Estimating the Parameters of an Exponentiated Inverted Weibull Distribution under Type-II Censoring

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Abstract

The exponentiated inverted Weibull distribution is a generalization of the exponentiated inverted exponential distribution as well as the inverted Weibull distribution. In this paper, Bayes and classical estimators have been obtained for two parameters exponentiated inverted Weibull distribution when sample is available from complete and type II censoring scheme. Several Bayesian estimates are obtained against different symmetric and asymmetric loss functions such as squared error and LINEX. This was done with respect to the conjugate priors for two shape parameters. As expected, when the shape parameters are unknown, the closed-form expressions of the Bayes estimators cannot be obtained. We used an approximation based on the Lindley and Markov chain Monte Carlo method (MCMC) methods for obtaining Bayes estimates under these loss functions. All Bayesian estimates are compared with the corresponding maximum likelihood estimates numerically in terms of their mean square error values. Finally, real data sets are analyzed for the purpose of illustration.

Keywords: Exponentiated inverted Weibull distribution; Type II censored sample; Bayesian and non-Bayesian estimations; Symmetric and asymmetric loss functions; Gibbs and Metropolis sampler

1 Introduction

Recently the two-parameter exponentiated inverted Weibull distribution (EIW) distribution has been proposed by Flaih et al. (2012). The two-parameter EIW distribution has the following density function

\[ f(x) = \theta \beta x^{-(\beta+1)}(e^{-x^{-\beta}})^\theta, \quad x > 0, \quad (\beta > 0, \theta > 0). \] (1)

and the distribution function

\[ F(x) = (e^{-x^{-\beta}})^\theta, \quad x > 0, \] (2)
Also, the reliability and hazard functions of the EIW distribution with two shape parameters $\theta$ and $\beta$ are given by
\[ R(t) = 1 - \left( e^{-t^{-\beta}} \right)^{\theta}, t > 0, \] (3)
and
\[ H(t) = \frac{\theta \beta t^{-(\beta-1)} \left( e^{-t^{-\beta}} \right)^{\theta}}{1 - \left( e^{-t^{-\beta}} \right)^{\theta}}. \] (4)

In the next section, we deal with the problem of estimating the parameters $\theta$, $\beta$ and the reliability function under squared error loss (SEL) and LINEX loss functions. The prior distribution for the parameters of the model has been taken as a natural conjugate prior. The SEL and LINEX loss functions for a parameter $\delta$ are as follows, respectively,
\[ L_1(\delta, \hat{\delta}) = (\delta - \hat{\delta})^2, \] (5)
\[ L_2(\Delta) = a \left( e^{h\Delta} - h\Delta - 1 \right), a > 0, h \neq 0. \] (6)

where $a$ and $h$ are shape and scale parameters of the loss function, respectively and $\Delta = \left( \delta - \hat{\delta} \right)$ denotes the scalar estimation error in using $\hat{\delta}$ to estimate $\delta$.

Generally, the sign and magnitude of $h$ in LINEX loss function reflect the direction and degree of asymmetry. This has been introduced by Varian (1975) and further properties of this loss function have been investigated by Zellner (1986). For small values of $h$ (near to zero), the LINEX loss function is almost the same as the SEL and for the choice of negative or positive values of $h$, the LINEX loss function gives more weight to overestimation or underestimation (for details, see Zellner 1986).

In Bayesian approach, we need to integrate over the posterior distribution and the problem is that the integrals are usually impossible to evaluate analytically. Markov chain Monte Carlo (MCMC) technique is a Monte Carlo integration method which draws samples from the target posterior distribution. MCMC methodology provided a convenient and efficient way to sample from complex, high-dimensional statistical distributions. Recently, application of the MCMC method to the estimation of parameters or some other vital properties about statistical models is very common. Pang et al. (2007) claimed that MCMC is quite versatile and flexible for use in parameter estimation of the three-parameter Weibull distribution. Soliman et al. (2012) considered the maximum likelihood and Bayesian inferences of the unknown parameters of modified Weibull distribution and they used MCMC technique to approximate the Bayes estimates.

The main objective of this article is to estimate the two unknown parameters of the EIW($\theta$, $\beta$). We use the maximum likelihood and Bayes methods to derive such estimates. The estimators are obtained by using the data of type II censoring. Also the asymptotic confidence intervals for the parameters are also derived from the Fisher Information matrix. It is observed that the Bayes estimators can not be expressed in explicit forms and they can be obtained by two dimensional numerical integrations only. We use the idea of Lindley to compute the approximate Bayes estimators of the unknown parameters and it is observed that the approximation works quite
well. We compute the approximate Bayes estimators under the assumption of independent gamma priors of the unknown parameters and compare them with the maximum likelihood estimators (MLEs) by Monte Carlo simulations. We also propose Markov Chain Monte Carlo (MCMC) techniques to generate samples from the posterior distributions and in turn computing the Bayes estimators. The posterior density functions match quite well with the histograms of the asymptotic confidence intervals of the samples obtained by MCMC methods.

The rest of the paper is organized as follows. In the next section, the ML estimators of the unknown parameters and reliability function are presented. In section 3, we propose the Bayes estimators of the unknown parameters. The approximate Bayes estimators based on Lindley and MCMC techniques, under SEL and LINEX loss functions are also considered in sections 4 and 5. Numerical results are provided in section 6. Finally conclusions appear in section 7.

2 The maximum likelihood estimation

Maximum likelihood estimation is one of the most popular methods for estimating the parameters of continuous distributions because of its attractive properties, such as consistency, asymptotic unbiased, asymptotic efficiency, and asymptotic normality. In this section we discuss the maximum likelihood estimators of the parameters of the EIW distribution and their asymptotic properties.

Suppose that \( X_1 < X_2 < \ldots < X_r \) is a type-II censored sample of size \( r \) obtained from a life test on \( n \) items whose lifetimes have the EIW(\( \theta, \beta \)) model. It is assumed that both parameters \( \theta \) and \( \beta \) are unknown, the likelihood function for the parameters \( \theta \) and \( \beta \) is then

\[
L(\theta, \beta; x) = \frac{n!}{n - r!} \prod_{i=1}^{r} f(x_i, \theta, \beta)[1 - F(x_r, \theta, \beta)]^{n-r}
\]

\[
= \frac{n!}{n - r!} \theta^r \beta^r U(\beta, \underline{x}) e^{-\theta \sum_{i=1}^{r} x_i^{-\beta} \left[ 1 - e^{-\theta x_r^{-\beta}} \right]^{(n-r)}},
\]

where, \( \underline{x} = (x_1, x_2, \ldots, x_r) \) and \( U(\beta, \underline{x}) = \prod_{i=1}^{r} x_i^{-(\beta+1)} \). If \( n = r \) then, equation (7) reduces to complete samples. By taking logarithm of Eq. (7), the log-likelihood function is

\[
\ell(\theta, \beta; \underline{x}) = \ln L(\theta, \beta; \underline{x})
\]

\[
= r \ln \theta + r \ln \beta - (\beta + 1) \sum_{i=1}^{r} \ln x_i - \theta \sum_{i=1}^{r} x_i^{-\beta} + (n - r) \ln \left[ 1 - e^{-\theta x_r^{-\beta}} \right].
\]

