Analysis to the Solutions of Abel’s
Differential Equation of the First Kind
under the Transformation $y = u(x)z(x) + v(x)$

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Abstract

In this work we study different analytical solutions which can be obtained from a new Abel equation of first kind, under the transformation
\( y = u(x)z(x) + v(x) \), changing the variable to \( z(x) \), where the coefficients of this equation allow the construction of a system of auxiliary equation with \( \phi_1(x) \), \( \phi_2(x) \) and \( \phi_3(x) \) as free functions to the system. From the form of the system, different cases are obtained, whose details are described in this work.

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**Keywords:** First kind Abel differential equation, Mapping, change of variables and analytical solutions

### 1 Introduction

Day by day there exists a need for solving solutions to Abel’s differential equations of first and second kind, since they frequently appear in different areas of knowledge, such as big picture modeling in oceanic circulation [1], in problems of magnetostatics [7, 6], control theory [10], cosmology [5], Fluids [9] and M-theory [16], just to name a few. There exist a great number of works which propose solutions and methods for solving these equation: see for example reference [2] which transform the general equation to a reduced form and proves that only a few particular functions are solutions of this equation. The work [8] shows a way to generate the general solution to Abel’s equations of first kind, once a particular solution. The article [4] expands the Abel’s equation in terms of Chevyshev polynomials giving a system soluble in matrix form. The paper [11] uses a certain type of parametrization and some particular results from Riccati’s equation, to get analytical solutions. It should be noted that the transformation used is similar to what we propose, the difference is that the proposal used by him is a product of functions where one of them is a composition, which differs from our proposal which also contains an additional term. Panayotounakos and Zarmpoutis give implicit solutions using first kind Bessel’s functions and second kind Newmann functions for the canonical form of Abel’s equation of first kind [12]. Salinas-Hernández et. al. get analitycal solutions for Abel’s differential equation of second kind using functional transformations [14]. The references [15, 3] use computational algorithms to solve such equation. It is worth to mention that in all these references above, where analytical solutions are given. These solutions are always restricted, since, as we know, the general solution to Abel’s differential equations is an open problem. Therefore, every interesting result is expected to enrich the wealth of solutions to this type of differential equations, as it is very well summarized in [13]. In this work we propose that it is possible to get the solution to Abel’s equation, under particular considerations on some functions \( \phi_1 \), \( \phi_2 \) and \( \phi_3 \) of an auxiliary system of equations. Such functions are constructed
from the coefficients in Abel’s equation. This system involves the introduction of functions \( u, v \) to be determined, and also \( f_0 \), the free function from the original equation. With no more preamble, we initiate the work describing the fundamental part of the proposal.

2 Construction of the functions \( \phi_1, \phi_2 \) and \( \phi_3 \)

From Abel’s equation of first kind

\[
y' = f_3 y'^3 + f_2 y'^2 + f_1 y + f_0, \tag{1}
\]

let \( y = uz + v \), where \( u \) and \( v \) are parameters to be determined. Upon substitution in the above equation, and after a few algebra, we get

\[
z' = u^2 f_3 z'^3 + (3 f_3 v + f_2) u z^2 + \\
+ (3 f_3 v^2 + 2 f_2 v + f_1 - \frac{u'}{u}) z + \\
+ \frac{f_0 - v' + f_3 v^3 + f_2 v^2 + f_1 v}{u}, \tag{2}
\]

or in the form

\[
z' = u^2 f_3 [z'^3 + \frac{1}{u f_3} (3 f_3 v + f_2) z^2 + \\
+ \frac{1}{u^2 f_3} (3 f_3 v^2 + 2 f_2 v + f_1 - \frac{u'}{u}) z + \\
+ \frac{f_0 - v' + f_3 v^3 + f_2 v^2 + f_1 v}{f_3 u^3}]. \tag{3}
\]

We assume that following is satisfied

\[
\frac{1}{u f_3} (3 f_3 v + f_2) = \phi_1(x), \tag{4}
\]

\[
\frac{1}{u^2 f_3} (3 f_3 v^2 + 2 f_2 v + f_1 - \frac{u'}{u}) = \phi_2(x), \tag{5}
\]

\[
\frac{f_0 - v' + f_3 v^3 + f_2 v^2 + f_1 v}{f_3 u^3} = \phi_3(x). \tag{6}
\]

