Option Pricing under Generalized Lévy Processes

with State Dependent Parameters

and the Volatility Surface

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Abstract

This paper presents a very general option pricing formula incorporating both the Lévy process methodology and the level dependent volatility approach. An approximate solution to the pricing problem is obtained throughout the construction of a parametrix by means of the pseudo differential calculus. Some examples are provided to illustrate the comprehensiveness of the framework. Finally, the implications in terms of the volatility smile are discussed.

Keywords: Lévy processes; pseudo-differential equations; volatility surface

1 Introduction

An impressive amount of financial literature is concerned with the theoretical modeling of the empirically observed implied volatilities and their term structures. Since a long time it
has been recognized that, when market option prices are used to estimate implied volatilities using the Black-Scholes model, at-the-money options typically exhibit lower volatilities than in or out-of-the-money options. In other words, when implied volatilities for options with the same expiration date are plotted against a moneyness ratio the resulting curve is U-shaped and this effect is termed "volatility smile". This effect is more pronounced for short maturities and the volatility surface tends to flatten out as time to maturity increases. Moreover smile asymmetries and volatility skews have been detected depending on the underlying security and the observation period. For example, the S&P500 implied volatilities followed a "smile" curve before the crash of 1987, while the post-crash pattern more resembles a "sneer" or "skew", with implied volatilities monotonically decreasing with increasing strikes (Andersen and Andreasen (2000)).

An extensive literature has been developed to capture such effects. Some models aims at offering an economic explanation of the emergence of smiles or skews (see Platen and Schweizer (1998), for example, where the volatility patterns are derived endogenously from agents' trading behavior), but usually volatility is modeled throughout an exogenously given function or stochastic process which is chosen on account of analytical tractability and good fitting of empirical data. Such extensions of the Black-Scholes model separate into two classes of models: the level dependent volatility approach and the stochastic volatility approach. The former approach, which was pioneered by Dupire (1994), models the volatility of the underlying asset as a deterministic function of time and the asset price. The stochastic volatility approach models the volatility as a further stochastic process, by introducing a new source of randomness and thus dismissing the market completeness of the Black-Scholes model. A comparison of the two approaches is contained in Hobson and Rogers (1998), where a new model is added to this literature. Given the computational complexity of stochastic volatility models, the local volatility approach is an attempt to offer a simpler practical way of pricing options consistently with the observed volatility skew and can be interpreted in the sense of “some kind of average over all the possible instantaneous volatility in a stochastic-volatility world” (Gatheral (2006)). On the other hand, the addition of jumps to the diffusion dynamics of the underlying asset or, more generally, the use of Lévy processes, can generate a multitude of volatility smiles and skews. Today models combining jumps and stochastic volatility have become common (see Carr et al. (2003), (2004), (2011)) and the empirical work following the theoretical research generally supports the need for both in the underlying asset: jumps are needed for the short-term behavior while stochastic volatility modeling works especially for long-term smiles, thus allowing for a better calibration of the implied volatility surface. The stochastic volatility effect can be engineered by incorporating stochastic time changes into
the price process: a second process is introduced that makes time stochastic and the price process is subordinated by this stochastic "clock" thus letting time run faster (slower) in periods with high (low) volatility. (See Carr et al. (2003)). Such models can also be accommodated to make the stochastic higher return moments significantly varying over time. Unfortunately, as Madan (2009) points out, these models tying volatility to randomness in time changes always predict a negative relationship between ATM implied volatility and the implied volatility skew, which conflicts with some observed data (see also Gatheral (2006)). Therefore Madan (2009) suggests to develop alternative models, for example, to scale innovation instead of time. On the other hand, the literature is not unanimous in favoring one class of models, while further empirical research has addressed some sophisticated patterns for the shape of implied volatility (See the literature review in Duque et al. (2003)). Thus a significant room is left to explore some alternatives to the most successful models.

