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On (m,n)-Ideals in LA-Semigroups

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Abstract

The purpose of this paper is to study (m,n)-ideals in LA-semigroups. Some properties of (m,n)-ideals in LA-semigroups and in locally associative LA-semigroups has been provided.

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1 Introduction

The concept of (m, n)-ideal of a semigroup was introduced by Lajos [2]. The left almost semigroup (LA-semigroup), was first introduced by Kazim and Naseerudin [1]. Later, Mushtaq [3, 4] and others investigated this idea further and added many useful results to the theory of LA-semigroups. Mushtaq and Yusuf produced useful results [4], on locally associative LA-semigroups in 1979. In locally associative LA-semigroups they defined powers of an element and congruences using these powers. In this paper, we discussed some properties of (m,n)-ideals in a locally associative LA-semigroup.

2 Preliminaries and basic definitions

Definition 2.1. *[1] A groupoid $(S, \cdot)$ is called an LA-semigroup or an AG-groupoid, if it satisfies left invertive law*

$$(a \cdot b) \cdot c = (c \cdot b) \cdot a, \quad \textit{for all } a, b, c \in S.$$

Definition 2.2. *[4] An LA-semigroup S is called a locally associative LA-semigroup if and only if $(aa)a = a(aa)$ for all $a \in S$.*

Example 2.3. *[4] The set $S = \{a, b, c\}$, under "$\cdot$" defined below in the form of Cayley table is a locally associative LA-semigroup.*

$\cdot$	a	b	c
a	c	c	b
b	b	b	b
c	b	b	b

In [4], the authors define powers of an element in a locally associative LA-semigroup S as follows: $a^1 = a$, $a^{n+1} = a^n a$, for $n \geq 1$. In a locally associative LA-semigroup S with left identity, the results $a^m a^n = a^{m+n}$, $(a^m)^n = a^{mn}$ and $(ab)^n = a^n b^n$ hold for all $a, b \in S$ and here m and n are positive integers. In an LA-semigroup S the following medial law holds $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in S$ [1]. In an LA-semigroup S with left identity, the following paramedial law holds $(ab)(cd) = (dc)(ba)$ for all $a, b, c, d \in S$.

If A and B are any subsets of a locally associative LA-semigroup S then it is easy to prove that $(AB)^n = A^n B^n$ for $n \geq 1$.

Lemma 2.4. *[4] If S is an LA-semigroup with left identity, then*

$$a(bc) = b(ac), \quad \textit{for all } a, b, c \in S.$$

Lemma 2.5. *[4] If S is an LA-semigroup with left identity e, then $SS = S$ and $S = eS = Se$.*

3 $(\mathbf{m}, \mathbf{n})$-ideals in LA-semigroups

A subset A of an LA-semigroup S is called a right (left) ideal of S if $AS \subseteq A$ ($SA \subseteq A$), and is called a two sided ideal if it is both left and right ideal of S. If A is any non-empty subset of an LA-semigroup S, then we define $A^n = (...((AA)A)...)A$, where AA is usual product and $n \in \mathbb{N}$.

Definition 3.1. *A subset A of an LA-semigroup S is called an $(m, 0)$-ideal ($(0, n)$-ideal) if $A^m S \subseteq A$ ($SA^n \subseteq A$), for $m, n \in \mathbb{N}$.*

Proposition 3.2. *Let S be an LA-semigroup with left identity e. Then*
(1) if A is a proper $(m, 0)$-ideal of S, then $e \notin A$.
(2) if A is a proper $(0, n)$-ideal of S, then $e \notin A$.

Proof. (1) Let A be a proper $(m, 0)$-ideal of S. Suppose on contrary that $e \in A$. Then $S = eS = e^m S \subseteq A^m S \subseteq A$, implies that $S \subseteq A$. This is a contradiction to the fact that A is proper. Hence $e \notin A$.

(2) Let A be a proper $(0, n)$-ideal of S. Suppose on contrary that $e \in A$. Then by left invertive law

$$S = (ee)S = (Se)e = (Se^n)e^n \subseteq (SA^n)A^n \subseteq AA^n \subseteq SA^n \subseteq A$$

imply that $S \subseteq A$. This is a contradiction to the fact that A is proper. Hence $e \notin A$. □

Definition 3.3. *Let S be an LA-semigroup. An LA-subsemigroup A of S is called an (m, n)-ideal of S, if A satisfies the condition*

$$(A^m S)A^n \subseteq A$$

where m, n are non-negative integers (A^m is suppressed if $m = 0$).

Let A be an LA-subsemigroup of S then we shall say that A is a bi-ideal or $(1, 1)$-ideal of S if $(AS)A \subseteq A$.

