On the Inclined Curves in Galilean 4-Space

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Abstract

In this paper, we give some characterizations for an inclined curve by
means of curvatures of a curve in a 4-dimensional Galilean space.

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1 Introduction

In Euclidean 3-space $R^3$ a inclined curve or general helix is a curve where the
tangent lines make a constant angle with a fixed direction. A inclined curve is
characterized by the fact that the ratio $\tau/\kappa$ is constant along the curve, where
$\kappa$ and $\tau$ denote the curvature and the torsion, respectively. In Minkowski 3-
space $R^3_1$ and Galilean 3-space $G^3$, one defines a inclined curve in $R^3_1$ and $G^3$
by a similar way. Although different expansions of the Frenet formula appear
depending on the form of the Frenet vectors, in many cases, a inclined curve is
characterized by the constancy of the function $\tau/\kappa$ again ([2],[5]) Recently, the
study of a inclined curve in 4-dimensional space $R^4$, $R^4_1$ and $Q^4$, quaternionic
space, examined by [3], [4, 6] and [8], respectively.

In this paper, we investigate the properties of a inclined curve in terms of
the curvatures of a curve in Galilean 4-space $G^4$. 
2 Preliminary Notes

Let \( \alpha : I \subset R \to G^4 \) be an arbitrary curve in a 4-dimensional Galilean space \( G^4 \) defined by
\[
\alpha(t) = (x(t), y(t), z(t), w(t)),
\]
where \( x(t), y(t), z(t), w(t) \) are smooth functions.

For any vectors \( \mathbf{x} = (x_1, y_1, z_1, w_1) \) and \( \mathbf{y} = (x_2, y_2, z_2, w_2) \) in Galilean space \( G^4 \) the Galilean norm of a vector \( \mathbf{x} \) is defined by
\[
||\mathbf{x}|| = \begin{cases} x_1, & \text{if } x_1 \neq 0 \\ \sqrt{y_1^2 + z_1^2 + w_1^2}, & \text{if } x_1 = 0. \end{cases}
\]

From this, the Galilean cross product on \( G^4 \) be the standard basis vectors.

On the other hand, a curve \( \alpha \) in \( G^4 \), parameterized by arc-length \( t = s \), given in coordinate form
\[
\alpha(s) = (s, y(s), z(s), w(s)).
\]

It follows that the tangent vector of \( \alpha \) is given by
\[
\mathbf{t} = \alpha'(s) = (1, y'(s), z'(s), w'(s)).
\]

From this we obtain the first curvature \( k_1 \) as follows:
\[
k_1(s) = ||\mathbf{t}'(s)|| = \sqrt{(y''(s))^2 + (z''(s))^2 + (w''(s))^2}.
\]

By the similar arguments as those of in the Euclidean differential geometry, we have the following the Frenet vectors \( \{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\} \) of \( \alpha(s) \) in \( G_4 \):
\[
\begin{align*}
\mathbf{t}(s) &= (1, y'(s), z'(s), w'(s)), \\
\mathbf{n}_1(s) &= \frac{1}{k_1(s)}(0, y''(s), z''(s), w''(s)), \\
\mathbf{n}_2(s) &= \frac{1}{k_2(s)} \left( 0, \left( \frac{1}{k_1(s)} y''(s) \right)', \left( \frac{1}{k_1(s)} z''(s) \right)', \left( \frac{1}{k_1(s)} w''(s) \right)' \right), \\
\mathbf{n}_3(s) &= \varepsilon \mathbf{t} \wedge \mathbf{n}_1 \wedge \mathbf{n}_2,
\end{align*}
\]
where \( \varepsilon \) is taken \( \pm 1 \) to make \( +1 \) the determinant \( |\mathbf{t}n_1n_2n_3| \) and \( k_2(s) = ||\mathbf{n}_1'(s)|| \), it is called the second curvature of \( \alpha(s) \).

For their derivatives the following Frenet formula satisfies ([cf. 7])

\[
\begin{align*}
\mathbf{t}'(s) &= k_1(s)\mathbf{n}_1(s), \\
\mathbf{n}_1'(s) &= k_2(s)\mathbf{n}_2(s), \\
\mathbf{n}_2'(s) &= -k_2(s)\mathbf{n}_1(s) + k_3(s)\mathbf{n}_3(s), \\
\mathbf{n}_3'(s) &= -k_3(s)\mathbf{n}_2(s).
\end{align*}
\]

(2)

Here \( k_3 = \langle \mathbf{n}_2', \mathbf{n}_3 \rangle \), it said to be the third curvature of \( \alpha(s) \).