This paper explores the potentialities of an approach that is the natural extension to a Lévy framework of the Black-Scholes models with variable coefficients, which are used to reduce the smile. In this method (see Agliardi (2011,a), Barndorff-Nielsen and Levendorskiĭ (2001), Boyarchenko and Levendorskiĭ (2002)) state-dependent functions are introduced in the generator of the process, which amounts to considering some pseudo differential operators of order in ]0,2[ as generators of Feller processes generalizing the classical Lévy case.

The main result is a general approximate pricing formula that generates several new valuation expressions and incorporates some historical models. Thanks to the great flexibility of the method one can obtain realistic volatility surfaces. Section 2 provides the notation on pseudo differential operators and the basic result, while the pricing formula is offered in Section 3. Section 4 concludes.

2 Notation and main result

Throughout the paper the price of any contingent claim is obtained as the solution of a generalized Black-Scholes equation of the form:

\[(2.1) \quad [\partial_t - r - \psi(x,D_x)]f(t,x) = 0\]

with terminal condition \(f(T,x)=g(x)\) which represents the terminal payoff. Here \(\psi(x,\xi)\) is assumed to be the characteristic exponent of a Lévy process for each fixed \(x \in \mathbb{R}^n\), and the
equivalent martingale measure (EMM) requirement \( r + \psi(x, -i) = 0 \) is supposed to hold. We refer to Agliardi (2009) and Agliardi (2011, b) for option pricing under a traditional Lévy framework. In this work we will employ the classical method of constructing an approximate solution to the Cauchy problem for (degenerate) pseudo-differential equations of parabolic type. To the purpose some assumptions are made on the symbol \( \psi(x, \xi) \), which are satisfied in most the cases of interest in the financial applications.

Let \( \lambda(x, \xi) \) denote a basic weight function in the sense of Iwasaki (1977), that is, it is a \( C^\infty(\mathbb{R}^{2n}) \) function such that \( 1 \leq \lambda(x+y, \xi) \leq A_0 \langle y \rangle^\tau \lambda(x, \xi), \tau \geq 0, A_0 \geq 1 \) and

\[
\left| \partial_x^\alpha \partial_{\xi}^\beta \lambda(x, \xi) \right| \leq A_{\alpha, \beta} \lambda(x, \xi)^{1-\rho \alpha + \delta \beta}
\]

for some \( \rho, \delta \in \mathbb{N}^n \) and \( A_{\alpha, \beta} \geq 0 \). Let us now define pseudo-differential operators of class \( S^m_{\lambda, \rho, \delta, q, \epsilon}(\mathbb{R}^{2n}) \) where \( m, q \in \mathbb{R}, \rho, \delta \in \mathbb{N}^n, \rho > \delta, \epsilon > 0 \). We say that a \( C^\infty \)-function \( \psi(x, \xi) \) defined in \( \mathbb{R}^{2n} \) is a symbol of class \( S^m_{\lambda, \rho, \delta, q, \epsilon}(\mathbb{R}^{2n}) \) if for any multi-index \( \alpha, \beta \) there is a constant \( C_{\alpha, \beta} \geq 0 \) (independent of \( \epsilon \)) such that:

\[
(2.2) \quad \sup_{x, \xi \in \mathbb{R}^n} \left| \partial_x^\alpha \partial_{\xi}^\beta \psi(x, \xi) \right| \leq C_{\alpha, \beta} \epsilon^{\min\{m + \rho \alpha - \delta \beta, 0\}}
\]

For pseudo-differential operators of this class one can prove the following expansion result.

**Proposition 1.** If \( \psi_j \in S^m_{\lambda, \rho, \delta, q, \epsilon}(\mathbb{R}^{2n}), j=1,2 \), then for any \( N \) one can write

\[
(2.3) \quad (\psi_1 \circ \psi_2)(x, \xi) = \frac{1}{|\Gamma|^N} \sum_{1 \leq j \leq N} \partial_x^\alpha \psi_1(x, \xi) \partial_{\xi}^\beta \psi_2(x, \xi) + r_N(x, \xi)
\]

where \( r_N \in S^m_{\lambda, \rho, \delta, q_1, q_2, N, \epsilon}(\mathbb{R}^{2n}) \) with \( \theta = \min_{1 \leq j \leq N} (\rho_j - \delta_j) \).