From the above, we can analyze different results obtained from concrete values for the functions \( \phi_1(x), \phi_2(x) \) and \( \phi_3(x) \). It is worth the effort to analyze the following three peculiar situations.
3 Abel equation of first kind in canonic form

Let’s consider the case in which the system (4,5,6) satisfies $\phi_1(x) = \phi_2(x) = 0$, $\phi_3(x) = \Phi(x)$. Obviously the corresponding system that We get is

\[
\frac{1}{uf_3}(3f_3v + f_2) = 0, 
\]

(7)

\[
\frac{1}{u^2 f_3}(3f_3v^2 + 2f_2v + f_1 - \frac{u'}{u}) = 0, 
\]

(8)

\[
\frac{f_0 - v' + f_3v^3 + f_2v^2 + f_1v}{f_3v^3} = \Phi(x). 
\]

(9)

Solving for $v$ in (7) and substituting on (8) and (9) the above system reduces to

\[
\frac{-f_2^2}{3f_3} + f_1 - \frac{u'}{u} = 0, 
\]

(10)

\[
f_0 - \frac{f_0 f_2}{3f_3} + \frac{2f_2^3}{3f_3^2} + \frac{1}{3} \frac{d}{dx}(\frac{f_2}{f_3}) = \Phi(x). 
\]

(11)

Upon solving for $u$ in (10) gives

\[
u(x) = e^{\int(f_1 - \frac{f_2^2}{3f_3})dx}. 
\]

(12)

With this result, equation (2) reduces to

\[
z' = u^2f_3(z^3 + \Phi). 
\]

(13)

Now, in order to solve this last equation, let $\xi = \int f_3u^2dx$, from which $z' = \xi'_x z'_x$. Moreover $\xi'_x = u^2f_3$, and equation (13) reduces to the well-known canonic form [13] for Abel’s equation of first kind

\[
z'_x(\xi) = z^3(\xi) + \Phi(\xi). 
\]

(14)

Even though the handbook does not show any solution for such equation, we propose two alternatives which allow analytic solutions.

3.1 First proposal of solution

Let’s take the canonic equation (14)

\[
z' = z^3 + \Phi. 
\]

Now, try the following simple trick: add and substract the term $\varphi z$ in which
φ is a free function, that is
\[ z' = z^3 + \varphi z - \varphi z + \Phi, \]
which also can be expressed as
\[ z' - z^3 - \varphi z = -\varphi z + \Phi. \]

Let us set the right-hand side to zero, which generates naturally the following system
\[
\begin{align*}
    z' - z^3 - \varphi z &= 0, \\
    -\varphi z + \Phi &= 0.
\end{align*}
\] (15)

Upon solving for \( z \) and \( \Phi \), gives
\[
\begin{align*}
    z(\xi) &= \frac{e^\int \varphi d\xi}{\sqrt{C - \int e^2 \int \varphi d\xi d\xi}}, \\
    \Phi(\xi) &= \frac{\varphi e^\int \varphi d\xi}{\sqrt{C - \int e^2 \int \varphi d\xi d\xi}}. \tag{16}
\end{align*}
\]

In other words, from the given differential equation
\[
    z' = z^3 + \frac{\varphi e^\int \varphi d\xi}{\sqrt{C - \int e^2 \int \varphi d\xi d\xi}}, \tag{17}
\]
the corresponding solution is
\[
    z(\xi) = \frac{e^\int \varphi d\xi}{\sqrt{C - \int e^2 \int \varphi d\xi d\xi}}. \tag{18}
\]

Now, using (12), (18) and \( v = -\frac{f_2}{3f_3} \) in \( y = uz + v \), it follows that
\[
    y = u(x) \frac{e^\int \varphi d\xi}{\sqrt{C - \int e^2 \int \varphi d\xi d\xi}} - \frac{f_2}{3f_3}. \tag{19}
\]
So of (9) and (16)

\[ \Phi(\xi) = \frac{\varphi e^{\int \varphi d\xi}}{\sqrt{C - \int e^{2 \int \varphi d\xi}} d\xi} = \]

\[ = \frac{1}{f_3 u^3} \left[ f_0 - \frac{f_1 f_2}{3 f_3} + \frac{2 f_2^3}{27 f_3^3} + \frac{1}{3} \frac{d}{dx} \frac{f_2}{f_3} \right]. \tag{20} \]

Let us now consider a few interesting cases, remembering that \( u(x) = e^{f(1 - \frac{f_2^2}{3 f_3})} \) and \( \xi = \int f_3 u^2 dx \). It is important to point that some results already have been reported, see [12] for example, for Abel equation of first kind in the canonical form.