The maximum likelihood estimates of \( \theta \) and \( \beta \) are obtained by setting the first partial derivatives of Eq. (8) to zero with respective to \( \theta \) and \( \beta \), respectively. These simultaneous equations are

\[
\frac{\partial \ell}{\partial \theta} = \frac{r}{\theta} - \sum_{i=1}^{r} x_i^{-\beta} - \frac{(n - r)x_r^{-\beta}}{1 - e^{-\theta x_r^{-\beta}}} = 0,
\]

\[
\frac{\partial \ell}{\partial \beta} = \frac{r}{\beta} - \sum_{i=1}^{r} \ln x_i + \theta \sum_{i=1}^{r} x_i^{-\beta} \ln x_i + \frac{(n - r)\theta x_r^{-\beta} e^{-\theta x_r^{-\beta}} \ln x_r}{1 - e^{-\theta x_r^{-\beta}}} = 0.
\]
It may be noted that this is an implicit equations in $\hat{\theta}_{ML}$ and $\hat{\beta}_{ML}$, so it can not be solved analytically. We propose to solve it by using numerical iteration method, particularly Newton–Raphson method. If $n = r$ the normal equations in (9) and (10) will reduce to the normal equations from complete sample in Flaih et al. (2012). Using the invariance property, the MLE $\hat{R}_{ML}$ of $R$ may be obtained by replacing $\theta$ and $\beta$ by its MLEs $\hat{\theta}$ and $\hat{\beta}$, in (3). Thus

$$\hat{R}_{ML}(t) = 1 - (e^{-t\hat{\beta}})^{\hat{\theta}}, t > 0,$$

(11)

The asymptotic variance covariance matrix of the estimators of the parameters $\theta$ and $\beta$ is obtained by inverting the Fisher information matrix in which elements are negatives of expected values of the second partial derivatives of the logarithm of the likelihood function. The elements of the sample information matrix, for censored schemes sample will be

$$I(\theta) = -E \left[ \begin{array}{cc} \hat{\ell}_{\theta \theta} & \hat{\ell}_{\theta \beta} \\ \hat{\ell}_{\beta \theta} & \hat{\ell}_{\beta \beta} \end{array} \right]^{-1} = \left[ \begin{array}{cc} \hat{\sigma}_{\theta}^2 & \hat{\sigma}_{\theta \beta} \\ \hat{\sigma}_{\beta \theta} & \hat{\sigma}_{\beta}^2 \end{array} \right],$$

(12)

with

$$\hat{\ell}_{\theta \theta} = \frac{-r}{\hat{\theta}^2} - \frac{(n - r)x_r^{-\hat{\beta}}e^{-\hat{\theta}x_r^{-\hat{\beta}}}}{(1 - e^{-\hat{\theta}x_r^{-\hat{\beta}}})^2},$$

(13)

$$\hat{\ell}_{\theta \beta} = \frac{\partial^2 \ell}{\partial \theta \partial \beta} \bigg|_{\theta = \hat{\theta}, \beta = \hat{\beta}} = \hat{\ell}_{\beta \theta},$$

$$\hat{\ell}_{\beta \beta} = \frac{\partial^2 \ell}{\partial \beta^2} \bigg|_{\theta = \hat{\theta}, \beta = \hat{\beta}} = \sum_{i=1}^{r} x_i^{-\beta} \ln x_i - (n - r)x_r^{-\hat{\beta}} \ln x_r \left[ \frac{1 - \hat{\theta}x_r^{-\hat{\beta}} - e^{\hat{\theta}x_r^{-\hat{\beta}}}}{(1 - e^{-\hat{\theta}x_r^{-\hat{\beta}}})^2} \right],$$

(14)

$$\hat{\ell}_{\beta \beta} = \frac{-r}{\hat{\beta}^2} - \hat{\theta} \sum_{i=1}^{r} x_i^{-\beta} \ln^2 x_i - \frac{(n - r)\hat{\theta}x_r^{-\hat{\beta}} e^{-\hat{\theta}x_r^{-\hat{\beta}}}}{\left(1 - e^{-\hat{\theta}x_r^{-\hat{\beta}}} \right)^2} \ln x_r \left[ \frac{1 - \hat{\theta}x_r^{-\hat{\beta}} - e^{\hat{\theta}x_r^{-\hat{\beta}}}}{(1 - e^{-\hat{\theta}x_r^{-\hat{\beta}}})^2} \right],$$

(15)

The asymptotic normality of the MLE can be used to compute the approximate confidence intervals for the parameters $\theta$ and $\beta$ become, respectively,

$$\hat{\theta} \pm z_{\alpha/2} \sqrt{\hat{\sigma}_{\theta}^2}, \text{ and } \hat{\beta} \pm z_{\alpha/2} \sqrt{\hat{\sigma}_{\beta}^2},$$

(16)

where $z_{\alpha/2}$ is a standard normal variate.
3 Bayes estimation

Now, we deal with the problem of estimating the parameters $\theta$ and $\beta$ and the reliability function $R(t)$ of EIW distribution under a SEL in (5) and LINEX loss functions in (6). Since the parameters $\theta$ and $\beta$ are assumed to be unknown, the prior distributions for $\theta$ and $\beta$ are taken to be Gamma($a, b$) and Gamma($c, d$) respectively. So, the joint prior distribution of $\theta$ and $\beta$ is of the form

$$\pi(\theta, \beta) \propto \theta^{a-1} \beta^{c-1} e^{-b\theta-d\beta}, \theta > 0, \beta > 0, a > 0, b > 0, c > 0, d > 0.$$  \tag{17}$$

Then the posterior distribution of $\theta$ and $\beta$ is obtained as

$$\pi^* (\theta, \beta | x) = k^{-1} \theta^{r+a-1} \beta^{r+c-1} U(\beta, x) e^{-[\theta(\sum_{i=1}^{r} x_i^{-\beta} + b) + d\beta]} \left[ 1 - e^{-\theta x_r^{-\beta}} \right]^{(n-r)},$$  \tag{18}$$

where

$$k = \int_{0}^{\infty} \int_{0}^{\infty} \theta^{r+a-1} \beta^{r+c-1} U(\beta, x) e^{-[\theta(\sum_{i=1}^{r} x_i^{-\beta} + b) + d\beta]} \left[ 1 - e^{-\theta x_r^{-\beta}} \right]^{(n-r)} d\theta d\beta. \tag{19}$$

Now, we obtain Bayesian estimates of $\theta$ and $\beta$ and $R(t)$ against the squared error loss function $L_1$ and the LINEX loss function $L_2$ when the prior distribution is given by (17).