**Remark 1.** A usual choice for the basic weight function is \( \lambda(x, \xi) = \langle \xi \rangle \). However this choice poses some restrictions on the growth of the level-dependent functions in the characteristic exponent. For example, if a diffusion is adopted, i.e.

\[
\psi(x, \xi) = i \left( \frac{\sigma^2(x)}{2} - r \right) \xi + \frac{\sigma^2(x)}{2} \xi^2,
\]

the choice of the basic weight function \( \langle \xi \rangle \) is allowed only if \( \sigma \) is bounded. Therefore this framework allows for more generality than Agliardi (2009) and Agliardi (2012).
In order to obtain the fundamental solution of the pseudo-differential equation (2.1) the following conditions are usually assumed:

(2.4) (i) There exist \( m' \in [0,m] \), \( c > 0 \) and \( c_0 \geq 0 \) such that \( \Re \psi (x, \xi) + c \geq c_0 \lambda(x, \xi)^{m'} \).

(ii) For any \( \alpha, \beta \in \mathbb{N}^n \) there is \( C_{\alpha, \beta} \geq 0 \) such that:

\[
\left| \partial_\xi^\alpha D_\xi^\beta \psi(x, \xi) / (\Re \psi(x, \xi) + c) \right| \leq C_{\alpha, \beta} e^{\Re \lambda(x, \xi)^{-\rho, \alpha+\beta}}
\]

In financial applications it is convenient to consider pseudo-differential operators whose symbol admit an analytic continuation with respect to \( \xi \) into \( \mathbb{I} \xi \in \Omega \), where \( \Omega \) is an open domain in \( \mathbb{R}^n \) whose closure contains the origin. If the symbol \( \psi \) admits an analytic continuation with respect to \( \xi \) into \( \mathbb{I} \xi \in \Omega \) and all its derivatives admit a continuous extension up to the boundary of \( \Omega \) satisfying (2.2), then we will write that \( \psi \in S^m_{\lambda, \rho, \delta, q, \varepsilon} (\mathbb{R}^n \times (\mathbb{R}^n + i\Omega)) \).

Let us now recall how an approximate solution to the pricing problem (2.1) can be constructed building on the method of Iwasaki (1977) and Kumano-go (1981).

**Theorem 1.** Assume that \( \psi \in S^m_{\lambda, \rho, \delta, q, \varepsilon} (\mathbb{R}^n \times (\mathbb{R}^n + i\Omega)) \) with \( m \leq 2 \) and \( \Omega = \prod\omega_j \cup [0] \), \( \omega_j < 1 < \omega_j' \), and suppose that (2.4) is satisfied in \( \mathbb{R}^n \times (\mathbb{R}^n + i\Omega) \). Let \( g \) be a piecewise continuous function such that \( \psi(x) \in L^1 (\mathbb{R}^n) \) for any \( \omega \in \Omega' \subset \Omega \). Then the Cauchy problem (2.1) with terminal condition \( f(T,x) = g(x) \) has a solution of the form:

\[
\frac{1}{(2\pi)^n} \int \cdots \int \exp(ix_1 \xi_1 - (T-t)(r + \psi(x, \xi))) \hat{g} (\xi) E_N (t, x, \xi) d\xi_1 \cdots d\xi_n
\]

for any integer \( N \), where \( \hat{g} (\xi) \) denotes the Fourier transform of \( g \) on \( \mathbb{I} \xi = \omega \),

\[
E_N (t, x, \xi) = \sum_{j=0}^{N-1} e_j (t, x, \xi) e^{(T-t)(r + \psi(x, \xi))} \in S^{m-\theta N}_{\lambda, \rho, \delta, q, \varepsilon} \quad , \quad e_0 (t, x, \xi) = e^{-(T-t)(r + \psi(x, \xi))}
\]

and the other \( e_j \)'s are obtained solving the following problems:

\[
[\partial_1 - r - \psi(x, \xi)] e_j (t, x, \xi) = - \sum_{k=0}^{N-1} \sum_{j+k=j} \frac{1}{j!} \partial_{\xi_1}^j \psi(x, \xi) D_{\xi_1}^k e_k (t, x, \xi)
\]

\[
e_j (T, x, \xi) = 0.
\]
Proof. This result is obtained by adapting the proof of Proposition 2 in Agliardi (2011,a) and employing Proposition 1 as the main tool in the symbolic calculus.

