On Some Inequalities for the Gamma Function

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Abstract

In this paper, we give some inequalities involving the gamma function using its integral representation and the concepts of neutrices.

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1 Introduction and Preliminaries

The classical gamma function $\Gamma(x)$ and its numerous variations have played major roles in research and applications in many branches; for example, statistics, physics, engineering and other mathematical sciences. The history of this function is described in detail in [2]. So, these functions have been studied extensively by many mathematicians, for example, see [7] and the references therein. New inequalities for the gamma function is established by using its integral representation [6, 8]. Recently, neutrix calculus, given by van der Corput, have been used to define the gamma function and its derivatives for all real values of $x$, see [4].
A neutrix \( \mathcal{N} \) is a commutative additive group of negligible functions \( f(x) \), defined on a domain \( \mathcal{N}' \) with values in an additive group \( \mathcal{N}'' \), which satisfies the condition that does not contain any constant except zero.

Let \( \mathcal{N}' \) be a set contained in a topological space with a limit point \( b \) which does not belong to \( \mathcal{N}' \). If \( f(x) \) is a real (or complex) valued function defined on \( \mathcal{N}' \) and if it is possible to find a constant \( c \) such that \( f(x) - c \) belongs to \( \mathcal{N} \) then \( c \) is called the neutrix limit of \( f(x) \) as \( x \) tends to \( b \) and we write

\[
\text{N} - \lim_{x \to b} f(x) = c.
\]

The reader may find the general definition of the neutrix limit with some examples in [1].

In this article, we would like to give some inequalities involving the gamma function for all real values of \( x \) via the theory of neutrices and a technique developed by Mercer.

In [6], Mercer considered a positive linear functional defined on a subspace \( C^*(I) \) of \( C(I) \) where \( I \) is the interval \( (0, a) \) with \( a > 0 \) or \( (0, \infty) \) and gave some inequalities involving some special functions such as the gamma function. For strictly positive functions \( f \) and \( g \) in \( C^*(I) \) such that \( f(x) \to 0 \), \( g(x) \to 0 \) as \( x \to 0^+ \) and \( \frac{L[f]}{g} \) is strictly increasing, he define the function \( \phi \) by \( \phi = g \cdot \frac{L(f)}{L(g)} \).

Then he proved the following lemma and theorem.

**Lemma 1.1** \( L(f^2) - L(\phi^2) \geq 0 \)

**Theorem 1.2** Let \( F \) be defined on the ranges of \( f \) and \( g \) such that the compositions \( F(f) \) and \( F(g) \) each belongs to \( C^*(I) \).

a) If \( F \) is convex then \( L[F(f)] \geq L[F(\phi)] \).

b) If \( F \) is concave then \( L[F(f)] \leq L[F(\phi)] \).

As an application of theorem 1.2, the results

\[
\frac{[\Gamma^{(2n)}(1+\delta)]^\alpha}{\Gamma(1+\alpha\delta)} > \frac{[\Gamma^{(2n)}(1+\beta)]^\alpha}{\Gamma^{2n}(1+\alpha\beta)},
\]

(1)

\[
\frac{[\Gamma^{(2n)}(1+\delta)]^\alpha}{\Gamma(1+\alpha\delta)} < \frac{[\Gamma^{(2n)}(1+\beta)]^\alpha}{\Gamma^{2n}(1+\alpha\beta)}
\]

(2)

are given for all \( \alpha < 0 \) or \( \alpha > 1 \) and \( 0 < \alpha < 1 \) respectively, in which \( \alpha\delta > -1, \alpha\beta > -1 \), [8].

## 2 Definitions of the gamma function

In this work, we let \( \mathcal{N} \) be a neutrix having domain \( \mathcal{N}' = \{ \epsilon : 0 < \epsilon < \frac{1}{2} \} \) and negligible functions finite linear sums of the functions

\[ \epsilon^\lambda \ln^{r-1} \epsilon, \ln^r \epsilon : \quad \lambda < 0, \quad r = 1, 2, \ldots \]
and all functions $f(\epsilon)$ which converge to zero in the usual sense as $\epsilon$ tends to zero.