### 3.1.1 Caso 1

If in (19) we let \( \varphi(\xi) = 0 \), then it reduces to

\[ y = \frac{u}{\sqrt{C - \int d\xi}} - \frac{f_2}{3 f_3}, \tag{21} \]

that is

\[ y = \frac{e^{f(1 - \frac{f_2^2}{3 f_3})}}{\sqrt{C - \int f_3 e^{2 f(1 - \frac{f_2^2}{3 f_3})} dx}} - \frac{f_2}{3 f_3}, \tag{22} \]

which corresponds to the solution published in [13].

### 3.1.2 Case 2

Let \( \varphi = k \), where \( k \) is a constant, then from (19), we get

\[ y = u \frac{e^{k \xi}}{\sqrt{C - \int e^{2k \xi} d\xi}} - \frac{f_2}{3 f_3}, \]

\[ y = e^{f(1 - \frac{f_2^2}{3 f_3})} \frac{e^{k \int f_3 e^{2 f(1 - \frac{f_2^2}{3 f_3})} dx}}{\sqrt{C - \int e^{2k \int f_3 e^{2f(1 - \frac{f_2^2}{3 f_3})} dx} dx}} + \]

\[ - \frac{f_2}{3 f_3}, \tag{23} \]
3.2 Second proposal for solution

From the canonic equation (14)

$$z' = z^3 + \Phi.$$  \hfill (24)

Let us make the following trick: add and subtract now \(\varphi z^3\) where \(\varphi\), again is a free function, that is

$$z' = z^3 + \varphi z^3 - \varphi z^3 + \Phi,$$  \hfill (25)

which can also be expressed as

$$z' - z^3 - \varphi z^3 = -\varphi z^3 + \Phi.$$  \hfill (26)

Let us impose that the right side is zero, which generates naturally the system of equations

$$z' - z^3 - \varphi z^3 = 0,$$
$$-\varphi z^3 + \Phi = 0.$$  \hfill (27)

Upon solving for \(z\) and \(\varphi\), gives

$$z = \frac{1}{\sqrt{C - 2 \int (1 + \varphi) d\xi}},$$
$$\Phi = \frac{\varphi}{[C - 2 \int (1 + \varphi) d\xi]^2}.$$  \hfill (28)

Using (28) and \(v = \frac{z}{y}\) in \(y = uz + v\), we get

$$y = u(x) \frac{1}{\sqrt{C - 2 \int (1 + \varphi) d\xi}} - \frac{f_2}{3f_3}.$$  \hfill (29)
then, from (9) and (28)

\[ \Phi = \frac{\varphi}{[C - 2 \int (1 + \varphi) d\xi]^{\frac{3}{2}}} = \]

\[ = \frac{1}{f_3 u^3} \left[ f_0 - \frac{f_1 f_2}{3 f_3} + \frac{2 f_2^3}{27 f_3^2} + \frac{1}{3} \frac{d}{dx} f_2 \right]. \]  

(30)

Let us now consider some cases which are obtained from the solution found from (14), which are obviously analytic. These are shown below.

3.2.1 Case 1

Now take \( \varphi(\xi) = 0 \), which implies that (29) reduces to

\[ y = u(x) \frac{1}{\sqrt{C - 2 \int d\xi}} - \frac{f_2}{3 f_3} = u(x) \frac{1}{\sqrt{C - 2 \xi}} - \frac{f_2}{3 f_3}. \]  

(31)

Then, since \( \xi = \int f_3 u^2 dx \) and \( u(x) = e^{f(f_1 - \frac{f_2^2}{3 f_3})} \), gives

\[ y = \frac{e^{f(f_1 - \frac{f_2^2}{3 f_3})}}{\sqrt{C - 2 \int f_3 e^{2 f(f_1 - \frac{f_2^2}{3 f_3})} dx}} - \frac{f_2}{3 f_3}, \]  

(32)

corresponding again to the result in [13].

