i+1} x, T^{n_i+2} x, T^{n_i+2} x) \right\}. $$

Since $0 \leq k < 1$, we get $G(T^{n_i+1} x, T^{n_i+2} x, T^{n_i+2} x) \leq k \cdot G(T^{n_i} x, T^{n_i+1} x, T^{n_i+1} x)$. Now for all $i > N_1$, by continuing the above process we obtain

$$G(T^{n_i} x, T^{n_i+1} x, T^{n_i+1} x) \leq k \cdot G(T^{n_i-1} x, T^{n_i-1} x) \leq k^2 \cdot G(T^{n_i-2} x, T^{n_i-2} x) \leq \ldots \leq k^{n_i-n_j} G(T^{n_j} x, T^{n_j+1} x, T^{n_j+1} x). \quad (2.2)$$

Taking the limit as $l \to \infty$ in (2.2) we get $\lim_{l \to \infty} G(T^{n_i} x, T^{n_i+1} x, T^{n_i+1} x) = 0$. It is a contradiction to (2.1). This proves that $Tu = u$.

Next we prove the uniqueness of the fixed point of $T$. Let $u, v$ be fixed points of $T$, that is, $Tu = u$ and $Tv = v$. It follows from the condition $(A_1)$ we have

$$G(u, v, v) = G(Tu, Tv, Tv) \leq k \cdot \max \left\{ G(u, v, v), G(u, Tu, Tu), G(v, Tv, Tv), G(v, Tv, Tv) \right\} = k \cdot G(u, v, v). $$

Since $0 \leq k < 1$, we get $G(u, v, v) = 0$. From the condition $(G_2)$ we obtain $u = v$. This proves that the fixed point of $T$ is unique. \qed

Remark 2.2. For all $x, y, z \in X$ and $a, b, c \geq 0$ with $0 \leq a + b + c < 1$ we have

$$G(T x, T y, T z) \leq a G(x, T x, T x) + b G(y, T y, T y) + c G(z, T z, T z) \leq (a + b + c) \max \left\{ G(x, T x, T x), G(y, T y, T y), G(z, T z, T z) \right\} \leq (a + b + c) \max \left\{ G(x, y, z), G(x, T x, T x), G(y, T y, T y), G(z, T z, T z) \right\} = k \cdot \max \left\{ G(x, y, z), G(x, T x, T x), G(y, T y, T y), G(z, T z, T z) \right\}$$

where $k = \frac{a}{a + b + c}$.
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with $k = a + b + c \in [0, 1)$. This proves that Theorem 1.2 is a consequence of Theorem 2.1.

The following example shows that Theorem 2.1 is a proper generalization of Theorem 1.2.

Example 2.3. Let $X = [0, 1)$ and $G : X \times X \times X \to \mathbb{R}^+$ be given by

$$G(x, y, z) = |x - y| + |y - z| + |z - x|$$

for all $x, y, z \in X$ and $T : X \to X$ be given by $T(x) = \frac{4}{5}x$ for all $x \in X$.

By [8, Example 1.2] we have $(X, G)$ is a $G$-metric space. Next we will show that $(X, G)$ is not complete. Indeed, we consider the sequence $\{x_n\}$ where $x_n = 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$ to get


g(x_m, x_n, x_l) \\
= |x_m - x_n| + |x_n - x_l| + |x_m - x_l| \\
= |(1 - \frac{1}{m}) - (1 - \frac{1}{n})| + |(1 - \frac{1}{n}) - (1 - \frac{1}{l})| + |(1 - \frac{1}{m}) - (1 - \frac{1}{l})| \\
= \frac{1}{n} - \frac{1}{m} + \frac{1}{l} - \frac{1}{n} + \frac{1}{l} - \frac{1}{n}.

Taking the limit as $m, n, l \to \infty$ in (2.3) we obtain $G(x_m, x_n, x_l) \to 0$. Then $\{x_n\}$ is a $G$-Cauchy sequence. Suppose to the contrary that $x_n \to x$ in $X$.

Then $G(x, x, x_n) = 2|x - 1 + \frac{1}{n}|$ which is convergent to 0 as $n \to \infty$, that implies $x = 1$. It is a contradiction since $1 \notin X$. Therefore, the sequence $\{x_n\}$ is not $G$-convergent in $X$. This proves that $(X, G)$ is not complete.