Bayesian estimates of $\theta$ and $\beta$ against the loss function $L_1$ are respectively obtained as

$$\hat{\theta}_{BS} = E[\theta | x] = \frac{1}{k} \int_{0}^{\infty} \int_{0}^{\infty} \theta^{r+a-1} \beta^{r+c-1} U(\beta, x) e^{-[\theta(\sum_{i=1}^{r} x_i^{-\beta} + b) + d\beta]} \left[ 1 - e^{-\theta x_r^{-\beta}} \right]^{(n-r)} d\theta d\beta, \tag{20}$$

and

$$\hat{\beta}_{BS} = E[\beta | x] = \frac{1}{k} \int_{0}^{\infty} \int_{0}^{\infty} \theta^{r+a-1} \beta^{r+c-1} U(\beta, x) e^{-[\theta(\sum_{i=1}^{r} x_i^{-\beta} + b) + d\beta]} \left[ 1 - e^{-\theta x_r^{-\beta}} \right]^{(n-r)} d\theta d\beta. \tag{21}$$

the Bayesian estimate of the reliability function $R(t)$ for the loss function $L_1$ is obtained as

$$\hat{R}_{BS}(t) = E[R(t) | x] = 1 - \frac{1}{k} \int_{0}^{\infty} \int_{0}^{\infty} \theta^{r+a-1} \beta^{r+c-1} U(\beta, x) e^{-[\theta(\sum_{i=1}^{r} x_i^{-\beta} + b) + d\beta]} \left[ 1 - e^{-\theta x_r^{-\beta}} \right]^{(n-r)} d\theta d\beta, \tag{22}$$

Also, against the loss function $L_2$, the Bayesian estimates of $\theta$ and $\beta$

$$\hat{\theta}_{BL} = E[e^{-h\theta} | x] = \frac{1}{k} \int_{0}^{\infty} \int_{0}^{\infty} \theta^{r+a-1} \beta^{r+c-1} U(\beta, x) e^{-[\theta(\sum_{i=1}^{r} x_i^{-\beta} + b + h) + d\beta]} \left[ 1 - e^{-\theta x_r^{-\beta}} \right]^{(n-r)} d\theta d\beta, \tag{23}$$

and
and
\[
\hat{\beta}_{BL} = E\left[e^{-h\beta|x}\right] = \frac{1}{k} \int_0^\infty \int_0^\infty g^{r+a-1}\beta^{r-1}U(\beta, x) e^{-\left[\theta\left(\sum_{i=1}^r x_i^{-\beta} + b\right) + (d+h)\beta\right]} \left[1 - e^{-\theta x^{-\beta}}\right]^{(n-r)} d\theta d\beta.
\]

The Bayesian estimate of the reliability function $R(t)$ for the loss function $L_2$ is obtained as
\[
\hat{R}_{BL}(t) = E\left[e^{-hR(t)|x}\right] = \frac{1}{k} \int_0^\infty \int_0^\infty g^{r+a-1}\beta^{r-1}U(\beta, x) e^{-\left[\theta\left(\sum_{i=1}^r x_i^{-\beta} + b\right) + (d+h)\beta\right]} \left[1 - e^{-\theta x^{-\beta}}\right]^{(n-r)} d\theta d\beta,
\]

In the next section all estimators considered in this section, are obtained using a well known approximation method.

4 Lindley approximation

In the previous section, based on a type II censored sample we obtained several Bayesian estimates of $\theta$, $\beta$ and the reliability function $R(t)$ of an EIW($\theta, \beta$) distribution. These Bayesian estimates are derived against squared error and LINEX loss functions. It is easily observed that all these estimates are in the form of ratio of two integrals for which simplified closed forms are not available. Thus to evaluate these estimates in practice intensive numerical techniques are required. Instead, one can apply approximation methods to evaluate these estimates. Here, we use Lindley’s method (see Lindley (1980)) to approximate all the Bayesian estimates discussed in the previous section. For our estimation problem we briefly describe this method below. As noticed the Bayesian estimates involve the ratio of two integrals, we consider $I(x)$ defined as
\[
I(x) = \frac{\int_0^\infty \int_0^\infty u(\theta, \beta) e^{\ell(\theta, \beta|x) + \rho(\theta, \beta)} d\theta d\beta}{\int_0^\infty \int_0^\infty e^{\ell(\theta, \beta|x) + \rho(\theta, \beta)} d\theta d\beta},
\]

where $u(\theta, \beta)$ is function of $\theta$ and $\beta$ only and $\ell(\theta, \beta|x)$ is the log-likelihood (defined by the Eq. (8)) and $\rho(\theta, \beta) = \log \pi(\theta, \beta)$. Utilizing the Lindley’s method $I(x)$ can be approximated as
\[
I(x) = u(\hat{\theta}, \hat{\beta}) + \frac{1}{2} \left[(\hat{u}_{\theta\theta} + 2\hat{u}_{\theta\beta}) \hat{\sigma}_{\theta\theta} + (\hat{u}_{\beta\theta} + 2\hat{u}_{\beta\beta}) \hat{\sigma}_{\beta\beta} + (\hat{u}_{\theta\beta} + 2\hat{u}_{\theta\beta}) \hat{\sigma}_{\theta\beta} + (\hat{u}_{\beta\theta} + 2\hat{u}_{\beta\beta}) \hat{\sigma}_{\theta\beta}\right] + \left[(\hat{u}_{\beta\beta} + 2\hat{u}_{\beta\beta}) \hat{\sigma}_{\beta\beta} + \frac{1}{2} \left[(\hat{u}_{\theta\theta} \hat{\sigma}_{\theta\theta} + \hat{u}_{\beta\theta} \hat{\sigma}_{\theta\beta} + \hat{u}_{\theta\beta} \hat{\sigma}_{\theta\beta} + \hat{u}_{\beta\theta} \hat{\sigma}_{\theta\beta} + \hat{u}_{\beta\beta} \hat{\sigma}_{\beta\beta} + \hat{u}_{\theta\beta} \hat{\sigma}_{\beta\beta} + \hat{u}_{\beta\beta} \hat{\sigma}_{\beta\beta} + \hat{u}_{\beta\beta} \hat{\sigma}_{\beta\beta}\right]ight],
\]
where \( \hat{\theta} \) and \( \hat{\beta} \) are the MLE's of \( \theta \) and \( \beta \) respectively. Also, \( u_{\theta\theta} \) is the second derivative of the function \( u(\theta, \beta) \) with respect to \( \theta \) and \( \hat{u}_{\theta\theta} \) is the same expression evaluated at \( (\hat{\theta}, \hat{\beta}) \). Other expressions can be interpreted exactly in similar manner with following definitions,

