On Magnetohydrodynamics with One Component of Magnetic Diffusion Near Equilibrium

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Abstract

In this paper, we consider the global existence and uniqueness of solutions to the incompressible magnetohydrodynamics with only one component of magnetic dissipation in two space dimensions. The initial data are supposed to be close to any given non-zero equilibrium. We also reveal the weak dissipative mechanism of MHD system with one component of magnetic diffusion.

Mathematics Subject Classification: 35A01, 35A09, 35Q60

Keywords: Magnetohydrodynamics, Partial magnetic diffusion, Global existence

1 Introduction

The equations of the incompressible MHD flows are a combination of the Navier-Stokes equations of fluid dynamics and Maxwell's equations of elec-
Itromagnetism which have the following form \([3, 6, 9]\)

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p &= \mu \Delta u + H \cdot \nabla H \\
\partial_t H + u \cdot \nabla H &= \nu \Delta H + H \cdot \nabla u \\
\nabla \cdot u &= 0, \quad \nabla \cdot H = 0
\end{aligned}
\] (1)

where \(u\) is the fluid velocity, \(h\) is the magnetic field, \(p\) is the pressure, \(\mu \geq 0\) is the viscosity of the fluid which is the inverse of the Reynolds number and \(\nu \geq 0\) is the magnetic diffusivity which is inversely proportional to the electrical conductivity constant. The system (1) describes the macroscopic behavior of electrically conducting incompressible fluids \([1]\). It is well-known that the electromagnetic fields are governed by the Maxwell’s equations.

In this paper, we are interested in the system (1) with one component of magnetic dissipation near equilibrium in two dimensions. To be precise, let us first fix the equilibrium state to be:

\[
\bar{u} = 0, \quad \bar{H} = (1, 0)^T
\] (2)

We are interested in the following MHD equation with one component of magnetic dissipation:

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p &= \Delta u + H \cdot \nabla H \\
\partial_t H + u \cdot \nabla H + \partial_1^2 h_1 &= H \cdot \nabla u \\
\nabla \cdot u &= 0, \quad \nabla \cdot H = 0
\end{aligned}
\] (3)

with the initial data:

\[
(u, H)|_{t=0} = (u_0(x), H_0(x)) = (u_0(x), h_0(x) + (1, 0)^T), \quad x \in \mathbb{R}^2
\] (4)

Let us denote

\[
H = h + (1, 0)^T
\]

and then the system (3) can be rewritten as

\[
\begin{aligned}
\partial_t u + u \cdot \nabla u + \nabla p &= \Delta u + h \cdot \nabla h + \partial_1 h \\
\partial_t h + u \cdot \nabla h + \partial_2^2 h_1 &= h \cdot \nabla u + \partial_1 u \\
\nabla \cdot u &= 0, \quad \nabla \cdot h = 0
\end{aligned}
\] (5)

We state our local well-posedness theorem as following

**Theorem 1.1** Suppose that the initial data satisfy

\[
u_0 \in H^2(\mathbb{R}^2), \quad h_0 \in H^2(\mathbb{R}^2)
\] (6)

Then, there exists a positive \(T\), which depends only on the initial data, such that the system (5)-(6) possesses a unique solution in the time interval \([0, T]\) with

\[
\begin{aligned}
(u, h) &\in L^\infty([0, T]; H^2(\mathbb{R}^2)) \\
(\partial_t u, \partial_t h) &\in L^\infty([0, T]; L^2(\mathbb{R}^2)) \cap L^2([0, T]; H^1(\mathbb{R}^2)) \\
\nabla u &\in L^2([0, T]; H^2(\mathbb{R}^2)), \quad \partial_2 h_1 \in L^2([0, T]; H^1(\mathbb{R}^2))
\end{aligned}
\] (7)
The local well-posedness of the Cauchy problem of the partially viscous magnetohydrodynamic systems (3) and (4) is rather standard and similar to the case of fully viscous magnetohydrodynamic system which is done in [17], [18]. There is an alternative method by introduced in [8]: For smooth initial data, we first solve the fully viscous MHD system, with replacement of the magnetic viscous dissipation $\nu \Delta H$ as $\nu \partial_1^2 H_1 + \gamma (\partial_1^2 H + \partial_2^2 H_2)$. The standard Galerkin's method provides a uniqueness local smooth solution to such full viscous MHD. we get the uniform a priori estimate to let $\gamma \to 0$.