Next we will show that Theorem 2.1 is applicable to $T$. For all $x, y, z \in X$ we have

$$G(x, Tx, Tx) = |x - Tx| + |Tx - Tx| + |Tx - x| = \frac{2}{5}x$$

and

$$G(y, Ty, Ty) = \frac{2}{5}y, \quad G(z, Tz, Tz) = \frac{2}{5}z.$$
Therefore,

\[ G(Tx, Ty, Tz) = |Tx - Ty| + |Ty - Tz| + |Tz - Tx| \]

\[ = \left| \frac{4}{5}x - \frac{4}{5}y \right| + \left| \frac{4}{5}y - \frac{4}{5}z \right| + \left| \frac{4}{5}z - \frac{4}{5}x \right| \]

\[ = \frac{4}{5}(|x - y| + |y - z| + |z - x|) \]

\[ \leq \frac{9}{10}(|x - y| + |y - z| + |z - x|) \]

\[ \leq \frac{9}{10} \max \left\{ |x - y| + |y - z| + |z - x|, \frac{2}{5}, \frac{2}{5}, \frac{2}{5} \right\} \]

\[ = k. \max \left\{ G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \right\} \]

where \( k = \frac{9}{10} \). This proves that \( T \) satisfies the condition \((A_1)\) in Theorem 2.1.

For each \( \varepsilon > 0, \delta = \varepsilon \) and \( G(0, y, z) = |0 - y| + |y - z| + |0 - z| < \delta \) we have

\[ G(T0, Ty, Tz) = |0 - \frac{4y}{5}| + \left| \frac{4y}{5} - \frac{4z}{5} \right| + |0 - \frac{4z}{5}| \]

\[ = \frac{4}{5}(|0 - y| + |y - z| + |0 - z|) = \frac{4}{5}G(0, y, z) \]

\[ < \varepsilon. \]

This proves that \( T \) is \( G \)-continuous at \( 0 \in X \), that is, \( T \) satisfies the condition \((A_2)\) of Theorem 2.1.

By choosing \( x = \frac{1}{4} \in X \) we have \( T^n(\frac{1}{4}) = \frac{1}{4}(\frac{4}{5})^n \) that is \( G \)-convergent to \( 0 \in X \). Then \( T \) satisfies the condition \((A_3)\) in Theorem 2.1.

Therefore, all assumptions of Theorem 2.1 are satisfied. Then Theorem 2.1 is applicable to \( T \). We see that \( x = 0 \) is the unique fixed point of \( T \).

Now we show that \( T \) does not satisfy the condition \((1)\) in Theorem 1.2. For all \( x, y, z \in X \) we may assume that \( x \geq y \geq z \), then

\[ G(Tx, Ty, Tz) = |Tx - Ty| + |Ty - Tz| + |Tz - Tx| \]

\[ = \left| \frac{4}{5}x - \frac{4}{5}y \right| + \left| \frac{4}{5}y - \frac{4}{5}z \right| + \left| \frac{4}{5}z - \frac{4}{5}x \right| \]

\[ = \frac{4}{5}(x - y + y - z - z + x) = \frac{8}{5}(x - z) \]

and

\[ \max \left\{ G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \right\} = \max \left\{ \frac{2}{5}x, \frac{2}{5}y, \frac{2}{5}z \right\} = \frac{2}{5}x. \]
If $T$ satisfies the condition (1) in Theorem 1.2, then
\begin{equation}
G(Tx, Ty, Tz) \leq aG(x, Tx, Tx) + bG(y, Ty, Ty) + cG(z, Tz, Tz)
\end{equation}
\begin{equation}
\leq (a + b + c) \max \{G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}
\end{equation}
for all $x, y, z \in X$ and $a, b, c \geq 0$ with $0 \leq a + b + c < 1$. By combining (2.4), (2.5) and (2.6) we have
\begin{equation}
\frac{8}{5}(x - z) \leq (a + b + c)\frac{2}{5}x
\end{equation}
which is equivalent to
\begin{equation}
4x - 4z \leq (a + b + c)x.
\end{equation}
This inequality does not hold if $x > y > z = \frac{3}{4}x$ because of $0 \leq (a + b + c) < 1$. That is, the condition (1) in Theorem 1.2 does not hold.

**Theorem 2.4.** Let $(X, G)$ be a $G$-metric space and $T : X \rightarrow X$ be a $G$-continuous map such that
\begin{itemize}
  \item[(B1)] $G(Tx, Ty, Tz) \leq k \cdot \max \{G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz)\}$
  \quad for all $x, y, z \in M$ where $M$ is an everywhere dense subset of $X$ with respect the $G$-metric topology and $0 \leq k < \frac{1}{2}$
  \item[(B2)] There exists $x \in X$ such that the sequence $\{T^n x\}$ is $G$-convergent to some $u \in X$.
\end{itemize}
Then $u$ is the unique fixed point of $T$.