\[
\hat{\ell}_{\theta\theta} = \left. \frac{\partial^2 \ell}{\partial \theta \partial \theta} \right|_{\theta=\hat{\theta}, \beta=\hat{\beta}} = \frac{2r}{\beta^3} + \frac{(n-r)x_r^{-3\hat{\beta}}e^{-\hat{\beta}x_r^{-\hat{\beta}}}(1 + e^{-\hat{\theta}x_r^{-\hat{\beta}}})}{(1 - e^{-\hat{\theta}x_r^{-\hat{\beta}}})^3},
\]

\[
\hat{\ell}_{\beta\beta} = \left. \frac{\partial^2 \ell}{\partial \beta \partial \beta} \right|_{\theta=\hat{\theta}, \beta=\hat{\beta}} = \frac{2r}{\beta^3} + \sum_{i=1}^{r} x_i^{-\hat{\beta}} \ln^3 x_i - (n-r)\hat{\theta}x_r^{-\hat{\beta}}e^{-\hat{\theta}x_r^{-\hat{\beta}}} \ln^3 x_r
\]
\[
\times \left\{ \frac{1 - 3\hat{\theta}x_r^{-\hat{\beta}} + \hat{\beta}^2 x_r^{-2\hat{\beta}}}{(1 - e^{-\hat{\theta}x_r^{-\hat{\beta}}})} - \frac{3\hat{\theta}x_r^{-\hat{\beta}}e^{-\hat{\theta}x_r^{-\hat{\beta}}} [1 - \hat{\theta}x_r^{-\hat{\beta}}e^{-\hat{\theta}x_r^{-\hat{\beta}}}]^2 + \frac{2\hat{\theta}^2 x_r^{-2\hat{\beta}}e^{-2\hat{\theta}x_r^{-\hat{\beta}}}}{(1 - e^{-\hat{\theta}x_r^{-\hat{\beta}}})^3} \right\},
\]

\[
\hat{\rho}_\theta = \left. \frac{\partial \log \pi(\theta, \beta)}{\partial \theta} \right|_{\theta=\hat{\theta}, \beta=\hat{\beta}} = \frac{(a-1)}{\theta} - a , \quad \hat{\rho}_\beta = \left. \frac{\partial \log \pi(\theta, \beta)}{\partial \beta} \right|_{\theta=\hat{\theta}, \beta=\hat{\beta}} = \frac{(c-1)}{\beta} - d.
\]

\( \sigma_{ij} \) is the \((i, j)\)th elements of matrix \( \left[ - \frac{\partial^2 \ell}{\partial \theta \partial \beta} \right]^{-1} \); \( i, j = 1, 2 \).

With the above defined expressions, we can deduce the values of the Bayes estimates under SEL of various parameters in what follows.

(I) If \( u(\theta, \beta) = \theta \), then \( u_{\theta\theta} = u_\beta = u_{\theta\beta} = u_{\beta\beta} = u_{\theta\beta} = 0 \),

\[
\hat{\theta}_{BS} = E(\theta|x) = \hat{\theta} + 0.5 \left( 2\hat{\rho}_\theta \hat{\sigma}_{\theta\theta} + 2\hat{\rho}_\beta \hat{\sigma}_{\theta\beta} \right) + 0.5 \hat{\sigma}_{\theta\theta} \left( \hat{\ell}_{\theta\theta\theta} \hat{\sigma}_{\theta\theta} + \hat{\ell}_{\theta\theta\beta} \hat{\sigma}_{\theta\beta} + \hat{\ell}_{\theta\beta\beta} \hat{\sigma}_{\beta\beta} + \hat{\ell}_{\beta\beta\beta} \hat{\sigma}_{\beta\beta} \right) + 0.5 \hat{\sigma}_{\theta\beta} \left( \hat{\ell}_{\theta\theta\beta} \hat{\sigma}_{\theta\beta} + \hat{\ell}_{\theta\beta\beta} \hat{\sigma}_{\beta\beta} + \hat{\ell}_{\beta\beta\beta} \hat{\sigma}_{\beta\beta} \right).
\]

(II) If \( u(\theta, \beta) = \beta \), then \( u_\beta = 1, u_{\beta\beta} = u_\theta = u_{\theta\theta} = u_{\theta\beta} = u_{\beta\beta} = 0 \), the Bayesian estimate of \( \beta \) under SEL is given by

\[
\hat{\beta}_{BS} = E(\beta|x) = \hat{\beta} + (\hat{\rho}_\theta \hat{\sigma}_{\beta\theta} + \hat{\rho}_\beta \hat{\sigma}_{\beta\beta} + \hat{\rho}_\beta \hat{\sigma}_{\beta\beta}) + 0.5 \hat{\sigma}_{\beta\theta} \left( \hat{\ell}_{\beta\theta\theta} \hat{\sigma}_{\beta\theta} + \hat{\ell}_{\beta\theta\beta} \hat{\sigma}_{\beta\beta} + \hat{\ell}_{\beta\beta\beta} \hat{\sigma}_{\beta\beta} \right) + 0.5 \hat{\sigma}_{\beta\beta} \left( \hat{\ell}_{\beta\theta\beta} \hat{\sigma}_{\beta\beta} + \hat{\ell}_{\beta\beta\beta} \hat{\sigma}_{\beta\beta} + \hat{\ell}_{\beta\beta\beta} \hat{\sigma}_{\beta\beta} \right),
\]

(III) If we put \( u(\theta, \beta) \equiv R(t) = 1 - e^{-\theta t^{-\beta} \ln(t)} \) in (27), \( u_{\theta\theta} = t^{-\beta}e^{-\theta t^{-\beta}}, u_{\theta\beta} = -t^{-2\beta}e^{-\theta t^{-\beta}}, u_{\beta\beta} = \theta t^{-3\beta} \ln(t) e^{-\theta t^{-\beta}}, u_{\beta\beta} = -\theta \ln(t) t^{-\beta}e^{-\theta t^{-\beta}}, u_{\beta\beta} = \theta \ln^2(t) t^{-2\beta}e^{-\theta t^{-\beta}} \), we obtain the Bayes estimate \( \hat{R}_{BS}(t) \) of the reliability function \( R(t) \), in which