The heating of high temperature plasmas by waves is one of the most interesting and challenging problems of plasma physics especially when the energy is injected into the system at length scales much larger than the dissipative ones. It has been conjectured that in 2-D MHD systems energy is dissipated at a rate that is independent of the ohmic resistivity (see [4] and the references therein). This also motivates to explore some dissipative mechanism of the iMHD using rigorous mathematical methods. Even though there is partial magnetic dissipation, the analysis seems still not trivial. Our observation is motivated by the work on partial MHD in [8]. The same idea is used in viscoelastic materials by Lin et al in [15] in two space dimensions and Lei et al in [11] in three space dimensions. We point out here that, independently, Chen and Zhang [5] also studied the similar result as that in [11].

2 Main Results

In this paper we intend to explore similar structures for the MHD system with one component of magnetic diffusion (5) as those for Oldroyd-B viscoelastic system. This seems not trivial. For example, to write the two-dimensional Oldroyd-B viscoelastic system as a system of Ericksen-Leslie type and explore its weak dissipative mechanism, the methods in [15] heavily depend on the identity $det F = 1$. Here the magnetic field $H$ is just a vector. This difficulty will be overcome by treating the derivatives along different directions in different ways with the help of one component of magnetic dissipation. We can obtain the following global existence result of (5) with small initial data:

**Theorem 2.1** There exist absolute positive constants $\varepsilon_0$ and $C_0$ depending only on $\mu$ and $\nu$ such that if the initial data satisfies

$$
\|u_0\|_{H^2(\mathbb{R}^2)} + \|h_0\|_{H^2(\mathbb{R}^2)} \leq \varepsilon_0
$$

then the MHD with one component of magnetic dissipation (3) is globally well-posed and the unique solution satisfies

$$
\|u\|_{H^2} + \|h\|_{L^2} + \int_0^\infty (\|\nabla u\|_{H^2} + \|\partial_1 h_1\|_{H^1})^2 dt \leq C(\|u_0\|_{H^2} + \|h_0\|_{H^2})
$$
The proof of Theorem 2.1 depends on introducing the following new auxiliary function \( v = u + \triangle^{-1} \partial_1 h \), which helps to explore a weak dissipative structure of the system (3). As long as one has established a local well-posedness theory, it suffices to have a priori estimate of (9) for extending the local solutions in time to get global ones. Hence, in the following section we shall provide the a priori estimate (9) of solutions to (5). We are not interested in the dependence of the constants on the viscosity coefficients \( \mu \) and \( \nu \), which are set to be one hereafter. Let us point out here that after the work of Lin et al in [15], there have been a lot of works on exploring weak dissipative mechanism for partial dissipative system. For instance, see [5, 10, 11, 12, 13, 16]. At last, let us cite some related works on MHD. Weak solutions of the initial-boundary value problem were constructed by Duvaut and Lions in [7]. See also [19] where the authors studied some properties of the weak solutions, and proved the global well-posedness of the Cauchy problem of the fully viscous MHD (1) in two space dimensions. For MHD with mixed partial dissipation and magnetic diffusion, global regularity can be obtained with the general initial data in 2-D in [2] and with small initial data in 3-D in [20]. A non-blowup criterion of Beale-Kato-Majda type was established in the case of zero resistivity in [14], where authors also proved the existence of global strong solution in the case of positive resistivity and zero viscosity, See also [21, 22].