The gamma function $\Gamma(x)$, introduced by Euler, is defined for $x > 0$ by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt. \quad (3)$$

The integral only converging for $x > 0$, see [3]. It follows from equation (3) that

$$\Gamma(x + 1) = x\Gamma(x)$$

for $x > 0$ and this equation is used to define $\Gamma(x)$ for negative, non-integer values of $x$. Using the regularization, Gelfand and Shilov in [5] define the gamma function

$$\Gamma(x) = \int_0^1 t^{x-1} \left[ e^{-t} - \sum_{i=0}^{n-1} (-1)^i \frac{t^i}{i!} \right] dt + \int_1^\infty t^{x-1} e^{-t} \, dt + \sum_{i=0}^{n-1} \frac{(-1)^i}{i!(x+i)}$$

for $x > -n$, $x \neq 0, -1, -2, \ldots, -n + 1$ and

$$\Gamma(x) = \int_0^\infty t^{x-1} \left[ e^{-t} - \sum_{i=0}^{n-1} (-1)^i \frac{t^i}{i!} \right] dt$$

for $-n < x < -n + 1$.

It has been shown in [4] that the gamma function is defined by the neutrix limit

$$\Gamma(x) = \lim_{\epsilon \to 0} \int_\epsilon^\infty t^{x-1} e^{-t} \, dt$$

and in general the existence of the $r$th derivative of the gamma function was proved and then it was defined by the neutrix limit

$$\Gamma^{(r)}(x) = \lim_{\epsilon \to 0} \int_\epsilon^\infty t^{x-1} \ln^r t e^{-t} \, dt$$

for all real values of $x$ and $r = 0, 1, 2, \ldots$.

### 3 Main Results

Now, we use the concept of neutrix and neutrix limit to give some inequalities which generalize the inequalities (1) and (2). Since one can see easily that the lemma 1.1 still holds for $f, g \in C^\infty(I)$ such that $f(x) \to 0$, $g(x) \to 0$ as $x \to 0^+$ or $f(x) \to 0$, $g(x) \to 0$ as $x \to \infty$, we can study for a larger class of functions of $f$ and $g$. In writing,
Lemma 3.1 Let $f$ and $g$ be strictly positive function in $C^*(I)$ such that $f(x) \to 0$, $g(x) \to 0$ as $x \to 0^+$ or $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$ and $f/g$ be strictly increasing. Then for $\phi = g^{L(f)}$ we have $L(f^2) - L(\phi^2) \geq 0$

Proof The proof is the same as in [6] also for the case $f(x) \to 0$ and $g(x) \to 0$ as $x \to \infty$.

Therefore we can study for a larger subspace $C^*(I)$ of $C(I)$. Then, theorem 1.2 still holds.

Note that, the function $F(t) = t^\alpha$ is convex if $\alpha < 0$ or $\alpha > 1$ and concave if $0 < \alpha < 1$. Now, take $f(x) = x^\beta$ and $g(x) = x^\delta$ with $\beta > \delta$ (not necessarily $\delta > 0$). Then we obtained by theorem 1.2 that

$$\frac{[L(x^\delta)]^\alpha}{L(x^{\alpha\delta})} > \frac{[L(x^\beta)]^\alpha}{L(x^{\alpha\beta})}$$

and

$$\frac{[L(x^\delta)]^\alpha}{L(x^{\alpha\delta})} < \frac{[L(x^\beta)]^\alpha}{L(x^{\alpha\beta})}$$

for $\alpha < 0$ or $\alpha > 1$ and $0 < \alpha < 1$ respectively. There is equality for the case $\alpha = 0$ and $\alpha = 1$ in the equations (4) and (5).