3.2.2 Case 2

Now if we let \( \varphi(\xi) = \xi \), and using that \( \xi = \int f_3 u^2 dx \) and \( u(x) = e^{f(f_1 - \frac{f_2^2}{3 f_3})} \), then (29) can be expressed as

\[ y = \frac{u(x)}{\sqrt{C - 2 \xi - \xi^2}} - \frac{f_2}{3 f_3} \]

\[ y = \frac{e^{f(f_1 - \frac{f_2^2}{3 f_3})}}{\sqrt{C - 2 \int f_3 e^{2 f(f_1 - \frac{f_2^2}{3 f_3})} dx - \left[ \int f_3 e^{2 f(f_1 - \frac{f_2^2}{3 f_3})} dx \right]^2}} - \frac{f_2}{3 f_3}. \]  

(33)
4 Total separation of variables

We must note that the total separation of variables in Abel’s equation (1) refers to the solution of an algebraic equation of third degree with variable coefficients, still assuming that analytic solution where found, the resulting integrals are impossible to get. In this section we propose a first approach to a solution upon demanding a particular condition to the equation (3), which allows the separation in point form: It is important to note that with this method we are not solving Abel’s equation with constant coefficients (1).

The studied here considers conditions for the free function that satisfy \( \phi_1(x) = \kappa_1, \phi_2(x) = \kappa_2 \) and \( \phi_3(x) = \kappa_3 \) with \( \kappa_1, \kappa_2, \kappa_3 \in \mathbb{R} \) constants. Then, the corresponding system is

\[
\begin{align*}
(3f_3v + f_2) &= \kappa_1, \\
\frac{1}{f_3u^2}(3f_3u^2 + 2f_2v + f_1 - \frac{v'}{u}) &= \kappa_2, \\
\frac{1}{f_3w^3}(f_0 - v' + f_3v^3 + f_2v^2 + f_1v) &= \kappa_3.
\end{align*}
\]

From equation (34) we get

\[
v = \frac{(\kappa_1 - f_2)}{3f_3},
\]

Upon substitution in (35) and (36) we obtain

\[
\begin{align*}
F_{\kappa_1,\kappa_2} &= \frac{e^{f\left(\frac{x^3}{f_3}+f_1-\frac{x^2}{f_3}\right)}dx}{\sqrt{C - 2\kappa_2 \int f_3e^{f\left(\frac{x^3}{f_3}+f_1-\frac{x^2}{f_3}\right)}dx}} ,
\end{align*}
\]

Then, equation (3) reduces to

\[
z' = u^3_{\kappa_1,\kappa_2}f_3[z^3 + \kappa_1z^2 + \kappa_2z + \kappa_3] = u^3_{\kappa_1,\kappa_2}f_3[(z - a)(z - b)(z - c)],
\]

where the values \( a, b \) and \( c \) are the roots of the cubic equation \( x^3 + \kappa_1x^2 + \kappa_2x + \kappa_3 = 0 \) which are given by Cardano-Tartaglia. In our case

\[
x^3 + a_1x^2 + b_1x + c_1 = 0,
\]
with the transformation
\[ x = w - \frac{a_1}{3}, \] (42)
we get another cubic equation
\[ w^3 + pw + q = 0, \] (43)
with solutions
\[ w_1 = 2\sqrt{-\frac{p}{3}} \cos\left(\frac{\theta}{3}\right), \] (44)
\[ w_2 = 2\sqrt{-\frac{p}{3}} \cos\left(60^\circ - \frac{\theta}{3}\right), \] (45)
\[ w_3 = 2\sqrt{-\frac{p}{3}} \cos\left(60^\circ + \frac{\theta}{3}\right). \] (46)
where
\[ p = b_1 - \frac{a_1^2}{3}, \quad q = c_1 + \frac{2a_1^3}{27} - \frac{b_1a_1}{3}, \] (47)
and
\[ \theta = \arccos\left(\frac{\sqrt{27}q}{2p\sqrt{-p}}\right). \] (48)
In our case we have the solutions give by
\[ a = z_1 = w_1 - \frac{\kappa_1}{3}, \]
\[ b = z_2 = w_2 - \frac{\kappa_1}{3}, \]
\[ c = z_3 = w_3 - \frac{\kappa_1}{3}, \] (49)
and
\[ p = \kappa_2 - \frac{\kappa_1^2}{3}, \quad q = \frac{2\kappa_1^3}{27} - \frac{\kappa_1\kappa_2}{3} + \kappa_3. \] (50)
Afterwards, making the separation of variables of (40)
\[ \int \frac{dz}{(z - a)(z - b)(z - c)} = \int u_{\kappa_1, \kappa_2}^2 f_3 dx, \] (51)
applying partial fractions
\[ \frac{1}{(z - a)(z - b)(z - c)} = \]
\[ = \frac{A}{(z - a)} + \frac{B}{(z - b)} + \frac{C}{(z - c)}, \] (52)
and after some rearrangements
\[
\ln \left[ (z - a) \frac{1}{(a-b)(a-c)} (z - b) \frac{1}{(b-a)(b-c)} \times \right. \\
\times (z - c) \frac{1}{(c-a)(c-b)} \right] = \int u_{\kappa_1, \kappa_2}^2 f_3 dx,
\]
(53)
or also in the form
\[
(z - a) \frac{1}{(a-b)(a-c)} (z - b) \frac{1}{(b-a)(b-c)} \times \\
\times (z - c) \frac{1}{(c-a)(c-b)} = \tau_1 e^f u_{\kappa_1, \kappa_2}^2 f_3 dx,
\]
(54)
with \( \tau_1 \) constant, then the solutions are given by
\[
y = u_{\kappa_1, \kappa_2} z - \frac{f_2}{3 f_3},
\]
(55)
where \( u_{\kappa_1, \kappa_2} \) and \( z \) are given by (38) and (54) for especific values of \( \beta \) and \( \gamma \). This result is general, in addition, we can’t find easily particular results, so it will be important to analyze some particular cases for specific values of \( k_1, k_2, k_3 \).