**Proof.** For all $x, y, z \in X$, since $\overline{M} = X$, there exist sequences $\{x_n\}, \{y_n\}, \{z_n\}$ in $M$ such that $x_n \rightarrow x, y_n \rightarrow y$ and $z_n \rightarrow z$. From the condition $(G_5)$ we have
\begin{equation}
G(Tz, Ty, Ty) \leq G(Tz, Tz_n, Tz_n) + G(Tz_n, Ty, Ty)
\leq G(Tz, Tz_n, Tz_n) + G(Tz_n, Ty_n, Ty_n) + G(Ty_n, Ty, Ty).
\end{equation}
Since $y_n, z_n \in M$ for all $n \in \mathbb{N}$, from the condition $(B_1)$ we have
\begin{equation}
G(Tz_n, Ty_n, Ty_n) \leq k \cdot \max \{G(z_n, y_n, y_n), G(z_n, Tz_n, Tz_n), G(y_n, Ty_n, Ty_n)\}
\end{equation}
Using the condition $(G_5)$ again we get
\begin{equation}
G(z_n, y_n, y_n) \leq G(z_n, z, z) + G(z, y_n, y_n)
= G(z_n, z, z) + G(y_n, y_n, z)
\leq G(z_n, z, z) + G(y_n, y_n, y) + G(y, y, z)
\leq G(z_n, z, z) + G(y_n, y_n, y) + G(x, y, z)
\end{equation}
and

\[(2.10) \quad G(z_n, Tz_n, Tz_n) \leq G(z_n, z, z) + G(z, Tz_n, Tz_n) \]

and

\[(2.11) \quad G(y_n, Ty_n, Ty_n) \leq G(y_n, y, y) + G(y, Ty_n, Ty_n) \]

From (2.8), (2.9), (2.10) and (2.11) we have

\[(2.12) \quad G(Tz_n, Ty_n, Ty_n) \leq k \cdot \max \{G(z_n, y_n, y_n), G(z_n, Tz_n, Tz_n), G(y_n, Ty_n, Ty_n)\} \]

From (2.7) and (2.12) we get

\[(2.13) \quad G(Tz, Ty, Ty) \]

Since \(T\) is \(G\)-continuous and \(y_n \to y, z_n \to z\), by using Lemma 1.6 we have

\[Ty_n \to Ty, Tz_n \to Tz.\]

On the other hand, by using Lemma 1.5 we get

\[y_n \to y, z_n \to z, Ty_n \to Ty, Tz_n \to Tz.\]

This implies that

\[G(y_n, y, y) \to 0, G(z_n, z, z) \to 0, G(Ty_n, Ty_n) \to 0, G(Tz_n, Tz_n) \to 0\]

as \(n \to \infty\). Therefore, by taking the limit as \(n \to \infty\) in (2.13) we get

\[(2.14) \quad G(Tz, Ty, Ty) \leq k \cdot \max \{G(x, y, z), G(y, Ty, Ty), G(z, Tz, Tz)\} \]

Since \(y_n, x_n \in M\) for all \(n \in \mathbb{N}\), by a similar argument we obtain

\[(2.15) \quad G(Tx, Ty, Ty) \leq k \cdot \max \{G(x, y, z), G(y, Ty, Ty), G(x, Tx, Tx)\} \].
It follows from \((G_5), (2.14)\) and \((2.15)\) we have
\[
G(Tx, Ty, Tz) \leq G(Tx, Ty, Ty) + G(Tz, Ty, Ty) \\
\leq k \cdot \max \left\{ G(x, y, z), G(y, Ty, Ty), G(z, Tz, Tz) \right\} \\
+ k \cdot \max \left\{ G(x, y, z), G(y, Ty, Ty), G(x, Tx, Tx) \right\} \\
\leq 2k \cdot \max \left\{ G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \right\}.
\]
Therefore,
\[
G(Tx, Ty, Tz) \leq 2k \cdot \max \left\{ G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \right\} \\
\text{for all } x, y, z \in X \text{ and } 0 \leq 2k < 1. \text{ It follows from Theorem 2.1 that } u \text{ is a unique fixed point of } T.
\]

**Remark 2.5.** For all \(x, y, z \in M\) and \(0 \leq k < \frac{1}{6}\) we have
\[
G(Tx, Ty, Tz) \leq k \cdot \max \left\{ G(x, Tz, Tz) + G(y, Ty, Ty) + G(z, Tz, Tz) \right\} \\
\leq 3k \cdot \max \left\{ G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \right\} \\
\leq 3k \cdot \max \left\{ G(x, y, z), G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \right\}.
\]
Since \(0 \leq k < \frac{1}{6}\) we get \(0 \leq 3k < \frac{1}{2}\). Therefore, Theorem 1.3 is a consequence of Theorem 2.4.

The following example shows that Theorem 2.4 is a proper generalization of Theorem 1.3.