**Theorem 3.2** Let $\beta$ and $\delta$ be the numbers such that $\beta > \delta$. Then the inequalities

$$\frac{[\Gamma(2n)(1 + \delta)]^\alpha}{\Gamma(2n)(1 + \alpha\delta)} > \frac{[\Gamma(2n)(1 + \beta)]^\alpha}{\Gamma(2n)(1 + \alpha\beta)},$$

and

$$\frac{[\Gamma(2n)(1 + \delta)]^\alpha}{\Gamma(2n)(1 + \alpha\delta)} < \frac{[\Gamma(2n)(1 + \beta)]^\alpha}{\Gamma(2n)(1 + \alpha\beta)}$$

hold for $\alpha < 0$ or $\alpha > 1$ and $0 < \alpha < 1$ respectively.

Proof Let $\epsilon > 0$. Then for $w \in C^*(I)$ define

$$L(w) = N - \lim_{\epsilon \to 0^+} \int_{\epsilon}^\infty w(x) \ln^r x e^{-x} dx$$

for $r = 0, 1, 2, \ldots$ where the members of $C^*(I)$ satisfies the properties of the following:

i) $w(x) = O(x^\theta)$ for any $\theta \in \mathbb{R}$ as $x \to 0^+$

ii) $w(x) = O(x^r)$ for any finite $\varphi$ as $x \to \infty$.

Firstly we need to show that $L$ is well defined. By i and ii, the integral

$$\int_{\epsilon}^\infty w(x) \ln^r x e^{-x} dx$$
is well-defined, and since we have the property that the neutrix limit is unique and its precisely the same as the ordinary limit, if it exist; then $L$ is well-defined on $C^*(I)$. Moreover $L$ is positive and linear. Then by applying the inequalities (4) and (5), the results follow.

\[ \frac{1}{\Gamma(1 + \alpha)} \leq \frac{[\Gamma(1 + x)]^\alpha}{\Gamma(1 + \alpha x)} \leq 1, \]
\[ 1 \leq \frac{[\Gamma(1 + x)]^\alpha}{\Gamma(1 + \alpha x)} \leq \frac{1}{\Gamma(1 + \alpha)} \]

Corollary 3.3 For all $x \in (0,1)$ we have,

\[ \frac{1}{\Gamma(1 + \alpha)} \leq \frac{[\Gamma(1 + x)]^\alpha}{\Gamma(1 + \alpha x)} \leq 1, \]
\[ 1 \leq \frac{[\Gamma(1 + x)]^\alpha}{\Gamma(1 + \alpha x)} \leq \frac{1}{\Gamma(1 + \alpha)} \]

if $\alpha \leq 0$ or $\alpha \geq 1$ and $0 \leq \alpha \leq 1$ respectively.

Proof Let $\alpha < 0$ or $\alpha > 1$. For $n = 0$ we have from equation (6) that

\[ \frac{[\Gamma(1 + \beta)]^\alpha}{\Gamma(1 + \alpha \beta)} < \frac{[\Gamma(1 + \delta)]^\alpha}{\Gamma(1 + \alpha \delta)} . \]

Now taking $\beta = 1$ and remembering that $\beta > \delta$ we can take $\delta = x$ for all $x \in (0,1)$ in the equation (10). Then we get

\[ \frac{[\Gamma(2)]^\alpha}{\Gamma(1 + \alpha)} < \frac{[\Gamma(1 + x)]^\alpha}{\Gamma(1 + \alpha x)} . \]

On the other hand, by taking $\beta = x \in (0,1)$ we can write $\delta = 0$ in the equation (10). Then we get

\[ \frac{[\Gamma(1 + x)]^\alpha}{\Gamma(1 + \alpha x)} < \frac{[\Gamma(1)]^\alpha}{\Gamma(1)} . \]

And since there is equality in case $\alpha = 0$ or $\alpha = 1$ in the equations (11) and (12) the inequality (8) follows.

For $0 \leq \alpha \leq 1$, similar argument leads us to the desired result.

References


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