3 An approximate pricing formula

Consider a model of a financial market where the risky securities follow a stochastic process $S_t = e^{X_t}$ whose generators are $\Psi$-do differential operators with symbols $-\psi(x, \xi)$ belonging to the class $S^m_{2,p,q,0,0}((\mathbb{R}^n \times (\mathbb{R}^n + i\Omega)))$ ($m \leq 2$ and $\Omega = \prod_{j=1}^{n} |\omega_{j}^-, \omega_{j}^+|$, for some $\omega_{j}^- < 0 < \omega_{j}^+$). In the one-dimensional case we will denote $\Omega = |\omega, \omega^+|$ for simplicity’s sake. Assume a deterministic saving account $e^{rt}$, $r \geq 0$, and suppose that the usual EMM requirement holds. Then, in this framework and under the condition (2.4), the following approximate pricing formula for standard European options holds.

Proposition 2. In our framework the current price of a European vanilla option whose $T$-payoff is $\max(0, \max(K - S_T - K))$, $w = \pm 1$, can be approximated as follows:

$$
\frac{-Ke^{-r(T-t)}}{2\pi} \int_{-w-iw\omega}^{w+iw\omega} \exp(i\xi \ln(S_T/K) - (T-t)\psi(\ln S_t, \xi)) \left[ 1 - \frac{i(T-t)\partial_{\xi} \psi(\ln S_t, \xi) \partial_{\xi} \psi(\ln S_T, \xi)}{2} \right] d\xi
$$

up to an error which is $o(\epsilon)$ for $\epsilon \to 0$.

Here $\omega \in \omega, \omega^+$, where $a_w = -\omega^-$ if $w=1$ and $a_w = \omega^+ - 1$ if $w=-1$.

Proof. The Fourier transform of the payoff $\max(w(e^x - K),0)$ is $\frac{-Ke^{-i\xi \ln K}}{\xi^2 + i\xi}$ with $\xi \in \mathbb{R}$.

Here some examples are offered to show how some historical models can be incorporated in our setting and, at the same time, new models can be easily generated to account for the smile/skewness effects in the empirical implied volatility.

Example 1. Platen and Schweizer (1998) offers an endogenous - economically motivated – explanation for the smile and skewness effect, in that the shape of the volatility function
depends on the trading behavior of the agents in the economy and, in particular, on feedback effects from hedging strategies. The resulting volatility function is of the form:

\[-\sqrt[\gamma]{\sum_{i,j} \frac{V_{ij}}{\sqrt{2\pi \tau_i \sigma_0}} \exp\left(-\frac{(x - \ln K_{ij} + 0.5\sigma_0^2 \tau_i)^2}{2\sigma_0^2 \tau_i}\right)}\]

where the sum runs over the number of types of hedged call options subdivided according to their maturities (\(\tau_i\)) and strikes (\(K_{ij}\)). Here \(V_{ij}\) represents the relative size of each type.