### 3 Characterizations of inclined curves in \( G^4 \)

**Theorem 3.1** Let \( \alpha = \alpha(s) \) be a unit speed curve in \( G^4 \) with non-zero curvatures \( k_1(s), k_2(s) \) and \( k_3(s) \). Then \( \alpha \) is a inclined curve in \( G^4 \) if and only if the function

\[
\left( \frac{k_1}{k_2} \right)^2 + \frac{1}{k_3^2} \left( \left( \frac{k_1}{k_2} \right)' \right)^2 = \text{constant}.
\]

(3)

**Proof.** Let \( \alpha(s) \) be a inclined curve in \( G^4 \) and the axis of the curve \( \alpha(s) \) be the unit vector \( \mathbf{u} \). Then, we have

\[
\langle \mathbf{t}, \mathbf{u} \rangle = \text{constant}
\]

(4)

along the curve \( \alpha \). By differentiating (4) with respect to \( s \) and using the Frenet formula (2) we have \( \langle k_1\mathbf{n}_1, \mathbf{u} \rangle = 0 \), which implies that the unit vector \( \mathbf{u} \) is in the subspace \( \text{span}\{\mathbf{t}, \mathbf{n}_2, \mathbf{n}_3\} \) and can be written as follows

\[
\mathbf{u} = a_1(s)\mathbf{t}(s) + a_2(s)\mathbf{n}_2(s) + a_3(s)\mathbf{n}_3(s),
\]

(5)

where

\[
a_1(s) = \langle \mathbf{t}, \mathbf{u} \rangle = \text{constant}, \quad a_2(s) = \langle \mathbf{n}_2, \mathbf{u} \rangle, \quad a_3(s) = \langle \mathbf{n}_3, \mathbf{u} \rangle, \quad a_1^2 + a_2^2 + a_3^2 = 1.
\]

The differentiation of (5) gives

\[
(a_1k_1 - a_2k_2)\mathbf{n}_1 + (a_2' - a_3k_3)\mathbf{n}_2 + (a_3' + a_2k_3)\mathbf{n}_3 = 0.
\]

Since the vectors \( \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \) are linearly independent, we yield

\[
a_1k_1 - a_2k_2 = 0, \quad a_2' - a_3k_3 = 0, \quad a_3' + a_2k_3 = 0,
\]

that is,

\[
a_2 = \frac{k_1}{k_2}a_1 = -\frac{1}{k_3}a_3', \quad a_2' = a_3k_3.
\]

(6)
Therefore, from (6) we have the ODE for \( a_3 \) as follows
\[
a'_3 - \frac{k_1'}{k_3} a'_3 + k_3^2 a_3 = 0. \tag{7}
\]
If we change variables in (7) as \( t = \int_0^s k_3 ds \), then (7) becomes
\[
\frac{d^2 a_3}{dt^2} + a_3 = 0. \tag{8}
\]
Solving this differential equation, we get the solution
\[
a_3 = A \cos t(s) + B \sin t(s), \tag{9}
\]
for some constants \( A \) and \( B \). From the first equation of (6) and (9) we find

\[
a_2 = \frac{k_1 a_1}{k_2} = A \sin t(s) - B \cos t(s), \quad a_3 = \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' a_1 = A \cos t(s) + B \sin t(s).
\]

From the above equations we obtain
\[
A = a_1 \left( \frac{k_1}{k_2} \sin t(s) + \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' \cos t(s) \right),
\]
\[
B = a_1 \left( \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' \sin t(s) - \frac{k_1}{k_2} \cos t(s) \right),
\]
which imply
\[
A^2 + B^2 = a_1^2 \left( \left( \frac{k_1}{k_2} \right)^2 + \frac{1}{k_3^2} \left( \left( \frac{k_1}{k_2} \right)' \right)^2 \right).
\]
Thus, we have
\[
\left( \frac{k_1}{k_2} \right)^2 + \frac{1}{k_3^2} \left( \left( \frac{k_1}{k_2} \right)' \right)^2 = \text{constant}. \tag{10}
\]

Conversely, if (4) holds, then we can always find a constant unit vector \( u \) satisfying \( \langle t, u \rangle = \text{constant} \). We consider the unit vector defined by
\[
u = t + \frac{k_1}{k_2} n_2 + \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)' n_3.
\]
Differentiation of \( u \) with the help of (10) gives \( u' = 0 \), this mean that \( u \) is a constant vector. Consequently, the curve \( \alpha(s) \) is a inclined curve in \( G^4 \).