\[
\hat{R}_{BS}(t) = \left( 1 - e^{-\hat{\theta} t^{-\hat{\beta}}} \right) + \frac{1}{2} \left( \hat{u}_{\theta\theta} + 2\hat{u}_{\theta \hat{\rho}_\theta} \hat{\sigma}_{\theta\theta} + \hat{u}_{\beta\beta} + 2\hat{u}_{\beta \hat{\rho}_\beta} \hat{\sigma}_{\beta\beta} + \hat{u}_{\theta\beta} + 2\hat{u}_{\theta \hat{\rho}_\beta} \hat{\sigma}_{\beta\beta} \right) + \hat{u}_{\beta\beta} \left( \hat{\rho}_\theta \hat{\sigma}_{\beta\theta} + \hat{\rho}_\beta \hat{\sigma}_{\beta\beta} \right) + \frac{1}{2} \left[ \hat{u}_{\theta\theta} \hat{\sigma}_{\theta\theta} + \hat{u}_{\beta\beta} \hat{\sigma}_{\beta\beta} \right] \left( \hat{\ell}_{\theta\theta\theta} \hat{\sigma}_{\theta\theta} + \hat{\ell}_{\theta\theta\beta} \hat{\sigma}_{\theta\beta} + \hat{\ell}_{\theta\beta\beta} \hat{\sigma}_{\beta\beta} + \hat{\ell}_{\beta\beta\beta} \hat{\sigma}_{\beta\beta} \right).
\]
With the same argument, we can obtain Bayes estimators under the LINEX loss function of the parameters $\theta$, $\beta$ and the reliability function from Eq. (27). They are obtained by the following forms:

(I) If $u(\theta, \beta) = e^{-h\theta}$, then $u_\theta = -he^{-h\theta}$, $u_{\theta\theta} = h^2e^{-h\theta}$, $u_\beta = u_{\theta\beta} = u_{\beta\beta} = u_{\beta\theta} = 0$, the Bayes estimate $\hat{\theta}_{BL}$ of the parameter $\theta$

\[
\hat{\theta}_{BL} = -\frac{1}{h} \ln \left[ E \left( e^{-h\theta} \right) \right], \tag{31}
\]

where

\[
E \left( e^{-h\theta} \right) = e^{-h\theta_{ML}} + \frac{1}{2} \left[ (\hat{\theta}_{\theta\theta} + 2\hat{\theta}_{\theta\beta}) \sigma_{\theta\theta} + (\hat{\theta}_{\theta\beta} + 2\hat{\theta}_{\beta\beta}) \sigma_{\theta\beta} \right] + \frac{1}{2} \left[ \hat{\theta}_{\beta\beta} \right] + \hat{\theta}_{\beta\beta} \left[ \hat{\theta}_{\beta\theta} + \hat{\theta}_{\beta\beta} \right] \left[ \hat{\theta}_{\theta\theta} + \hat{\theta}_{\beta\beta} \right]. \tag{32}
\]

(II) If $u(\theta, \beta) = e^{-h\beta}$, with $u_\beta = -he^{-h\beta}$, $u_{\beta\beta} = h^2e^{-h\beta}$, $u_\theta = u_{\theta\beta} = u_{\beta\theta} = u_{\theta\theta} = 0$, we obtain the Bayes estimate $\hat{\beta}_{BL}$ of the parameter $\beta$

\[
\hat{\beta}_{BL} = -\frac{1}{h} \ln \left[ E \left( e^{-h\beta} \right) \right], \tag{33}
\]

where

\[
E \left( e^{-h\beta} \right) = e^{-h\beta_{ML}} + \frac{1}{2} \left[ 2\hat{\beta}_{\beta\beta} \sigma_{\beta\beta} + \hat{\beta}_{\beta\beta} \sigma_{\beta\beta} \right] + \frac{1}{2} \left[ \hat{\beta}_{\beta\beta} \right] + \hat{\beta}_{\beta\beta} \left[ \hat{\beta}_{\beta\beta} \right] \left[ \hat{\beta}_{\beta\beta} \right] \left[ \hat{\beta}_{\beta\beta} \right]. \tag{34}
\]

(II) If $u(\theta, \beta) = e^{-hR(t)}$ in (27),

\[
\begin{align*}
&\left\{ \\
&u_\theta = -ht^{-\beta}e^{-hR(t)+\theta t^{-\beta}}, \\
&u_{\theta\theta} = h^2t^{-2\beta} e^{-hR(t)+2\theta t^{-\beta}}, \\
&u_\beta = -cht^{-\beta} \ln(t) e^{-hR(t)+\theta t^{-\beta}}, \\
&u_{\beta\beta} = -ht^{-2\beta} \ln(t) e^{-hR(t)+2\theta t^{-\beta}}, \\
&u_{\theta\beta} = u_{\beta\theta} = -ht^{-2\beta} \ln(t) e^{-hR(t)+2\theta t^{-\beta}}, \\
&u_{\theta\theta} = -ht^{-2\beta} \ln(t) e^{-hR(t)+2\theta t^{-\beta}}
\end{align*} \tag{35}
\]

The Bayes estimate of the reliability function $R(t)$ under a LINEX loss function is given by

\[
\hat{R}_{BL} = -\frac{1}{h} \ln \left[ E \left( e^{-hR(t)} \right) \right] \tag{36}
\]

where

\[
E \left( e^{-hR(t)} \right) = e^{-cR_{ML}} + \frac{1}{2} \left[ (\hat{\theta}_{\theta\theta} + 2\hat{\theta}_{\theta\beta}) \sigma_{\theta\theta} + (\hat{\theta}_{\theta\beta} + 2\hat{\theta}_{\beta\beta}) \sigma_{\theta\beta} + (\hat{\theta}_{\beta\beta} + 2\hat{\theta}_{\beta\theta}) \sigma_{\beta\beta} \right] + \frac{1}{2} \left[ \hat{\theta}_{\beta\beta} \right] + \hat{\theta}_{\beta\beta} \left[ \hat{\theta}_{\beta\beta} \right] \left[ \hat{\theta}_{\beta\beta} \right] \left[ \hat{\theta}_{\beta\beta} \right] \left[ \hat{\theta}_{\beta\beta} \right]. \tag{37}
\]
Markov Chain Monte Carlo Estimation

The MCMC algorithm is used for computing the Bayes estimates of the parameters $\theta$ and $\beta$ as well as the reliability function. We consider the Metropolis-Hastings algorithm, to generate samples from the conditional posterior distributions and then compute the Bayes estimates. For more details about the MCMC methods see, for example, Upadhyaya et al. (2001) and Upadhyaya and Gupta (2010). The Metropolis-Hastings (Metropolis et al. (1953)) algorithm generate samples from an arbitrary proposal distribution (i.e a Markov transition kernel). From (18), the marginal posterior density of $\theta$ is proportional to

$$
\pi_1^*(\theta|\beta, x) \propto \theta^{r+a-1} e^{-\theta\left(\sum_{i=1}^{r} x_i^{-\beta} + b\right)}
$$

