An Identity of the Twisted \((h, q)\)-Euler Polynomials
-associated with the \(p\)-adic \(q\)-Integrals on \(\mathbb{Z}_p\)

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Abstract

In this paper, we investigate the alternating sums of powers of consecutive integers. By applying the symmetry of the fermionic \(p\)-adic \(q\)-integral on \(\mathbb{Z}_p\), we give recurrence identities the twisted \((h, q)\)-Euler polynomials.

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1 Introduction

Throughout this paper, we always make use of the following notations: \(\mathbb{C}\) denotes the set of complex numbers, \(\mathbb{Z}_p\) denotes the ring of \(p\)-adic rational integers, \(\mathbb{Q}_p\) denotes the field of \(p\)-adic rational numbers, and \(\mathbb{C}_p\) denotes the completion of algebraic closure of \(\mathbb{Q}_p\). Let \(\nu_p\) be the normalized exponential valuation of \(\mathbb{C}_p\) with \(|p|_p = p^{-\nu_p(p)} = p^{-1}\). When one talks of \(q\)-extension, \(q\) is considered in many ways such as an indeterminate, a complex number \(q \in \mathbb{C}\), or \(p\)-adic number \(q \in \mathbb{C}_p\). If \(q \in \mathbb{C}\) one normally assume that \(|q| < 1\). If \(q \in \mathbb{C}_p\), we normally assume that \(|q - 1|_p < p^{-\frac{1}{p-1}}\) so that \(q^x = \exp(x \log_q)\) for \(|x|_p \leq 1\). Throughout this paper we use the notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q}.
\]
Hence, \(\lim_{q \to 1} [x] = x\) for any \(x\) with \(|x|_p \leq 1\) in the present \(p\)-adic case. For 
\[g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},\]
the \(p\)-adic \(q\)-integral was defined by 
\[
I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \to \infty} \frac{[2]_q}{1 + q^{pN}} \sum_{x=0}^{pN-1} g(x)(-q)^x, \quad \text{see [1-8]}. \quad (1.1)
\]
If we take \(g_1(x) = g(x+1)\) in (1.1), then we easily see that 
\[
qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \quad (1.2)
\]
Let \(T_p = \bigcup_{N \geq 1} C_{pN} = \lim_{N \to \infty} C_{pN}\), where \(C_{pN} = \{\zeta | \zeta^{pN} = 1\}\) is the cyclic group of order \(p^N\). For \(\zeta \in T_p\), we denote by \(\phi_{\zeta} : \mathbb{Z}_p \to \mathbb{C}_p\) the locally constant function \(x \mapsto \zeta^x\).

For \(h \in \mathbb{Z}, q \in \mathbb{C}_p\) with \(|1 - q|_p \leq 1\), and \(\zeta \in T_p\), the twisted \((h, q)\)-Euler polynomials \(\widetilde{E}^{(h)}_{n,q,\zeta}(x)\) are defined by 
\[
\widetilde{F}^{(h)}_{q,\zeta}(x, t) = \sum_{n=0}^{\infty} \frac{\widetilde{E}^{(h)}_{n,q,\zeta}(x) t^n}{n!} = \frac{[2]_q}{\zeta q^h e^t + 1} e^{xt}. \quad (1.3)
\]
The twisted \((h, q)\)-Euler numbers \(\widetilde{E}^{(h)}_{n,q,\zeta}\) are defined by the generating function:
\[
\widetilde{F}^{(h)}_{q,\zeta}(t) = \sum_{n=0}^{\infty} \frac{\widetilde{E}^{(h)}_{n,q,\zeta} t^n}{n!} = \frac{[2]_q}{\zeta q^h e^t + 1}. \quad (1.4)
\]
The following elementary properties of the \((h, q)\)-Euler numbers \(\widetilde{E}^{(h)}_{n,q,\zeta}\) and polynomials \(\widetilde{E}^{(h)}_{n,q,\zeta}(x)\) are readily derived from (1.1), (1.2), (1.3) and (1.4).

**Theorem 1.1** (Witt formula). For \(h \in \mathbb{Z}, q \in \mathbb{C}_p\) with \(|1 - q|_p < 1\), and \(\zeta \in T_p\), we have 
\[
\widetilde{E}^{(h)}_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \zeta^x q^{(h-1)x} x^n d\mu_{-q}(x),
\]
\[
\widetilde{E}^{(h)}_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \zeta^y q^{(h-1)y} (x + y)^n d\mu_{-q}(y).
\]

**Theorem 1.2** For any positive integer \(n\), we have 
\[
\widetilde{E}^{(h)}_{n,q,\zeta}(x) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{E}^{(h)}_{k,q,\zeta} x^{n-k}.
\]
2 The alternating sums of powers of consecutive $q$-integers