**Example 2.6.** Let \(X = [0; 1]\), \(M = (0; 1)\) be an everywhere dense subset of \(X\) and \(G : X \times X \times X \to \mathbb{R}^+\) be a map given by
\[
G(x, y, z) = |x - y| + |y - z| + |z - x|
\]
for all \(x, y, z \in X\); \(T : X \to X\) be a map given by \(T(x) = \frac{2}{5}x\) for all \(x \in X\).

As in Example 2.3 we have \((X, G)\) is a \(G\)-metric space. For each \(a \in X\) and \(\varepsilon > 0\) we put \(\delta = \varepsilon\). Then for all \(x, y \in X\) with
\[
G(a, y, z) = |a - y| + |y - z| + |a - z| < \delta
\]
we have
\[
G(T(a), T(y), T(z)) = \left| \frac{2a}{5} - \frac{2y}{5} \right| + \left| \frac{2y}{5} - \frac{2z}{5} \right| + \left| \frac{2a}{5} - \frac{2z}{5} \right| = \frac{2}{5} G(a, y, z) < \varepsilon.
\]
This proves that \(T\) is \(G\)-continuous at \(a\).

Next, for all \(x, y, z \in M\) we have
\[
G(x, y, z) = |x - y| + |y - z| + |z - x|
\]
\[ G(x, Tx, Tx) = |x - Tx| + |Tx - Tx| + |Tx - x| = \frac{6}{5}x. \]

Similarly we get
\[ G(y, Ty, Ty) = \frac{6}{5}y, \text{ and } G(z, Tz, Tz) = \frac{6}{5}z. \]

Therefore, for all \( x, y, z \in M \) we have
\[ G(Tx, Ty, Tz) = |Tx - Ty| + |Ty - Tz| + |Tz - Tx| = \frac{6}{5}
\]
\[ \frac{2}{5}x - \frac{2}{5}y \]
\[ \frac{2}{5}y - \frac{2}{5}z \]
\[ \frac{2}{5}z - \frac{2}{5}x \]
\[ = \frac{2}{5} (|x - y| + |y - z| + |z - x|) \]
\[ \leq \frac{7}{15} (|x - y| + |y - z| + |z - x|) \]
\[ \leq \frac{7}{15} \max \left\{ \frac{6}{5} |x|, \frac{6}{5} |y|, \frac{6}{5} |z| \right\} \]
\[ = k. \max \{ G(x, T x, T x), G(y, T y, T y), G(z, T z, T z) \} \]

where \( 0 \leq k = \frac{7}{15} < \frac{1}{2} \). This proves that the condition \((B_1)\) in Theorem 2.4 is satisfied.

By choosing \( x = \frac{1}{2} \in X \) we have the sequence \( \left\{ T^n \left( \frac{1}{2} \right) \right\} = \left\{ \frac{1}{2} \left( \frac{2}{3} \right)^n \right\} \) is G-convergent to \( 0 \in X \). This proves that \( T \) satisfies the condition \((B_2)\) in Theorem 2.4.

Therefore, all assumptions of Theorem 2.4 are satisfied. Then Theorem 2.4 is applicable to \( T \). We see that \( x = 0 \) is the unique fixed point of \( T \).

Now we prove that \( T \) does not satisfy the condition (1) in Theorem 1.3. For all \( x, y, z \in M \), we may assume that \( x \geq y \geq z \). Then we have
\[
(2.16) \quad G(Tx, Ty, Tz) = |Tx - Ty| + |Ty - Tz| + |Tz - Tx| = \frac{2}{5}x - \frac{2}{5}y + \frac{2}{5}y - \frac{2}{5}z + \frac{2}{5}z - \frac{2}{5}x = \frac{2}{5}(x - y + y - z - z + x) = \frac{4}{5}(x - z).
\]

and
\[
(2.17) \quad \max \{ G(x, Tx, Tx), G(y, Ty, Ty), G(z, Tz, Tz) \} = \max \left\{ \frac{6}{5}x, \frac{6}{5}y, \frac{6}{5}z \right\} = \frac{6}{5}x.
\]
Two fixed point theorems for maps on incomplete $G$-metric spaces

Suppose to the contrary that $T$ satisfies the condition (1) in Theorem 1.3. Then we have

$$G(Tx, Ty, Tz) \leq k \{ G(x, Tx, Tx) + G(y, Ty, Ty) + G(z, Tz, Tz) \}$$

for all $x, y, z \in M$ and $0 < k < \frac{1}{6}$. Combining (2.16) and (2.17) to give

$$\frac{4}{5}(x - z) \leq 3k\frac{6}{5}x$$

which is equivalent to $2x - 2z \leq 9k.x$. By choosing $x > y > z = \frac{1}{4}x$ we get $k \geq \frac{1}{6}$. It is a contradiction. Therefore, Theorem 1.3 is not applicable to $T$.

REFERENCES

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