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On the Inclined Curves in Galilean 4-Space

Dae Won Yoon

Department of Mathematics Education and RINS
Gyeongsang National University, Jinju 660-701, South Korea
dwyoon@gnu.ac.kr

Abstract

In this paper, we give some characterizations for a inclined curve by means of curvatures of a curve in a 4-dimensional Galilean space.

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1 Introduction

In Euclidean 3-space R^3 a inclined curve or general helix is a curve where the tangent lines make a constant angle with a fixed direction. A inclined curve is characterized by the fact that the ratio τ/κ is constant along the curve, where κ and τ denote the curvature and the torsion, respectively. In Minkowski 3-space R^3_1 and Galilean 3-space G^3, one defines a inclined curve in R^3_1 and G^3 by a similar way. Although different expansions of the Frenet formula appear depending on the form of the Frenet vectors, in many cases, a inclined curve is characterized by the constancy of the function τ/κ again ([2],[5]) Recently, the study of a inclined curve in 4-dimensional space R^4, R^4_1 and Q^4, quaternionic space, examined by [3], [4, 6] and [8], respectively.

In this paper, we investigate the properties of a inclined curve in terms of the curvatures of a curve in Galilean 4-space G^4.

2 Preliminary Notes

Let $\alpha : I \subset R \to G^4$ be an arbitrary curve in a 4-dimensional Galilean space G^4 defined by

$$\alpha(t) = (x(t), y(t), z(t), w(t)),$$

where $x(t), y(t), z(t), w(t)$ are smooth functions.
For any vectors $\mathbf{x} = (x_1, y_1, z_1, w_1)$ and $\mathbf{y} = (x_2, y_2, z_2, w_2)$ in Galilean space G^4 the Galilean scalar product of $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \begin{cases} x_1 x_2, & \text{if } x_1 \neq 0 \text{ or } x_2 \neq 0 \\ y_1 y_2 + z_1 z_2 + w_1 w_2, & \text{if } x_1 = 0 \text{ and } x_2 = 0. \end{cases}$$

From this, the Galilean norm of a vector $\mathbf{x}$ in G^4 is given by

$$||\mathbf{x}|| = \begin{cases} x_1, & \text{if } x_1 \neq 0 \\ \sqrt{y_1^2 + z_1^2 + w_1^2}, & \text{if } x_1 = 0. \end{cases}$$

Let $\mathbf{x} = (x_1, y_1, z_1, w_1)$, $\mathbf{y} = (x_2, y_2, z_2, w_2)$ and $\mathbf{z} = (x_3, y_3, z_3, w_3)$ be the vectors in G^4. Then the Galilean cross product on G^4 defined by

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \begin{vmatrix} 0 & e_2 & e_3 & e_4 \\ x_1 & y_1 & z_1 & w_1 \\ x_2 & y_2 & z_2 & w_2 \\ x_3 & y_3 & z_3 & w_3 \end{vmatrix},$$

where e_i are the standard basis vectors.

On the other hand, a curve α in G^4, parameterized by arc-length $t = s$, given in coordinate form

$$\alpha(s) = (s, y(s), z(s), w(s)). \tag{1}$$

It follows that the tangent vector of α is given by

$$\mathbf{t} = \alpha'(s) = (1, y'(s), z'(s), w'(s)).$$

From this we obtain the first curvature k_1 as follows:

$$k_1(s) = ||\mathbf{t}'(s)|| = \sqrt{(y'')^2 + (z'')^2 + (w'')^2}.$$

By the similar arguments as those of in the Euclidean differential geometry, we have the following the Frenet vectors $\{\mathbf{t}, \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ of $\alpha(s)$ in G_4:

$$\begin{aligned} &\mathbf{t}(s) = (1, y'(s), z'(s), w'(s)), \\ &\mathbf{n}_1(s) = \tfrac{1}{k_1(s)}(0, y''(s), z''(s), w''(s)), \\ &\mathbf{n}_2(s) = \tfrac{1}{k_2(s)}\left(0, \left(\tfrac{1}{k_1(s)} y''(s)\right)', \left(\tfrac{1}{k_1(s)} z''(s)\right)', \left(\tfrac{1}{k_1(s)} w''(s)\right)'\right), \\ &\mathbf{n}_3(s) = \varepsilon \mathbf{t} \wedge \mathbf{n}_1 \wedge \mathbf{n}_2, \end{aligned}$$

where ε is taken ± 1 to make $+1$ the determinant $|\mathbf{t}\mathbf{n}_1\mathbf{n}_2\mathbf{n}_3|$ and $k_2(s) = ||\mathbf{n}_1'(s)||$, it is called the second curvature of $\alpha(s)$.