Case 1

In this case we choose conveniently \( \kappa_1 = -(\alpha_1 + \beta_1 + \gamma_1), \kappa_2 = (\alpha \beta + \alpha \gamma + \beta \gamma) \) and \( \kappa_3 = \alpha_1 \beta_1 \gamma_1 \) with \( \alpha_1, \beta_1, \gamma_1 \in \mathbb{Q} \). Then, the formula (37), (38) and (39) are reduced to
\[
v = -\frac{(\alpha_1 + \beta_1 + \gamma_1) + f_2}{3 f_3},
\]
(56)
\[
u_{\alpha_1, \beta_1, \gamma_1} = \frac{e \frac{f((\alpha_1+\beta_1+\gamma_1)^2 + f_1 - \frac{f_2}{3 f_3})}{3 f_3}}{W_{\alpha_1, \beta_1, \gamma_1}(x)},
\]
(57)
\[
f_{\alpha_1, \beta_1, \gamma_1} = \frac{\alpha_1 \beta_1 \gamma_1}{3 f_3} \frac{e \frac{f((\alpha_1+\beta_1+\gamma_1)^2 + f_1 - \frac{f_2}{3 f_3})}{3 f_3}}{3 f_3} + \\
\frac{4 f_1 [(\alpha_1+\beta_1+\gamma_1) + f_2]}{3 f_3} - [\frac{2 f_2 + (\alpha_1+\beta_1+\gamma_1)^3}{3 f_3}] \\
\frac{1}{3} \frac{d}{dx} f_3 + (\alpha \beta + \alpha \gamma + \beta \gamma) \frac{d}{dx} \frac{1}{3 f_3} \frac{(\alpha_1+\beta_1+\gamma_1) f_2}{3 f_3},
\]
(58)
where
\[
W_{\alpha_1, \beta_1, \gamma_1}(x) = \\
C - 2(\alpha \beta + \alpha \gamma + \beta \gamma) \int f_3 \times \\
\times e^f \frac{f((\alpha_1+\beta_1+\gamma_1)^2 + f_1 - \frac{f_2}{3 f_3})}{3 f_3} dx.
\]
(59)
So the equation (3) reduces to

\[ z' = u_{\alpha_1, \beta_1, \gamma_1}^3 \left[ z^3 - (\alpha_1 + \beta_1 + \gamma_1)z^2 + (\alpha_1 \beta_1 + \alpha_1 \gamma_1 + \beta_1 \gamma_1)z + \alpha_1 \beta_1 \gamma_1 \right]. \quad (60) \]