Let $q$ be a complex number with $|q| < 1$ and $\zeta$ be the $p^N$-th root of unity. By using (1.3), we give the alternating sums of powers of consecutive $(h, q)$-integers as follows:

$$\sum_{n=0}^{\infty} \bar{E}_{n,q,\zeta}^{(h)} \frac{t^n}{n!} = \frac{[2]_q}{\zeta q^h e^t + 1} = [2]_q \sum_{n=0}^{\infty} (-1)^n \zeta^n q^{hn} e^{nt}.$$

From the above, we have

$$- [2]_q \sum_{n=0}^{\infty} (-1)^n \zeta^n q^{hn} e^{(n+k)t} + [2]_q (-1)^{-k} \zeta^{-k} q^{-hk} \sum_{n=0}^{\infty} (-1)^n \zeta^n q^{hn} e^{nt}$$

$$= [2]_q (-1)^{-k} \zeta^{-k} q^{-hk} \sum_{n=0}^{k-1} (-1)^n \zeta^n q^{hn} e^{nt}.$$ (2.1)

By using (1.3) and (1.4), and (2.1), we obtain

$$\sum_{n=0}^{k-1} (-1)^n \zeta^n q^{hn} n^j = \frac{(-1)^{k+1} \zeta^k q^{hk} \bar{E}_{j,q,\zeta}^{(h)}(k) + \bar{E}_{j,q,\zeta}^{(h)}}{[2]_q}.$$

By using the above equation we arrive at the following theorem:

**Theorem 2.1** Let $k$ be a positive integer and $q \in \mathbb{C}$ with $|q| < 1$. Then we obtain

$$\tilde{T}_{j,q,\zeta}^{(h)}(k-1) = \sum_{n=0}^{k-1} (-1)^n \zeta^n q^{hn} n^j = \frac{(-1)^{k+1} \zeta^k q^{hk} \bar{E}_{j,q,\zeta}^{(h)}(k) + \bar{E}_{j,q,\zeta}^{(h)}}{[2]_q}.$$

**Corollary 2.2** For $\zeta = 1$, we have

$$\lim_{q \to 1} \tilde{T}_{j,q,\zeta}^{(h)}(k-1) = \sum_{n=0}^{k-1} (-1)^n n^j = \frac{(-1)^{k+1} E_j(k) + E_j}{2},$$

where $E_j(x)$ and $E_j$ denote the Euler polynomials and Euler numbers, respectively.

Next, we assume that $q \in \mathbb{C}_p$ and $\zeta \in T_p$. We obtain recurrence identities the $(h, q)$-Euler polynomials and the $q$-analogue of alternating sums of powers of consecutive integers. By using (1.1), we have

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l),$$
where \( g_n(x) = g(x + n) \). If \( n \) is odd from the above, we obtain

\[
q^n I_q(g_n) + I_q(g) = [2]q \sum_{l=0}^{n-1} (-1)^{n-l-1} q^l g(l) \quad (\text{cf. [1-5]}).
\]

(2.2)

Substituting \( g(x) = \zeta x q^{(h-1)x} e^{xt} \) into the above, we have

\[
\zeta^n q^{hn} \int_{\mathbb{Z}_p} \zeta x q^{(h-1)x} e^{(x+n)t} d\mu_q(x) + \int_{\mathbb{Z}_p} \zeta x q^{(h-1)x} e^{xt} d\mu_q(x) = \frac{[2]q(1 + \zeta^n q^{hn} e^{nt})}{\zeta q^h e^t + 1}.
\]

From the above, we get

\[
\frac{[2]q(1 + \zeta^n q^{hn} e^{nt})}{\zeta q^h e^t + 1} = \frac{2 \int_{\mathbb{Z}_p} \zeta x q^{(h-1)x} e^{xt} d\mu_q(x)}{\int_{\mathbb{Z}_p} \zeta x q^{hn} e^{nt} d\mu_{-1}(x)}.
\]

(2.3)

By substituting Taylor series of \( e^{xt} \), we obtain

\[
\sum_{m=0}^{\infty} \left( \zeta^n q^{hn} \int_{\mathbb{Z}_p} \zeta x q^{(h-1)x} (x+n)^m d\mu_q(x) + \int_{\mathbb{Z}_p} \zeta x q^{(h-1)x} x^m d\mu_q(x) \right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left( [2]q \sum_{j=0}^{n-1} (-1)^j \zeta^j q^{hj} j^m \right) \frac{t^m}{m!}.
\]

(2.4)

By using Theorem 2.1 and (2.4), we have

\[
\zeta^n q^{hn} \sum_{k=0}^{m} \binom{m}{k} n^{m-k} \int_{\mathbb{Z}_p} \zeta x q^{(h-1)x} x^k d\mu_q(x) + \int_{\mathbb{Z}_p} \zeta x q^{(h-1)x} x^m d\mu_q(x) = [2]q T_{m,q,\zeta}(n - 1).
\]