(38)

Similarly, the full posterior conditional distributions for $\beta$ is

$$
\pi_2^*(\beta|\theta, x) \propto \beta^{r+c-1} e^{-d\beta} \prod_{i=1}^{r} x_i^{-(\beta+1)} \left[1 - e^{-\theta x_i^{-\beta}}\right]^{(n-r)},
$$

(39)

It can be seen that Equation (42) is a gamma density with shape parameter $(r + a)$ and scale parameter $(\sum_{i=1}^{r} x_i^{-\beta} + b)$ and, therefore, samples of $\theta$ can be easily generated using any gamma generating routine. However, in our case, both conditional posterior distributions of $\beta$ cannot be reduced analytically to well known distributions and therefore it is not possible to sample directly by standard methods, but the plots of them show that they are similar to normal distribution. So, as suggested by Tierney (1994), a common way to solve this problem is to use the hybrid algorithm by combined a Metropolis sampling with the Gibbs sampling scheme using normal proposal distribution. To sample from (43) we generated a proposal value from a normal distribution $N\left(\beta^{(j-1)}, K_\beta \hat{\sigma}_{\beta\beta}\right)$, $\hat{\sigma}_{\beta\beta}$ is variances-covariances matrix, and $K_\beta$ is scaling factors. The hybrid Metropolis-Hasting and Gibbs sampler works as follows:

1. Start with initial guess $\left(\theta^{(0)}, \beta^{(0)}\right)$.
2. set $j = 1$.
3. Generate $\theta^{(j)}$ from Gamma($r + a, b + \sum_{i=1}^{r} x_i^{-\beta^{(j)}}$).
4. Using Metropolis–Hastings, generate $\beta^{(j)}$ from $\pi_2^*(\beta^{(j-1)}|\theta^{(j)}, x)$ with normal proposal distribution, $N\left(\beta^{(j-1)}, K_\beta \hat{\sigma}_{\beta\beta}\right)$.
5. Compute $R(t)$ from (3).
7. Repeat steps 3–6, $N$ times.

It well known that rapid convergence is facilitated by choosing appropriate starting values. In order to guarantee the convergence and to remove the affection of the selection of initial value, the first $M$ simulated variates are discarded. Then the selected sample are $\theta^{(j)}$ and $\beta^{(j)}$, $j = M + 1, ..., N$, for sufficiently large $N$, forms an approximate posterior sample which
can be used to develop the Bayesian inference. Furthermore, \((\theta_{[\alpha/2]}, \theta_{[1-\alpha/2]}), (\beta_{[\alpha/2]}, \beta_{[1-\alpha/2]})\) and \((R_{[\alpha/2]}, R_{[1-\alpha/2]})\) yields the approximate 100\((1-\gamma)%\) credible intervals for \(\theta\), \(\beta\), and \(R(t)\), respectively.

Based on squared error loss (SEL), given by (5), the Bayes estimates of the unknown parameters as \(\theta\), \(\beta\) as well as reliability functions are given, respectively by

\[
\hat{\theta}_{BSMC} = \frac{1}{N - M} \sum_{j=M+1}^{N} \theta^{(j)},
\]

(40)

\[
\hat{\beta}_{BSMC} = \frac{1}{N - M} \sum_{j=M+1}^{N} \beta^{(j)},
\]

(41)

and

\[
\hat{R}(t)_{BSMC} = \frac{1}{N - M} \sum_{j=M+1}^{N} R^{(j)}(t).
\]

(42)

Also, the approximate Bayes estimates for \(\theta\), \(\beta\), and \(R(t)\), under LINEX loss function, from (6) are then given by

\[
\hat{\theta}_{BMC} = \frac{-1}{h} \log \left[ \frac{1}{N - M} \sum_{j=M+1}^{N} e^{-h\theta^{(j)}} \right],
\]

(43)

\[
\hat{\beta}_{BMC} = \frac{-1}{h} \log \left[ \frac{1}{N - M} \sum_{j=M+1}^{N} e^{-h\beta^{(j)}} \right],
\]

(44)

and

\[
\hat{R}(t)_{BMC} = \frac{-1}{h} \log \left[ \frac{1}{N - M} \sum_{j=M+1}^{N} e^{-hR^{(j)}(t)} \right].
\]

(45)

5 Data analysis and comparison study

In this section, a Monte Carlo simulation study and a real data set are presented to illustrate all the estimation methods described in the preceding sections. All the computations are performed using mathematica code.

Numerical comparison study

In this subsection, we present some results based on Monte Carlo simulations to compare the performance of the different methods for different censoring schemes, and for different parameter
values. We compare the performances of the MLEs and the Bayes estimates (with respect to the squared error loss function and LINEX loss function) in terms of mean squares errors (MSEs).

In order to assess the statistical performances of these estimates, a simulation study is conducted. The random samples are generated as follows:

1. For given values of the prior parameters \((a,b,c,d)\), generate random values for \(\theta\) and \(\beta\) from the gamma distributions.

2. Using \(\theta\) and \(\beta\), obtained in step (1), and generate random samples of different sizes: \(n = 15, 40\) and \(50\) from the EIW(\(\theta,\beta\)) distribution. The computations are carried out for such sample sizes and censored samples of sizes: \(r = 10, 35\) and \(45\), respectively.

3. The MLE of the parameters \(\theta\) and \(\beta\) are obtained by iteratively solving the Equations (9) and (10). The estimator \(\hat{R}_{ML}(t)\) of the functions \(R(t)\) is then computed at some values of \(t\).

4. Based on Lindley approximation, the Bayes estimates relative to squared error loss, \(\hat{\theta}_{BS}\), \(\hat{\beta}_{BS}\) and \(\hat{R}_{BS}(t)\), given, respectively, by Equations (28), (29)and (30), and relative to LINEX loss \(\hat{\theta}_{BI}, \hat{\beta}_{BI}\) and \(\hat{R}_{BI}(t)\), given, respectively, by Equations (31), (33) and (36), are all computed.

5- We run the Gibbs sampler with in M-H algorithm to generate a Markov chain with 10,000 observations. Discarding the first 500 values as ‘burn-in’ and taking every tenth variate as i.i.d. observations. Based on MCMC samples, the Bayes estimates relative to squared error loss, \(\hat{\theta}_{BSCM}, \hat{\beta}_{BSCM}\) and \(\hat{R}_{BSCM}(t)\), given, respectively, by Equations (40), (41)and (42), and relative to LINEX loss \(\hat{\theta}_{BMI}, \hat{\beta}_{BMI}\) and \(\hat{R}_{BMI}(t)\), given, respectively, by Equations (43), (44) and (45), are all computed.