Afterwards, making the separation of variables

\[ \int \frac{dz}{(z - \alpha_1)(z - \beta_1)(z - \gamma_1)} = \int u_{\alpha_1, \gamma_1, \beta_1}^2 f_3 dx, \quad (61) \]

applying partial fractions

\[ \frac{1}{(z - \alpha_1)(z - \beta_1)(z - \gamma_1)} = \frac{A}{(z - \alpha_1)} + \frac{B}{(z - \beta_1)} + \frac{C}{(z - \gamma_1)}, \quad (62) \]

and after some rearrangements

\[ \ln \left[ (z - \alpha_1)^{\alpha_1 - \beta_1}(z - \beta_1)^{\beta_1 - \gamma_1}(z - \gamma_1)^{\gamma_1 - \alpha_1} \times (z - \gamma_1)^{\gamma_1 - \alpha_1} \right] = \int u_{\alpha_1, \gamma_1, \beta_1}^2 f_3 dx, \quad (63) \]

or also in the form

\[ (z - \alpha_1)^{\alpha_1 - \beta_1}(z - \beta_1)^{\beta_1 - \gamma_1}(z - \gamma_1)^{\gamma_1 - \alpha_1} \times (z - \gamma_1)^{\gamma_1 - \alpha_1}(z - \beta_1)^{\beta_1 - \gamma_1} = \tau_2 e^{\int u_{\alpha_1, \gamma_1, \beta_1}^2 f_3 dx}, \quad (64) \]

with \( \tau_2 \) constant, so the solutions are given by

\[ y = u_{\alpha_1, \beta_1, \gamma_1}^3 z - \frac{f_2}{3 f_3}, \quad (65) \]

where \( u_{\alpha_1, \beta_1, \gamma_1} \) is given by (57) and \( z \) is given by (64) for specific values of \( \alpha_1, \beta_1, \gamma_1 \) and the condition \( \alpha_1 \neq \beta_1 \neq \gamma_1 \). Here we illustrate the method with two particular examples.

**Example 1** let \( \alpha_1 = 0, \beta_1 = 1, \gamma_1 = 1 \), so (64) we obtain that

\[ z^{\frac{1}{2}} (z - 2)^{\frac{1}{2}} (z - 1)^{-1} = \lambda_1 e^{\int u_{[0, 2, 1]}^2 f_3 dx} \quad (66) \]

whose solution to \( z \) is

\[ z = -1 \pm \sqrt{1 - \frac{\lambda_1^2 e^{2 \int u_{[0, 2, 1]}^2 f_3 dx}}{1 - \lambda_1^2 e^{2 \int u_{[0, 2, 1]}^2 f_3 dx}}} \quad (67) \]
Now according to (65)

\[ y = u_{(0,2,1)} \left[ -1 \pm \sqrt{1 - \frac{\lambda^2 e^2 \int u^2_{(0,2,1)} f_3 \, dx}{1 - \lambda^2 e^2 \int u^2_{(0,2,1)} f_3 \, dx}} \right] - \frac{f_2}{3f_3}, \tag{68} \]

with \( u_{(0,2,1)} \) given by

\[ u_{(0,2,1)} = \frac{e^{\int (\frac{3}{f_3} + f_1 - \frac{f_2^2}{f_3}) \, dx}}{\sqrt{C - 4 \int f_3 e^2 \int (\frac{3}{f_3} + f_1 - \frac{f_2^2}{f_3}) \, dx \, dx}}, \tag{69} \]

and

\[ f_{0_{(0,2,1)}} = \frac{f_1(3 + f_2)}{3f_3} - \frac{(2f_3^2 + 27)}{27f_3^2} - \frac{1}{3} \frac{df_2}{dx} \frac{f_2}{f_3} + \frac{2}{3} \frac{df_2^2}{dx} \frac{f_3}{f_3}. \tag{70} \]

**Case 2**

Now we choose \( \kappa_1 = 0, \kappa_2 = -(\beta_2 + \gamma_2)^2 + \beta_2 \gamma_2 \) and \( \kappa_3 = \beta_2 \gamma_2 (\beta_2 + \gamma_2) \) with \( \gamma_2, \beta_2 \in \mathbb{Q} \). Then, the formula (37), (38) and (39) are reduced to

\[ v = \frac{f_2}{3f_3}. \tag{71} \]

\[ u_{\gamma_2, \beta_2} = \frac{e^{\int (f_1 - \frac{f_2^2}{f_3}) \, dx}}{\sqrt{C - 2[(- (\beta_2 + \gamma_2)^2 + \beta_2 \gamma_2)] \int f_3 e^2 \int (\frac{3}{f_3} + f_1 - \frac{f_2^2}{f_3}) \, dx \, dx}}. \tag{72} \]

\[ f_{\gamma_2, \beta_2} = \frac{\beta_2 \gamma_2 (\beta_2 + \gamma_2) e^{\int (f_1 - \frac{f_2^2}{f_3}) \, dx}}{|C - 2((- (\beta_2 + \gamma_2)^2 + \beta_2 \gamma_2) \int f_3 e^2 \int (\frac{3}{f_3} + f_1 - \frac{f_2^2}{f_3}) \, dx \, dx)|^{1/2}} + \frac{f_1 f_2}{3f_3} - \frac{2f_2^3}{27f_3^2} - \frac{1}{3} \frac{df_2}{dx} \frac{f_2}{f_3}. \tag{73} \]