In the following, we will give the proof of Theorem 2.1. For notational convenience, we will write \( L^p \) for \( L^p(\mathbb{R}^2) \), \( H^s \) for \( H^s(\mathbb{R}^2) \). First of all, for classical solution to (5), one has the following basic energy law using integration by parts and divergence free of \( u \) and \( h \)

\[
\frac{d}{dt}(\|u\|_{L^2}^2 + \|h\|_{L^2}^2) + 2(\|\nabla u\|_{L^2}^2 + \|\partial_2 h_1\|_{L^2}^2) = 0
\]

(10)

Then, we can obtain

\[
\|u\|_{L^2}^2 + \|h\|_{L^2}^2 + 2 \int_0^t \|\nabla u\|_{L^2}^2 + \|\partial_2 h_1\|_{L^2}^2 ds \leq \|u_0\|_{L^2}^2 + \|h_0\|_{L^2}^2 \leq \varepsilon_0
\]

(11)

Next, let us apply the Laplacian operator to the equations in (5) and then taking the \( L^2 \) inner product with \( \triangle u \) and \( \triangle h \) respectively, to obtain

\[
\frac{1}{2} \frac{d}{dt}(\|\triangle u\|_{L^2}^2 + \|\triangle h\|_{L^2}^2) + \|\nabla \triangle u\|_{L^2}^2 + \|\triangle \partial_2 h_1\|_{L^2}^2 = 0
\]

(12)
By the divergence-free of $u$ and $h$, we can get
\[
R_1 = \int_{R^2} [(h \cdot \nabla)(\Delta u \cdot \Delta h) + \partial_1(\Delta u \cdot \Delta h)] dx
\]
\[- \frac{1}{2} \int_{R^2} (u \cdot \nabla)(|\Delta u|^2 + |\Delta h|^2) dx - \int_{R^2} \Delta u \cdot \nabla \Delta p dx = 0
\] (13)

In the following we will estimate $R_2$
\[
R_2 = \int_{R^2} \Delta u[(\Delta h \cdot \nabla)h + 2(\nabla h \cdot \nabla) \nabla h - (\Delta u \cdot \nabla) u - 2(\nabla u \cdot \nabla) \nabla u] dx
\]

It is easy to get
\[
\int_{R^2} \Delta u[(\Delta u \cdot \nabla) u - 2(\nabla u \cdot \nabla) \nabla u] dx \leq C\|\nabla u\|_{L^2} \|\Delta u\|_{H^1}^2
\] (14)

Noting the Sobolev inequality and $\partial_2 h_2 = -\partial_1 h_1$, we have
\[
\int_{R^2} \Delta u(\Delta h \cdot \nabla) h dx = \int_{R^2} \Delta u[\partial_1^2 h \cdot \nabla) h dx
\]
\[+ \int_{R^2} \Delta u[\partial_2^2 h \partial_1 h] dx + \int_{R^2} \Delta u[\partial_2^2 h \partial_2 h] dx
\]
\[\leq C\|\nabla h\|_{L^4}(\|\nabla \partial_1 h\|_{L^2} + \|\nabla \partial_2 h_1\|_{L^2})\|\Delta u\|_{L^4}
\]
\[\leq C\|h\|_{H^2}(\|\nabla \partial_1 h\|_{L^2} + \|\nabla \partial_2 h_1\|_{L^2})\|\Delta u\|_{H^1}
\] (15)

Using the similar way,
\[
\int_{R^2} \Delta u(\nabla h \cdot \nabla) \nabla h dx
\]
\[\leq C\|h\|_{H^2}(\|\nabla \partial_1 h\|_{L^2} + \|\partial_2 h_1\|_{H^2})\|\nabla u\|_{H^2}
\] (16)

Then
\[
R_2 \leq C[\|\nabla u\|_{L^2}^2 \|\nabla u\|_{H^2}^2 + \|h\|_{H^2}(\|\nabla \partial_1 h\|_{L^2} + \|\partial_2 h_1\|_{H^2})\|\nabla u\|_{H^2}]
\] (17)