For their derivatives the following Frenet formula satisfies([cf. 7])

$$\begin{aligned} \mathbf{t}'(s) &= k_1(s)\mathbf{n}_1(s), \\ \mathbf{n}_1'(s) &= k_2(s)\mathbf{n}_2(s), \\ \mathbf{n}_2'(s) &= -k_2(s)\mathbf{n}_1(s) + k_3(s)\mathbf{n}_3(s), \\ \mathbf{n}_3'(s) &= -k_3(s)\mathbf{n}_2(s). \end{aligned} \tag{2}$$

Here $k_3 = \langle \mathbf{n}_2', \mathbf{n}_3 \rangle$, it said to be the third curvature of $\alpha(s)$.

3 Characterizations of inclined curves in G^4

Theorem 3.1 *Let $\alpha = \alpha(s)$ be a unit speed curve in G^4 with non-zero curvatures $k_1(s), k_2(s))$ and $k_3(s)$. Then α is a inclined curve in G^4 if and only if the function*

$$\left(\frac{k_1}{k_2}\right)^2 + \frac{1}{k_3^2}\left(\left(\frac{k_1}{k_2}\right)'\right)^2 \tag{3}$$

is a constant.

Proof. Let $\alpha(s)$ be a inclined curve in G^4 and the axis of the curve $\alpha(s)$ be the unit vector $\mathbf{u}$. Then, we have

$$\langle \mathbf{t}, \mathbf{u} \rangle = \text{constant} \tag{4}$$

along the curve α. By differentiating (4) with respect to s and using the Frenet formula (2) we have $\langle k_1\mathbf{n}_1, \mathbf{u} \rangle = 0$, which implies that the unit vector $\mathbf{u}$ is in the subspace span$\{\mathbf{t}, \mathbf{n}_2, \mathbf{n}_3\}$ and can be written as follows

$$\mathbf{u} = a_1(s)\mathbf{t}(s) + a_2(s)\mathbf{n}_2(s) + a_3(s)\mathbf{n}_3(s), \tag{5}$$

where

$$a_1(s) = \langle \mathbf{t}, \mathbf{u} \rangle = \text{constant},\ a_2(s) = \langle \mathbf{n}_2, \mathbf{u} \rangle,\ a_3(s) = \langle \mathbf{n}_3, \mathbf{u} \rangle, a_1^2 + a_2^2 + a_3^2 = 1.$$

The differentiation of (5) gives

$$(a_1k_1 - a_2k_2)\mathbf{n}_1 + (a_2' - a_3k_3)\mathbf{n}_2 + (a_3' + a_2k_3)\mathbf{n}_3 = 0.$$

Since the vectors $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are linearly independent, we yield

$$a_1k_1 - a_2k_2 = 0,\ a_2' - a_3k_3 = 0,\ a_3' + a_2k_3 = 0,$$

that is,

$$a_2 = \frac{k_1}{k_2}a_1 = -\frac{1}{k_3}a_3', \quad a_2' = a_3k_3. \tag{6}$$

Therefore, from (6) we have the ODE for a_3 as follows

$$a_3'' - \frac{k_3'}{k_3}a_3' + k_3^2 a_3 = 0. \tag{7}$$

If we change variables in (7) as $t = \int_0^s k_3 ds$, then (7) becomes

$$\frac{d^2 a_3}{dt^2} + a_3 = 0. \tag{8}$$

Solving this differential equation, we get the solution

$$a_3 = A\cos t(s) + B\sin t(s), \tag{9}$$

for some constants A and B. From the first equation of (6) and (9) we find

$$a_2 = \frac{k_1 a_1}{k_2} = A\sin t(s) - B\cos t(s),\ a_3 = \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)' a_1 = A\cos t(s) + B\sin t(s).$$

From the above equations we obtain

$$\begin{aligned} A &= a_1\left(\tfrac{k_1}{k_2}\sin t(s) + \tfrac{1}{k_3}\left(\tfrac{k_1}{k_2}\right)'\cos t(s)\right), \\ B &= a_1\left(\tfrac{1}{k_3}\left(\tfrac{k_1}{k_2}\right)'\sin t(s) - \tfrac{k_1}{k_2}\cos t(s)\right), \end{aligned}$$

which imply

$$A^2 + B^2 = a_1^2\left(\left(\frac{k_1}{k_2}\right)^2 + \frac{1}{k_3^2}\left(\left(\frac{k_1}{k_2}\right)'\right)^2\right).$$