So the equation (3) reduces to

\[ z' = u_{\gamma_2, \beta_2}^2 f_3 [z^3 + (\beta_2 \gamma_2 - (\beta_2 + \gamma_2)^2)z + \beta_2 \gamma_2 (\beta_2 + \gamma_2)]. \tag{74} \]

Afterwards, making the separation of variables

\[ \int \frac{dz}{(z + \beta_2 + \gamma_2)(z - \beta_2)(z - \gamma_2)} = \int u_{\gamma_2, \beta_2}^2 f_3 \, dx, \tag{75} \]
applying partial fractions

\[
\frac{1}{(z + \beta_2 + \gamma)(z - \beta_2)(z - \gamma_2)} = \frac{A}{(z + \beta_2 + \gamma_2)} + \frac{B}{(z - \beta_2)} + \frac{C}{(z - \gamma_2)},
\]

and after some rearrangements

\[
\ln[(z + \beta_2 + \gamma_2)^{(2\beta_2 + \gamma_2)(2\gamma_2 + \beta_2)}(z - \beta_2)^{(2\beta_2 + \gamma_2)(\gamma_2 - \beta_2)} \times (z - \gamma_2)^{(2\gamma_2 + \beta_2)(\gamma_2 - \beta_2)}] = \int u_{\gamma_2, \beta_2}^2 f_3 dx,
\]

or also in the form

\[
(z + \beta_2 + \gamma_2)^{(2\beta_2 + \gamma_2)(2\gamma_2 + \beta_2)}(z - \beta_2)^{(2\beta_2 + \gamma_2)(\gamma_2 - \beta_2)} \times (z - \gamma_2)^{(2\gamma_2 + \beta_2)(\gamma_2 - \beta_2)} = \tau_2 e^\int u_{\gamma_2, \beta_2}^2 f_3 dx,
\]

where \(\tau_2\) is a constant, so the solutions are given by

\[
y = u_{\gamma_2, \beta_2}z - \frac{f_2}{3f_3},
\]

in this case \(u_{\gamma_2, \beta_2}\) is given by (72) and \(z\) is given by (78) for especific values of \(\beta_2\) and \(\gamma_2\). Here we illustrate the method with two particular examples.

**Example 1**

Let \(\beta_2 = 1, \gamma_2 = -1\), so (78) we obtain that

\[
z^{-1}(z - 1)^{\frac{1}{2}}(z + 1)^{\frac{1}{2}} = \tau_2 e^\int u_{(1,-1)}^2 f_3 dx,
\]

whose solution to \(z\) is

\[
z = \frac{1}{\sqrt{1 - \tau_2^2 e^{2 \int u_{(1,-1)}^2 f_3 dx}}},
\]

Now according to (79)

\[
y = \frac{u_{(1,-1)}}{\sqrt{1 - \tau_2^2 e^{2 \int u_{(1,-1)}^2 f_3 dx}}} - \frac{f_2}{3f_3},
\]

with \(u_{(1,-1)}\) given by

\[
u_{(1,-1)} = \frac{e^\int (f_1 - \frac{f_2}{f_3}) dx}{\sqrt{C - 2 \int f_3 e^{2 \int (f_1 - \frac{f_2}{f_3}) dx} dx}},
\]
and

\[ f_{0(1,-1)} = \frac{f_1 f_2}{3 f_3} - \frac{2 f_3^2}{27 f_3^2} - \frac{1}{3} \frac{d}{dx} \frac{f_2}{f_3}. \] (84)

**Example 2**

Let \( \beta_2 = \frac{1}{2}, \gamma_2 = 0 \), so (78) we obtain that

\[ (z + \frac{1}{2})^2 (z - \frac{1}{2})^2 (z)^{-4} = \tau_2 e^{\int u^2_{(\frac{1}{2},0)} f_3 dx}, \] (85)

whose solution to \( z \) is

\[ z = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2 \sqrt{\tau_2} e^{\frac{1}{2} \int u^2_{(\frac{1}{2},0)} f_3 dx}}}{2 \sqrt{\tau_2} e^{\frac{1}{2} \int u^2_{(\frac{1}{2},0)} f_3 dx} - 1}. \] (86)