Similarly, we can estimate the reminder term of $R_3$ as following
\[
R_3 \leq C\|h\|_{H^2}(\|\nabla \partial_1 h\|_{L^2} + \|\partial_2 h_1\|_{H^2})\|\nabla u\|_{H^2}
\] (18)

Noting the above estimates, we can obtain
\[
\frac{1}{2} \frac{d}{dt}(\|\Delta u\|_{L^2}^2 + \|\Delta h\|_{L^2}^2) + \|\nabla \Delta u\|_{L^2}^2 + \|\Delta \partial_2 h_1\|_{L^2}^2
\]
\[\leq C(\|\nabla u\|_{L^2} + \|h\|_{H^2})(\|\nabla \partial_1 h\|_{L^2}^2 + \|\partial_2 h_1\|_{H^2}) + \|\nabla u\|_{H^2}^2)
\] (19)
In the following, we will use the auxiliary function \( v = u + \Delta^{-1}\partial_1 h \) to consider the dissipative mechanism for \( \nabla \partial_1 h \). Applying the Laplacian operator to the momentum equation in (5), we can get

\[
\partial_t \Delta u + (u \cdot \nabla) \Delta u + \nabla \Delta p = \Delta \Delta v + \Delta u[h \cdot \nabla h] + f
\]  

(20)

where \( f = -(\Delta u \cdot \nabla) u - (\nabla u \cdot \nabla) \nabla u \).

Applying the operator \( \partial_1 \) to the magnetic equation in (5), one has

\[
\partial_t \partial_1 h + (u \cdot \nabla) \partial_1 h - \partial_2^2 \partial_1 h_1 = \partial_1[h \cdot \nabla u + \partial_1 u] - (\partial_1 u \cdot \nabla) h
\]

(21)

Then, we can get the equation of \( \Delta v \) as following

\[
\partial_t \Delta v + (u \cdot \nabla) \Delta v + \nabla \Delta p = \Delta \Delta v + \partial_1 \partial_2^2 h_1 + \Delta u[h \cdot \nabla h] + g
\]

(22)

where \( g = f + \partial_1[h \cdot \nabla u + \partial_1 u] - (\partial_1 u \cdot \nabla) h \).

Multiply \( \Delta v \) to (22) and take the \( L^2 \) inner product, which gives that

\[
\frac{1}{2} \frac{d}{dt} \| \Delta v \|^2_{L^2} + \| \nabla \Delta v \|^2_{L^2} = - \int_{\mathbb{R}^2} \Delta v \cdot \nabla \Delta p dx + \int_{\mathbb{R}^2} \Delta v \partial_1 \partial_2^2 h_1 dx \tag{23}
\]

\[+ \int_{\mathbb{R}^2} \Delta v \Delta [h \cdot \nabla h] dx + \int_{\mathbb{R}^2} (\Delta v) g dx = I_1 + I_2 + I_3
\]

Noting the divergence free of \( u \) and \( h \), we have \( \nabla \cdot v = 0 \). Then, by Sobolev inequality, we can easily get

\[
I_1 = - \int_{\mathbb{R}^2} \Delta v \cdot \nabla \Delta p dx + \int_{\mathbb{R}^2} \Delta v \partial_1 \partial_2^2 h_1 dx
\]

\[= \int_{\mathbb{R}^2} \nabla \cdot \Delta v \Delta p dx - (\partial_2 \Delta v) \partial_1 \partial_2 h_1 dx
\]

\[\leq \frac{1}{4} \| \nabla \Delta v \|^2_{L^2} + \| \nabla \partial_2 h_1 \|^2_{L^2}
\]