(2.5)

By using (2.3) and (2.5), we arrive at the following theorem:

**Theorem 2.3** Let \( n \) be odd positive integer and \( h \in \mathbb{Z} \). Then we have

\[
\frac{2 \int_{\mathbb{Z}_p} \zeta x q^{(h-1)x} e^{xt} d\mu_q(x)}{\int_{\mathbb{Z}_p} \zeta x q^{hn} e^{nt} d\mu_{-1}(x)} = [2]q \sum_{m=0}^{\infty} \left( T_{m,q,\zeta}(n - 1) \right) \frac{t^m}{m!}.
\]

Let \( w_1 \) and \( w_2 \) be odd positive integers. By Theorem 2.3, and after some elementary calculations, we obtain the following theorem.

**Theorem 2.4** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have

\[
\frac{2 \int_{\mathbb{Z}_p} \zeta^{w_1} q^{(w_2-1)x} e^{w_1 x_2} d\mu_q(x_2)}{\int_{\mathbb{Z}_p} \zeta^{w_1} q^{w_2} e^{w_1 x_2} d\mu_{-1}(x)} = [2]q \sum_{m=0}^{\infty} \left( T_{m,q,\zeta}(w_1 - 1)w_2 \right) \frac{t^m}{m!}.
\]

(2.6)
By (1.1), we obtain
\[
\int_{\mathbb{Z}} \int_{\mathbb{Z}} \zeta^{w_1 x_1 + w_2 x_2} q^{(w_1 - 1) x_1} q^{(w_2 - 1) x_2} e^{(w_1 x_1 + w_2 x_2 + w_1 x_2) t} d\mu_{-q}(x_1) d\mu_{-q}(x_2)
\]
\[
= e^{w_1 x_2 t} \int_{\mathbb{Z}} \zeta^{w_1 x_1} q^{(w_1 - 1) x_1} e^{w_1 x_1 t} d\mu_{-q}(x_1) \int_{\mathbb{Z}} \zeta^{w_2 x_2} q^{(w_2 - 1) x_2} e^{w_2 x_2 t} d\mu_{-q}(x_2)
\]
\[
= \frac{e^{w_1 x_2 t}}{\int_{\mathbb{Z}} \zeta^{w_1 x_1} q^{(w_1 - 1) x_1} e^{w_1 x_1 t} d\mu_{-1}(x)} \int_{\mathbb{Z}} \zeta^{w_2 x_2} q^{(w_2 - 1) x_2} e^{w_2 x_2 t} d\mu_{-1}(x).
\]  \tag{2.7}

By using (2.6) and (2.7), after elementary calculations, we obtain
\[
a = \frac{[2]_q}{2} \left( \sum_{m=0}^{\infty} \tilde{E}_{m,q,\zeta}^{(w_1)}(w_2 x) w_1^m \frac{t^m}{m!} \right) \left( \sum_{m=0}^{\infty} \tilde{T}_{m,q,\zeta}^{(w_2)}(w_1 - 1) w_2^m \frac{t^m}{m!} \right). \tag{2.8}
\]

By using Cauchy product in the above, we have
\[
a = \frac{[2]_q}{2} \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q,\zeta}^{(w_1)}(w_2 x) w_2^j \tilde{T}_{m-j,q,\zeta}^{(w_2)}(w_1 - 1) w_1^{m-j} \right) \frac{t^m}{m!}. \tag{2.9}
\]

By using the symmetry in (2.8), we obtain
\[
a = \frac{[2]_q}{2} \left( \sum_{m=0}^{\infty} \tilde{E}_{m,q,\zeta}^{(w_2)}(w_1 x) w_1^m \frac{t^m}{m!} \right) \left( \sum_{m=0}^{\infty} \tilde{T}_{m,q,\zeta}^{(w_1)}(w_2 - 1) w_2^m \frac{t^m}{m!} \right).
\]

Thus we obtain
\[
a = \frac{[2]_q}{2} \sum_{m=0}^{\infty} \left( \sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q,\zeta}^{(w_2)}(w_1 x) w_1^j \tilde{T}_{m-j,q,\zeta}^{(w_1)}(w_2 - 1) w_2^{m-j} \right) \frac{t^m}{m!}. \tag{2.10}
\]

By comparing coefficients \( \frac{t^m}{m!} \) in the both sides of (2.9) and (2.10), we arrive at the following theorem.