Now according to (86)

\[ y = u_{(\frac{1}{2},0)} \left[ \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 2 \sqrt{\tau_2} e^{\frac{1}{2} \int u^2_{(\frac{1}{2},0)} f_3 dx}}}{2 \sqrt{\tau_2} e^{\frac{1}{2} \int u^2_{(\frac{1}{2},0)} f_3 dx} - 1} \right] - \frac{f_2}{3 f_3}, \] (87)

with \( u_{(\frac{1}{2},0)} \) given by

\[ u_{(\frac{1}{2},0)} = \frac{e^{\int (f_1 - f_2) f_3 dx}}{\sqrt{C - \frac{1}{2} \int f_3 e^{\frac{1}{2} \int (f_1 - f_2) f_3 dx} dx}}, \] (88)

and

\[ f_{0(\frac{1}{2},0)} = \frac{f_1 f_2}{3 f_3} - \frac{2 f_3^2}{27 f_3^2} - \frac{1}{3} \frac{d}{dx} \frac{f_2}{f_3}. \] (89)

## 5 Reduction to Bernoulli

In this case we assume that system (4-6) satisfies \( \phi_1(x) = 0, \phi_2(x) = \zeta(x) \) and \( \phi_3(x) = 0 \). Then, the resulting system is

\[ (3 f_3 v + f_2) = 0, \] (90)
\[ (3 f_3 v^2 + 2 f_2 v + f_1 - v') = \zeta f_3 u^2, \] (91)
\[ (f_0 - v' + f_3 v^3 + f_2 v^2 + f_1 v) = 0. \] (92)

From the first expression, we get

\[ v = -\frac{f_2}{f_3}. \] (93)
Which upon substituting in (91), we obtain the following

\[ u' = (f_1 - \frac{f_2}{3f_3})u = -f_3\zeta u^3. \]  

(94)

The particular election of the \( \phi(x) \)'s allowed us to get the previous expression, which is easily identified as of Bernoulli type: Its direct solution is

\[ u_\zeta = \frac{e^{\int (f_1 - \frac{f_2}{3f_3})dx}}{\sqrt{C + 2 \int f_3\zeta e^{2\int (f_1 - \frac{f_2}{3f_3})dx} dx}}. \]  

(95)

In this case, also \( f_0 \) satisfies

\[ f_{0,\zeta} = \frac{f_1f_2}{3f_3} - \frac{2f_3^2}{27f_3^2} - \frac{1}{3 f_3^2} \frac{d}{dx} f_2, \]  

(96)

likewise, (3) reduces to

\[ z' = u_\zeta^2 f_3 z^3 + u_\zeta^2 f_3 \zeta z, \]  

(97)

which is also an equation of Bernoulli type, whose solution is

\[ z = \frac{e^{\int u_\zeta^2 f_3 dx}}{\sqrt{C - 2 \int f_3 u_\zeta^2 e^{2\int u_\zeta^2 f_3 dx} dx}}. \]  

(98)

From this we summarized that the Abel’s equation of first kind

\[ y' = f_3y^3 + f_2y^2 + f_1y + f_0, \]

with \( f_{0,(\zeta)} \) given by (96), has the solution

\[ y = u_\zeta z - \frac{f_2}{3f_3}. \]

Where \( u_\zeta \) is given by (95) and \( z \) by (98)

6 Conclusions

In this work, it is shown as widely as possible the different ways to approach the Abel’s equation of first kind and the possibility for obtaining analytic solutions under certain conditions by the introduction of a system of differential equations where the free functions \( \phi_1(x), \phi_2(x) \) and \( \phi_3(x) \) can be chosen conveniently, in such a way that allow the construction of the already mentioned solutions. As a particular result, it is achieved, under adequate manipulation of the \( \phi \)'s, to take Abel’s equation of first kind to a well-know canonic form.
Moreover, two ways of solving it are suggested. In general, during the process several solutions are obtained as are reported in the handbook, confirming the success of the proposal. Unedited solutions are obtained in concrete cases that enrich the literature, this was achieved choosing the adequate form of the φ’s.

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References


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