For \( I_2 \), using the integration by parts and Sobolev inequalities, we have

\[
\int_{\mathbb{R}^2} \Delta v \Delta [h \cdot \nabla h] dx = - \int_{\mathbb{R}^2} \nabla \cdot \Delta v : \partial_2 (h \otimes h) dx - \int_{\mathbb{R}^2} \partial_1 \Delta v_1 : \partial_2^2 (h_1 h_2)
\]

\[= \int_{\mathbb{R}^2} \partial_2 \Delta v_1 : \partial_2^2 (h_1 h_2) - \int_{\mathbb{R}^2} \partial_1 \Delta v_2 : \partial_2^2 (h_2 h_1) - \int_{\mathbb{R}^2} \partial_2 \Delta v_2 : \partial_2^2 (h_2 h_2)
\]

\[\leq C[(\| \partial_1 h \|^2_{L^4} + \| h \|^4_{L^\infty} \| \nabla \partial_1 h \|^2_{L^2}) + (\| \partial_2 h_1 \|^2_{L^4} + \| h \|^4_{L^\infty} \| \nabla \partial_2 h_1 \|^2_{L^2}) + (\| \partial_2 h_2 \|^2_{L^4} + \| h \|^4_{L^\infty} \| \nabla \partial_2 h_2 \|^2_{L^2})]
\]

\[\leq C \| h \|^2_{H^2} \| \nabla \Delta v \|^2_{L^2} (\| \nabla \partial_1 h \|^2_{L^2} + \| \partial_2 \nabla h_1 \|^2_{L^2})
\]
Using the same method, we can get

\[
I_3 = \int_{R^2} (\triangle v)(-(\triangle u \cdot \nabla)u - (\nabla u \cdot \nabla)v + \partial_1 [h \cdot \nabla u + \partial_1 u] - (\partial_1 u \cdot \nabla)h) dx
\]

\[
\leq \|\nabla \triangle v\|_{L^2} \|u\|_{\triangle} + \|\nabla u\|_2^2 + |h|_{\partial_1 u} + |\partial_1 h|_{u} + \|\nabla u\|_{L^2}
\]

\[
\leq C(\|u\|_{H^2} + \|h\|_{H^2})(\|\nabla \triangle v\|_{L^2}^2 + \|\nabla u\|_{H^2}^2 + \|\nabla \partial_1 h\|_{L^2}^2) + \frac{1}{4}(\|\nabla u\|_{L^2}^2 + \|\nabla \triangle v\|_{L^2}^2)
\]

Note that

\[
\|\nabla \partial_1 h\|_{L^2} \leq \|\nabla \triangle u\|_{L^2} + \|\nabla \triangle v\|_{L^2}
\]

(26)

Then, we can get

\[
\frac{1}{2} \frac{d}{dt} \|\triangle v\|_{L^2}^2 + \|\nabla \triangle v\|_{L^2}^2 \leq \|\nabla u\|_{L^2}^2 + \|\nabla \partial_1 h\|_{L^2}^2
\]

\[
+ C(\|u\|_{H^2} + \|h\|_{H^2})(\|\nabla \triangle v\|_{L^2}^2 + \|\nabla u\|_{H^2}^2 + \|\partial_2 h\|_{H^2}^2)
\]

(27)

Combining (10), (19), (26) and (27), we can get

\[
\frac{1}{2} \frac{d}{dt} (\|u\|_{L^2}^2 + \|h\|_{L^2}^2 + \|\triangle h\|_{L^2}^2 + \|\triangle u\|_{L^2}^2 + \|\triangle v\|_{L^2}^2)
\]

\[
+ \|\nabla \triangle u\|_{L^2}^2 + \|\triangle \partial_2 h\|_{L^2}^2 + \|\partial_2 h\|_{L^2}^2 + \|\nabla \triangle v\|_{L^2}^2
\]

\[
\leq C(\|u\|_{L^2} + \|h\|_{L^2}) + (\|\partial_2 h\|_{H^2} + \|\nabla \triangle v\|_{L^2}^2 + \|\nabla u\|_{H^2}^2) + \|\nabla u\|_{L^2}^2
\]

Note (8) and (11), we can get the desired estimate (9). Then, the conclusion of Theorem 2.1 is proved.

ACKNOWLEDGEMENTS.

The second author was supported by NSFC (No.11126058), Excellent Young Teachers Program of Shanghai and Shanghai Leading Academic Discipline Project (No.J50101).

References


Received: December 3, 2012