**Theorem 2.5** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain
\[
\sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q,\zeta}^{(w_1)}(w_2 x) w_1^j \tilde{T}_{m-j,q,\zeta}^{(w_2)}(w_1 - 1) w_2^{m-j}
\]
\[
= \sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q,\zeta}^{(w_2)}(w_1 x) w_2^j \tilde{T}_{m-j,q,\zeta}^{(w_1)}(w_2 - 1) w_1^{m-j},
\]

where \( \tilde{E}_{k,q,\zeta}^{(h)}(x) \) and \( \tilde{T}_{m,q,\zeta}^{(h)}(k) \) denote the twisted \((h,q)\)-Euler polynomials and the \( q \)-analogue of alternating sums of powers of consecutive integers, respectively.
By using Theorem 1.2, we have the following corollary:

**Corollary 2.6** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain

\[
\sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j x^j \bar{E}_{k,q,\xi}^{(w_2)} \bar{T}_{m-j,q,\xi}^{(w_1)} (w_2 - 1)
= \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^j w_2^{m-k} x^j \bar{E}_{k,q,\xi}^{(w_1)} \bar{T}_{m-j,q,\xi}^{(w_2)} (w_1 - 1).
\]

By using (2.9), we have

\[
a = \left( \frac{1}{2} e^{w_1 w_2 x t} \int_{\mathbb{Z}_p} \zeta^{w_1 x_1} q^{(w_1-1)x_1} e^{x_1 w_1 t} d\mu_q(x_1) \right) \left( \frac{2 \int_{\mathbb{Z}_p} \zeta^{w_2 x_2} q^{(w_2-1)x_2} e^{x_2 w_2 t} d\mu_q(x_2)}{\int_{\mathbb{Z}_p} \zeta^{w_1 w_2 x q w_1 w_2 x} e^{x w_1 w_2 t} d\mu_q(x)} \right)
= \frac{[q]}{2} \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_1 j} q^{w_1 j} \int_{\mathbb{Z}_p} \zeta^{w_1 x_1} q^{(w_1-1)x_1} e^{x_1 w_1 t} \left( \frac{w_2}{w_1} \right)^{(w_2 t)} d\mu_q(x_1)
= \frac{[q]}{2} \sum_{n=0}^{\infty} \left( \frac{w_1-1}{w_1} \right) \frac{n!}{n!}.
\]

By using the symmetry property in (2.11), we also have

\[
a = \left( \frac{1}{2} e^{w_1 w_2 x t} \int_{\mathbb{Z}_p} \zeta^{w_2 x_2} q^{(w_2-1)x_2} e^{x_2 w_2 t} d\mu_q(x_2) \right) \left( \frac{2 \int_{\mathbb{Z}_p} \zeta^{w_1 x_1} q^{(w_1-1)x_1} e^{x_1 w_1 t} d\mu_q(x_1)}{\int_{\mathbb{Z}_p} \zeta^{w_1 w_2 x q w_1 w_2 x} e^{x w_1 w_2 t} d\mu_q(x)} \right)
= \frac{[q]}{2} \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_2 j} q^{w_2 j} \int_{\mathbb{Z}_p} \zeta^{w_2 x_2} q^{(w_2-1)x_2} e^{x_2 w_2 t} \left( \frac{w_1}{w_2} \right)^{(w_1 t)} d\mu_q(x_2)
= \frac{[q]}{2} \sum_{n=0}^{\infty} \left( \frac{w_2-1}{w_2} \right) \frac{n!}{n!}.
\]

By comparing coefficients \( \frac{t^n}{n!} \) in the both sides of (2.11) and (2.12), we have the following theorem:

**Theorem 2.7** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have

\[
\sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j} q^{w_2 j} \tilde{E}_{n,q,\xi}^{(w_1)} \left( w_2 x + j \frac{w_2}{w_1} \right) w_1^n
= \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j} q^{w_1 j} \tilde{E}_{n,q,\xi}^{(w_2)} \left( w_1 x + j \frac{w_1}{w_2} \right) w_2^n.
\]
Corollary 2.8  Let $w_1$ and $w_2$ be odd positive integers. If $q \to 1$ and $\zeta = 1$, we have
\[ \sum_{j=0}^{w_1-1} (-1)^j E_n \left( w_2 x + j \frac{w_2}{w_1} \right) w_1^n = \sum_{j=0}^{w_2-1} (-1)^j E_n \left( w_1 x + j \frac{w_1}{w_2} \right) w_2^n. \]

Substituting $w_1 = 1$ into (2.13), we arrive at the following corollary.

Corollary 2.9  Let $w_2$ be odd positive integer. Then we obtain
\[ \tilde{E}_{n,q,\zeta}(x) = \sum_{j=0}^{w_2-1} (-1)^j \zeta^j q^j \tilde{E}_{n,q,\zeta}(w_2) \left( \frac{x + j}{w_2} \right) w_2^n, \]
where $\tilde{E}_{n,q,\zeta}(x)$ denotes the twisted $q$-Euler polynomials (see [6], [7]).

References


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