**Theorem 3.2** A unit speed curve \( \alpha(s) \) in \( G^4 \) is a inclined curve if and only if there exists a \( C^2 \)-function \( f \) such that
\[
k_3 f(s) = \frac{d}{ds} \left( \frac{k_1}{k_2} \right), \quad \frac{d}{ds} f(s) = -k_3 \left( \frac{k_1}{k_2} \right). \tag{11}
\]
**Proof.** We assume that $\gamma$ is a inclined curve. Differentiation of (10) gives
\[
\left( \frac{k_1}{k_2} \right) \left( \frac{k_1}{k_2} \right)' + \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)'' \left( -\frac{k_3'}{k_3^2} + \frac{1}{k_3} \left( \frac{k_1}{k_2} \right)'' \right) = 0,
\]
or equivalently,
\[
\left( \frac{k_1}{k_2} \right) \frac{d}{ds} \left( \frac{k_1}{k_2} \right) + \frac{1}{k_3} \frac{d}{ds} \left( \frac{k_1}{k_2} \right) \frac{d}{ds} \left( \frac{1}{k_3} \frac{d}{ds} \left( \frac{k_1}{k_2} \right) \right) = 0. \tag{12}
\]
Therefore, we have
\[
\frac{1}{k_3} \frac{d}{ds} \left( \frac{k_1}{k_2} \right) = -\frac{\left( \frac{k_1}{k_2} \right) \frac{d}{ds} \left( \frac{k_1}{k_2} \right)}{\frac{d}{ds} \left( \frac{1}{k_3} \frac{d}{ds} \left( \frac{k_1}{k_2} \right) \right)}. \tag{13}
\]
If we define $f = f(s)$ by
\[
f(s) = -\frac{\left( \frac{k_1}{k_2} \right) \frac{d}{ds} \left( \frac{k_1}{k_2} \right)}{\frac{d}{ds} \left( \frac{1}{k_3} \frac{d}{ds} \left( \frac{k_1}{k_2} \right) \right)},
\]
then (13) becomes
\[
k_3 f(s) = \frac{d}{ds} \left( \frac{k_1}{k_2} \right). \tag{14}
\]
From (13) it can be written
\[
\frac{d}{ds} \left( \frac{1}{k_3} \frac{d}{ds} \left( \frac{k_1}{k_2} \right) \right) = -k_3 \left( \frac{k_1}{k_2} \right) \tag{15}
\]
By combining (14) and (15), we find
\[
\frac{d}{ds} f(s) = -k_3 \left( \frac{k_1}{k_2} \right). \tag{16}
\]

Conversely, if (11) holds, we define a unit constant vector $u$ by
\[
u = t + \frac{k_1}{k_2} n_2 + f(s) n_3.
\]
It follows $\langle t, u \rangle = 1$. Thus, $\alpha$ is a inclined curve.

**Theorem 3.3** Let $\alpha$ be a unit speed curve in $G^4$. Then $\alpha$ is a inclined curve if and only if the following condition holds;
\[
\frac{k_1}{k_2} = C_1 \cos t + C_2 \sin t, \tag{17}
\]
where $C_1, C_2$ are constants and $t(s) = \int_0^s k_3 ds$. 

Proof. Suppose that $\alpha$ is a inclined curve. By using Theorem 3.1, let define the $C^2$-function $t(s)$ and the $C^1$-functions $m(s)$ and $n(s)$ by

$$t(s) = \int_0^s k_3 ds,$$

$$m(s) = \frac{k_1}{k_2} \cos t - f(s) \sin t,$$

$$n(s) = \frac{k_1}{k_2} \sin t + f(s) \cos t. \quad (19)$$

If we differentiate equation (19) with respect to $s$ and take account of (18), (14) and (16), we have $m' = 0$ and $n' = 0$. Therefore, $m = C_1$ and $n = C_2$ are constants. Thus, from (19) we obtain

$$\frac{k_1}{k_2} = C_1 \cos t + C_2 \sin t.$$

Conversely, if the equation (17) holds. Then from (19) we have

$$f = -C_1 \cos t + C_2 \sin t,$$

it satisfies the condition of Theorem 3.3. Thus, $\alpha$ is a inclined curve in $G^4$.

References


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