6- The above steps are repeated 1000 times and the biases and the mean square errors are computed for different sample sizes \(n\) and censoring sizes \(r\),

In all above cases the prior parameters chosen as \(a = 2, b = 1, c = 2\) and \(d = 1\), which yield the generated values of \(\theta\) and \(\beta\) are 0.7013 and0.9129 as the true values. Also The true values of \(R(t)\) and, when \(t = 0.5\) is computed to be \(R(0.9) = 0.5379\). The average Bayes estimates (first entries) and mean squared errors (MSEs) (second entries) are displayed in Tables 1-3. The computations are achieved under complete and censored samples.
Table 1: Average estimates, and the associated MSE's of the ML and the Bayes estimates of $\theta$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r$</th>
<th>Criteria</th>
<th>ML</th>
<th>$\theta_{BS}$</th>
<th>$\theta_{BL}$</th>
<th>Lindley</th>
<th>$\theta_{BS}$</th>
<th>$\theta_{BL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
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<td>$\hat{\theta}_{BS}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
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<td>$h = 2$</td>
<td>$h = 1$</td>
<td>$h = 2$</td>
<td>$h = 1$</td>
<td>$h = 2$</td>
</tr>
<tr>
<td>15</td>
<td>10</td>
<td>mean</td>
<td>0.7028</td>
<td>0.7136</td>
<td>0.7092</td>
<td>0.7205</td>
<td>0.7116</td>
<td>0.7097</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
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<td>0.0324</td>
<td>0.0303</td>
<td>0.0305</td>
<td>0.0285</td>
<td>0.0279</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>mean</td>
<td>0.7001</td>
<td>0.7211</td>
<td>0.7123</td>
<td>0.7116</td>
<td>0.7033</td>
<td>0.7066</td>
</tr>
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<td></td>
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<td>MSE</td>
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<td>0.0301</td>
<td>0.0285</td>
<td>0.0288</td>
<td>0.0267</td>
<td>0.0261</td>
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<tr>
<td>40</td>
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<tr>
<td></td>
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<td>MSE</td>
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<td>0.0196</td>
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<td>mean</td>
<td>0.7013</td>
<td>0.7152</td>
<td>0.7067</td>
<td>0.7065</td>
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<td>50</td>
<td>45</td>
<td>mean</td>
<td>0.7053</td>
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<td>MSE</td>
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<td>0.0146</td>
<td>0.0140</td>
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<td>0.0145</td>
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<tr>
<td>50</td>
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<td>mean</td>
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<td>0.7122</td>
<td>0.7274</td>
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<td></td>
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<td>MSE</td>
<td>0.0139</td>
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<td>0.0137</td>
<td>0.0138</td>
<td>0.0142</td>
<td>0.0136</td>
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</table>

Table 2: Average estimates, and the associated MSEs of the ML and the Bayes estimates of $\beta$.

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<th>$n$</th>
<th>$r$</th>
<th>Criteria</th>
<th>ML</th>
<th>$\beta_{BS}$</th>
<th>$\beta_{BL}$</th>
<th>Lindley</th>
<th>$\beta_{BS}$</th>
<th>$\beta_{BL}$</th>
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<tbody>
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<td></td>
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<td></td>
<td>$\hat{\beta}_{BS}$</td>
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<td></td>
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<td>$h = 1$</td>
<td>$h = 2$</td>
<td>$h = 1$</td>
<td>$h = 2$</td>
</tr>
<tr>
<td>15</td>
<td>10</td>
<td>mean</td>
<td>0.9147</td>
<td>0.9203</td>
<td>0.9092</td>
<td>0.8997</td>
<td>0.9147</td>
<td>0.9327</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
<td>0.1927</td>
<td>0.1169</td>
<td>0.1140</td>
<td>0.0689</td>
<td>0.1955</td>
<td>0.1822</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>mean</td>
<td>0.9266</td>
<td>0.9128</td>
<td>0.9152</td>
<td>0.8962</td>
<td>0.8833</td>
<td>0.9012</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
<td>0.0816</td>
<td>0.0526</td>
<td>0.0511</td>
<td>0.0437</td>
<td>0.0656</td>
<td>0.0521</td>
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<tr>
<td>40</td>
<td>35</td>
<td>mean</td>
<td>0.9111</td>
<td>0.9245</td>
<td>0.8898</td>
<td>0.9035</td>
<td>0.9212</td>
<td>0.9302</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
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<td>0.0389</td>
<td>0.0362</td>
<td>0.0279</td>
<td>0.0615</td>
<td>0.0528</td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>mean</td>
<td>0.9332</td>
<td>0.9087</td>
<td>0.9024</td>
<td>0.8938</td>
<td>0.9294</td>
<td>0.9232</td>
</tr>
<tr>
<td></td>
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<td>MSE</td>
<td>0.0154</td>
<td>0.0132</td>
<td>0.0128</td>
<td>0.0114</td>
<td>0.0144</td>
<td>0.0137</td>
</tr>
<tr>
<td>50</td>
<td>45</td>
<td>mean</td>
<td>0.9206</td>
<td>0.9005</td>
<td>0.8952</td>
<td>0.8885</td>
<td>0.9340</td>
<td>0.9297</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
<td>0.0123</td>
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<td>0.0100</td>
<td>0.0116</td>
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<tr>
<td>50</td>
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<td>mean</td>
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<td>0.8907</td>
<td>0.9189</td>
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<tr>
<td></td>
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<td>MSE</td>
<td>0.0114</td>
<td>0.0103</td>
<td>0.0101</td>
<td>0.0093</td>
<td>0.0111</td>
<td>0.0107</td>
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</table>
Table 3: Average estimates, and the associated MSE’s of the ML and the Bayes estimates of $R(t)$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r$</th>
<th>Criteria</th>
<th>ML $R_{ML}$</th>
<th>Lindley $R_{BL}$</th>
<th>MCMC $R_{BL}$</th>
</tr>
</thead>
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<td>$h=1$</td>
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<td>$h=1$</td>
</tr>
<tr>
<td>15</td>
<td>10</td>
<td>mean</td>
<td>0.5325</td>
<td>0.5385</td>
<td>0.5391</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
<td>0.0131</td>
<td>0.0114</td>
<td>0.0113</td>
</tr>
<tr>
<td>15</td>
<td></td>
<td>mean</td>
<td>0.5344</td>
<td>0.5351</td>
<td>0.5351</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
<td>0.0105</td>
<td>0.0089</td>
<td>0.0089</td>
</tr>
<tr>
<td>40</td>
<td>35</td>
<td>mean</td>
<td>0.5358</td>
<td>0.5354</td>
<td>0.5354</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
<td>0.0043</td>
<td>0.0004</td>
<td>0.0042</td>
</tr>
<tr>
<td>40</td>
<td></td>
<td>mean</td>
<td>0.5335</td>
<td>0.5329</td>
<td>0.5329</td>
</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
<td>0.0045</td>
<td>0.0043</td>
<td>0.0043</td>
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<tr>
<td>40</td>
<td>35</td>
<td>mean</td>
<td>0.5362</td>
<td>0.5359</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
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<td>0.0033</td>
<td>0.0033</td>
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<tr>
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<td></td>
<td>mean</td>
<td>0.5378</td>
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</tr>
<tr>
<td></td>
<td></td>
<td>MSE</td>
<td>0.0033</td>
<td>0.0032</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

From Tables 1 - 3, the Bayes estimates based on Type-II censored data relative to SEL and LINEX loss functions are better than their corresponding ML estimates, for most cases of $n$ and $r$. When the effective sample sizes $(n, r)$ are increase the MSEs of the all estimates based on Type-II censored data are decrease.