Proposition 3.4. *Let S be an LA-semigroup, B be an LA-subsemigroup of S and let A be an (m, n)-ideal of S. Then the intersection $A \cap B$ is an (m, n)-ideal of the LA-semigroup B.*

Proof. The intersection $A \cap B$ evidently is an LA-subsemigroup of S. We have to show that $A \cap B$ is an (m, n)-ideal of B, for this

$$((A \cap B)^m B)(A \cap B)^n \subseteq (A^m S)A^n \subseteq A, \tag{1}$$

because A is an (m, n)-ideal of S. Secondly

$$((A \cap B)^m B)(A \cap B)^n \subseteq (B^m B)B^n \subseteq B. \tag{2}$$

Therefore (1) and (2) implies that $((A \cap B)^m B)(A \cap B)^n \subseteq A \cap B$. Thus the intersection $A \cap B$ is an (m, n)-ideal of B. □

Theorem 3.5. *Let $\{A_i : i \in I\}$ be a family of (m,n)-ideals of an LA-semigroup S. Then $B = \bigcap_{i=1}^{n} A_i \neq \emptyset$ is an (m,n)-ideal of S.*

Proof. Let $\{A_i : i \in I\}$ be a family of (m,n)-ideals of an LA-semigroup S. We know that the intersection of LA-subsemigroups is an LA-subsemigroup, now to show that $B = \bigcap_{i=1}^{n} A_i$ is an (m,n)-ideal of S. Here we need only to show that $(B^m S)B^n \subseteq B$. Let $x \in (B^m S)B^n$. Then $x = (a_1^m s)a_2^n$ for some $a_1^m, a_2^n \in B$ and $s \in S$. Thus for any arbitrary $i \in I$ as $a_1^m, a_2^n \in B_i$. So $x \in (B_i^m S)B_i^n$. Since B_i is an (m,n)-ideal so $(B_i^m S)B_i^n \subseteq B_i$ and therefore $x \in B_i$. Since i was chosen arbitrarily so $x \in B_i$ for all $i \in I$ and hence $x \in B$. So, $(B^m S)B^n \subseteq B$ and hence $B = \bigcap_{i=1}^{n} A_i$ is an (m,n)-ideal of S. □ □

Definition 3.6. *An element a of an LA-semigroup S is called idempotent if $aa = a$. A subset I of an LA-semigroup S is called idempotent if all of its elements are idempotent.*

Theorem 3.7. *If B is an idempotent (m,m)-ideal of a locally associative LA-semigroup S with left identity, then B is an $(m,0)$-ideal and a $(0,m)$-ideal of S.*

Proof. By using left invertive law and paramedial law, we obtain

$$\begin{aligned} B^m S &= (B^2)^m S = (B^m)^2 S = (B^m B^m)S = (SB^m)B^m = (SB^m)(B^m B^m) \\ &= (B^m B^m)(B^m S) = ((B^m S)B^m)B^m \subseteq BB^m \subseteq B. \end{aligned}$$

This shows that B is an $(m,0)$-ideal of S. Similarly, by using Lemma 2.4 and medial law, we get

$$SB^m = S(B^m B^m) = (B^m B^m)(SB^m) = (B^m S)(B^m B^m) = (B^m S)B^m \subseteq B.$$

This shows that B is a $(0,m)$-ideal of S. Hence B is an $(m,0)$-ideal and a $(0,m)$-ideal of S. □

Proposition 3.8. *Let A be an LA-subsemigroup of a locally associative LA-semigroup S with left identity. If A is a $(0,m)$-ideal and B is an (m,n)-ideal of S, then BA is an (m,n)-ideal of S.*

Proof. Using medial law, we get $(BA)(BA) = (BB)(AA) \subseteq BA$. Now by left invertive law

$$\begin{aligned} ((BA)^m S)(BA)^n &= ((B^m A^m)S)(B^n A^n) = ((SA^m)B^m)(B^n A^n) \\ &= ((B^n A^n)B^m)(SA^m) = ((B^m A^n)B^n)(SA^m) \\ &\subseteq ((B^m S)B^n)(SA^m) \subseteq BA. \end{aligned}$$

Hence BA is an (m,n)-ideal of S. □

Proposition 3.9. *The product of two (m,n)-ideals of a locally associative LA-semigroup S with left identity is an (m,n)-ideal of S.*

Proof. By medial law, we get $(AB)(AB) = (AA)(BB) \subseteq AB$. Now using medial law and Lemma 2.5, we get

$$\begin{aligned}((AB)^m S)(AB)^n &= ((A^m B^m)S)(A^n B^n) = ((A^m B^m)(SS))(A^n B^n) \\ &= ((A^m S)(B^m S))(A^n B^n) = ((A^m S)A^n)((B^m S)B^n) \subseteq AB.\end{aligned}$$

Hence AB is an (m,n)-ideal of S. □

Theorem 3.10. *Let A and B be LA-subsemigroups of a locally associative LA-semigroup S. If A is an $(m,0)$-ideal and B is a $(0,n)$-ideal of S, then the product AB is an (m,n)-ideal of S if $AB \subseteq A$.*

Proof. By medial law we get $(AB)(AB) = (AA)(BB) \subseteq AB$. This shows that AB is an LA-subsemigroup. Now

$$((AB)^m S)(AB)^n \subseteq (A^m S)(A^n B^n) \subseteq A(SB^n) \subseteq AB.$$

Hence the product AB is an (m,n)-ideal of S. □

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