The implied volatility is obtained throughout numerically computed option prices (see (1.18) in Platen and Schweizer (1998)). By employing the approximation of Proposition 2 we are able to obtain an analytical expression for option prices which holds, at least, for small parameter values (e.g. small \(V_{ij}\)). The approximate pricing formula for a call option in the simplified case (2.1) adopted in Platen-Schweizer (1998) and \(r=0\) is written as follows:

\[
C(x, \tau) = \frac{K(x-k)\sigma(x)}{4\pi v} e^{-(x-k-0.5\sigma_0^2 \tau_i)^2/(2\sigma_0^2 \tau_i)} \left( x-k + \frac{\sigma_0^2 \tau_i}{2} \sum_{j=1}^{N} \frac{V_{ij}}{\sigma_0^2 \tau_j} e^{-(x-k_j+0.5\sigma_0^2 \tau_j)/(2\sigma_0^2 \tau_j)} (x-k_j + \frac{\sigma_0^2 \hat{\tau}_j}{2}) \right)
\]

where \(x=\ln S_t\), \(k = \ln K\), being K the strike of the call option, \(k_j = \ln K_{ij}\), \(\tau\) is the maturity of the option and \(\hat{\tau}\) an average maturity of the hedged option, \(C(x,\tau)\) is Black-Scholes formula with \(\sigma(x)\) replacing \(\sigma\).

**Example 2.** The NIG-like process defined in Barndorff-Nielsen and Levendorskiĭ (2001) is included in this framework. The symbol of the generator is:

\[
\psi(x, \xi) = -i\mu(x)\xi + \delta(x)((\alpha(x)^2 - (\beta(x) + i\xi)^2)^{1/2} - (\alpha(x)^2 - \beta(x)^2)^{1/2})
\]

where \(\alpha, \beta, \delta, \mu\) are smooth and bounded, \(\alpha \pm \beta, \delta\) are bounded away from 0, and the derivatives of these functions are bounded by small parameters. As a paradigmatic example, \(\alpha, \delta, \mu\) are taken to be constant and \(\beta(x) = \beta_0 - \frac{2\chi}{\pi} \arctan(\epsilon(x-x_0))\) with a small \(\epsilon > 0\). In view of Proposition 2 we can afford also unbounded functions. For example, the implied volatility surface in Figure 1 results from a NIG-like process with \(\delta(x)\) of the form \(\delta_0 \ln(3 + e^{-\epsilon x})\) with a small parameter \(\epsilon > 0\) by employing Proposition 2.
4 Computing the at-the-money volatility skew

Let \( y = \ln \frac{K}{S} \) and let \( \sigma_{BS} \) denote the implied volatility calculated from our model.

Then the at-the-money volatility skew, \( \frac{\partial \sigma_{BS}}{\partial y} \), can be computed by means of Dini’s theorem. In view of Proposition 2 we get the following approximate expression:

\[
e^{\frac{\tau^2}{2}} (A_0 - A_1 - 2\pi N(d_-))
\]

where

\[
d_\pm = \frac{-y + (r \pm 0.5\sigma_{BS}^2)\tau}{\sigma_{BS} \sqrt{\tau}}, \quad A_0 = \int_{-\infty}^{+\infty} \exp[-\tau(r + \psi(k, \xi))] \left[ 1 + \frac{i\tau^2}{2} \partial_x \psi(k, \xi) \partial_y \psi(k, \xi) \right] d\xi,
\]

\[
A_1 = \int_{-\infty}^{+\infty} \frac{e^{-\tau(r + \psi(k, \xi))}}{i\xi + \sigma^2} \left[ \partial_x \psi(k, \xi)(\tau + \frac{i\tau^3}{2} \partial_y \psi(k, \xi)) \partial_y \psi(k, \xi) - \frac{i\tau^2}{2} \partial_x \psi(k, \xi) \partial_y \psi(k, \xi) \right] d\xi
\]
5 Summary

This paper builds on a combination of the options pricing under Lévy processes and of the level-dependent volatility approach. It follows the recent suggestions to explore some alternatives to the more traditional modeling of stochastic volatility, that is, the models allowing for jumps, which are needed for the short term behavior, and those generalizations of the diffusion models aiming at fitting the implied volatility over long term. As the literature is not unanimous in favouring one class of models, this work proposes a framework to combine both methods. The resulting approximate formulas are easily numerically implemented, as the examples show. Further research might aim at extending the pricing method to such exotic options as the ones studied in Agliardi (2007, 2011, 2012).

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