Example (real data set)

In our real data analysis we consider the data set from Nicholas and Padgett (2006) and later reported in Flaïh et al. (2012). The data concerning tensile strength of 100 observations of carbon fibers, they are:

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 2.93, 3.33, 2.56, 2.59, 2.83, 1.36, 1.84, 5.56, 1.12, 2.48, 1.25, 2.48, 2.03, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, 3.65.

From these uncensored data we applied our data selection technique for type-II censoring technique with two failure times $r = 70$ and 90 respectively. We estimated the parameters $\theta$, $\beta$ and reliability function $R(t)$ and 95% confidence intervals for the sample size 100 and number of failure 70 and 90 respectively. We run the Gibbs sampler with in MH algorithm to generate a Markov chain with 30,000 observations. Discarding the first 5000 values as ‘burn-in’ and taking every tenth variate as i.i.d. observations, starting from 5001. This is done to minimize the auto correlation among the generated deviates. The convergence is monitored using trace plots.
We used the non-informative gamma priors for $\theta$, $\beta$, that is, when the hyperparameters are 0, $a = b = c = d = 0$. The estimated results are given at Table 4, 5 and 6.

Table 4: MLE and Bayes estimates of $\theta$, $\beta$ and reliability function $R(t)$, $r = 70$.

<table>
<thead>
<tr>
<th>Estimate</th>
<th>MLE</th>
<th>Lindley</th>
<th>MCMC</th>
<th>Lindley</th>
<th>MCMC</th>
<th>Lindley</th>
<th>MCMC</th>
</tr>
</thead>
<tbody>
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<td>$\theta$</td>
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<td>$\beta$</td>
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<td>1.5181</td>
<td>1.4373</td>
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<td>1.4204</td>
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<td>$R(t = 2)$</td>
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<td>0.6405</td>
<td>0.6432</td>
<td>0.6417</td>
<td>0.6451</td>
</tr>
</tbody>
</table>

Table 5: MLE and Bayes estimates of $\theta$, $\beta$ and reliability function $R(t)$, $r = 90$.

<table>
<thead>
<tr>
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<th>MLE</th>
<th>Lindley</th>
<th>MCMC</th>
<th>Lindley</th>
<th>MCMC</th>
<th>Lindley</th>
<th>MCMC</th>
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<td>$\beta$</td>
<td>1.7117</td>
<td>1.6972</td>
<td>1.6799</td>
<td>1.6847</td>
<td>1.6650</td>
<td>1.7117</td>
<td>1.6944</td>
</tr>
<tr>
<td>$R(t = 2)$</td>
<td>0.6057</td>
<td>0.6065</td>
<td>0.6058</td>
<td>0.6049</td>
<td>0.6041</td>
<td>0.6058</td>
<td>0.6074</td>
</tr>
</tbody>
</table>

Table 6: MLE and Bayes estimates of $\theta$, $\beta$ and reliability function $R(t)$, $r = 100$.

<table>
<thead>
<tr>
<th>Estimate</th>
<th>MLE</th>
<th>Lindley</th>
<th>MCMC</th>
<th>Lindley</th>
<th>MCMC</th>
<th>Lindley</th>
<th>MCMC</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta$</td>
<td>3.0856</td>
<td>3.0796</td>
<td>3.0246</td>
<td>2.9839</td>
<td>2.9297</td>
<td>3.0856</td>
<td>3.1296</td>
</tr>
<tr>
<td>$\beta$</td>
<td>1.7737</td>
<td>1.7598</td>
<td>1.7499</td>
<td>1.7479</td>
<td>1.7372</td>
<td>1.7738</td>
<td>1.7626</td>
</tr>
<tr>
<td>$R(t = 2)$</td>
<td>0.5944</td>
<td>0.5954</td>
<td>0.5914</td>
<td>0.5938</td>
<td>0.5897</td>
<td>0.5944</td>
<td>0.5930</td>
</tr>
</tbody>
</table>

Confidence intervals can be computed using several different approaches. The usual method assumes that the sampling distribution of the estimator is approximately normal (MLEs are asymptotically normal). Thus, based on the complete samples ($r = 100$), from (16) the 95% probability intervals for $\theta$ and $\beta$ are $(2.4452, 3.7259)$ and $(1.554, 1.9935)$. Based on MCMC samples, a $100(1 - \gamma)\%$ probability interval for the unknown parameter may be estimated by taking the $(100\gamma)th$ and $(100(1 - \gamma))th$ percentiles of the sample as the lower (LL) and upper (UL) bounds, respectively. Hence, the 95% probability intervals for $\theta$, $\beta$ and reliability function $R(t)$ are given, respectively by $(2.4479, 3.6896)$, $(1.5511, 1.9870)$ and $(0.5071, 0.6678)$.

6 Conclusion

Estimation of the parameters and reliability function are obtained when data are drawn from the two parameters exponentiated inverted Weibull distribution. Type II censoring is imposed. The likelihood and Bayes methods are used in estimation. In the Bayes case, the estimators are obtained under squared-error and LINEX loss functions. The MCMC method provides an alternative method for parameter estimation of the EIW model. It is more flexible as compared to the traditional methods such as MLE method and Lindley approximation. Indeed, the MCMC
sample may be used to completely summarize posterior distribution about the parameters, through a kernel estimate. This is also true for any function of the parameters such as reliability function, hazard function, etc. The MCMC procedure can easily be applied to complex Bayesian modeling relating to EIW model. The methods are compared by computing the mean squared errors. From Tables 1-3, it may be noticed that the Bayes estimates are, generally, better than the MLEs against the proposed prior in the sense of having smaller MSEs. Even for sample size as small as $n = 15$, good Bayes estimates (with smaller MSEs), are obtained under LINEX loss function as well as SEL with the same censoring level. All estimates improve by increasing sample size.

References


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