Thus, we have

$$\left(\frac{k_1}{k_2}\right)^2 + \frac{1}{k_3^2}\left(\left(\frac{k_1}{k_2}\right)'\right)^2 = \text{constant}. \tag{10}$$

Conversely, if (4) holds, then we can always find a constant unit vector $\mathbf{u}$ satisfying $\langle \mathbf{t}, \mathbf{u}\rangle$=constant. We consider the unit vector defined by

$$\mathbf{u} = \mathbf{t} + \frac{k_1}{k_2}\mathbf{n}_2 + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\mathbf{n}_3.$$

Differentiation of $\mathbf{u}$ with the help of (10) gives $\mathbf{u}' = 0$, this mean that $\mathbf{u}$ is a constant vector. Consequently, the curve $\alpha(s)$ is a inclined curve in G^4.

Theorem 3.2 *A unit speed curve $\alpha(s)$ in G^4 is a inclined curve if and only if there exists a C^2-function f such that*

$$k_3 f(s) = \frac{d}{ds}\left(\frac{k_1}{k_2}\right), \quad \frac{d}{ds}f(s) = -k_3\left(\frac{k_1}{k_2}\right). \tag{11}$$

Proof. We assume that γ is a inclined curve. Differentiation of (10) gives

$$\left(\frac{k_1}{k_2}\right)\left(\frac{k_1}{k_2}\right)' + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)'\left(-\frac{k_3'}{k_3^2} + \frac{1}{k_3}\left(\frac{k_1}{k_2}\right)''\right) = 0,$$

or equivalently,

$$\left(\frac{k_1}{k_2}\right)\frac{d}{ds}\left(\frac{k_1}{k_2}\right) + \frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\frac{d}{ds}\left(\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right) = 0. \tag{12}$$

Therefore, we have

$$\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right) = -\frac{\left(\frac{k_1}{k_2}\right)\frac{d}{ds}\left(\frac{k_1}{k_2}\right)}{\frac{d}{ds}\left(\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right)}. \tag{13}$$

If we define $f = f(s)$ by

$$f(s) = -\frac{\left(\frac{k_1}{k_2}\right)\frac{d}{ds}\left(\frac{k_1}{k_2}\right)}{\frac{d}{ds}\left(\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right)},$$

then (13) becomes

$$k_3 f(s) = \frac{d}{ds}\left(\frac{k_1}{k_2}\right). \tag{14}$$

From (13) it can be written

$$\frac{d}{ds}\left(\frac{1}{k_3}\frac{d}{ds}\left(\frac{k_1}{k_2}\right)\right) = -k_3\left(\frac{k_1}{k_2}\right). \tag{15}$$

By combining (14) and (15), we find

$$\frac{d}{ds}f(s) = -k_3\left(\frac{k_1}{k_2}\right). \tag{16}$$

Conversely, if (11) holds, we define a unit constant vector $\mathbf{u}$ by

$$\mathbf{u} = \mathbf{t} + \frac{k_1}{k_2}\mathbf{n}_2 + f(s)\mathbf{n}_3.$$

It follows $\langle \mathbf{t}, \mathbf{u}\rangle = 1$. Thus, α is a inclined curve.

Theorem 3.3 *Let α be a unit speed curve in G^4. Then α is a inclined curve if and only if the following condition holds;*

$$\frac{k_1}{k_2} = C_1 \cos t + C_2 \sin t, \tag{17}$$

where C_1, C_2 are constants and $t(s) = \int_0^s k_3 ds$.

Proof. Suppose that α is a inclined curve. By using Theorem 3.1, let define the C^2-function $t(s)$ and the C^1-functions $m(s)$ and $n(s)$ by

$$t(s) = \int_0^s k_3 ds, \tag{18}$$

$$\begin{aligned} m(s) &= \tfrac{k_1}{k_2}\cos t - f(s)\sin t, \\ n(s) &= \tfrac{k_1}{k_2}\sin t + f(s)\cos t. \end{aligned} \tag{19}$$

If we differentiate equation (19) with respect to s and take account of (18), (14) and (16), we have $m' = 0$ and $n' = 0$. Therefore, $m = C_1$ and $n = C_2$ are constants. Thus, from (19) we obtain

$$\frac{k_1}{k_2} = C_1 \cos t + C_2 \sin t.$$

Conversely, if the equation (17) holds. Then from (19) we have

$$f = -C_1 \cos t + C_2 \sin t,$$

it satisfies the condition of Theorem 3.3. Thus, α is a inclined curve in G^4.

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