The Modular of $\Gamma^2$ Defined by Asymptotically $
abla$-Invariant Modulus of Fuzzy Numbers

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Abstract. In this paper, we introduce the modular sequence space of $\Gamma^2$ of fuzzy numbers using modulus function and examine some topological properties of these space also establish some duals results among them. Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to define the sequence space $\ell_M$ which is called an Orlicz sequence space. Another generalization of Orlicz sequence spaces is due to Woo [9]. We define the sequence spaces $\Gamma^2_{f\lambda}$ and $\Gamma^2_{g\lambda}$, where $f = (f_{mn})$ and $g = (g_{mn})$ are sequences of modulus functions such that $f_{mn}$ and $g_{mn}$ be mutually complementary for each $m, n$. This paper deals with the modular of $\Gamma^2$ defined by asymptotically $\lambda$-- and $\sigma$ invariant modulus of fuzzy numbers. If $f_{mn}$ and $g_{mn}$ are mutually complementary function then the
two sequences $X$ and $Y$ of fuzzy numbers are said to be asymptotically $\lambda-$invariant statistical equivalent of multiple $\bar{0}$ provided that for every $\epsilon > 0$,

$$X_{f,\sigma,\mu,\lambda}^{\mathbb{S}_{\Gamma}^{2\mu}} \approx Y = \frac{1}{\mu_{rs}} \left\{ m, n \in I_{rs} : f \left[ \frac{1}{\lambda_{mn}} \left( \frac{X_{\sigma,\mu,\lambda}(m,n)}{Y_{\sigma,\mu,\lambda}(m,n)} \right)^{1/m+n}, \bar{0} \right] \geq \epsilon \right\} \rightarrow 0 \text{ as } r,s \rightarrow \infty, \text{ uniformly in } \eta$$

$$X_{g,\sigma,\mu}^{\mathbb{S}_{\Gamma}^{2\mu}} \approx Y = \frac{1}{\mu_{rs}} \left\{ m, n \in I_{rs} : g \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{\sigma,\mu,\lambda}(m,n)}{Y_{\sigma,\mu,\lambda}(m,n)} \right)^{1/m+n}, \bar{0} \right) \geq \epsilon \right\} \rightarrow 0 \text{ as } r,s \rightarrow \infty, \text{ uniformly in } \eta.$$ 

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### 1. Introduction

The concept of fuzzy sets and fuzzy set operations were first introduced by Zadeh [30], fuzzy logic has become an important area of research in various branches of Mathematics such as metric and topological spaces, theory of functions, approximation theory etc. Subsequently several authors have discussed various aspects of the theory and applications of fuzzy sets. The concept of fuzzyiness has been applied in various fields such as Statistics, Cybernetics, Artificial intelligence, Operation research, Decision making, Agriculture, Weather forecasting, Quantum physics. Similarity relations of fuzzy orderings, fuzzy measures of fuzzy events, fuzzy mathematical programming etc.

In 1993, Marouf presented definitions for asymptotically equivalent sequences of real numbers and asymptotic regular matrices. In 2003, Patterson extended these concepts by presenting an asymptotically statistical equivalent analog of these definitions and natural regularity conditions for nonnegative summability matrices. In 2006, Savas and Basarir introduced and studied the concept of $(\sigma, \lambda)-$asymptotically statistical equivalent sequences. In 2008, Esi introduced and studied the concept of asymptotically equivalent difference sequences of fuzzy numbers. In 2009, sequences of fuzzy numbers, Savas introduced and studied the concepts of strongly $\lambda-$summable $\lambda-$statistical convergence and asymptotically $\lambda-$statistical equivalent sequences respectively. Recently, Braha defined asymptotically generalized difference lacunary sequences. This paper is extend to the modular of $\Gamma^2$ defined by asymptotically $\lambda-$invariant modulus of fuzzy numbers.

Throughout $w, \chi$ and $\Lambda$ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write $w_2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w_2$ is a linear space under the coordinate wise addition and scalar multiplication.
Some initial works on double sequence spaces is found in Bromwich [4]. Later on, they were investigated by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basar and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor

Let us define the following sets of double sequences:
\[
M_u (t) := \{(x_{mn}) \in w^2 : \sup_{m,n \in N} |x_{mn}| < \infty \},
\]
\[
C_p (t) := \{(x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn} - l|^t_{mn} = 1 \text{ for some } l \in \mathbb{C} \},
\]
\[
C_{0p} (t) := \{(x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn}| = 1 \},
\]
\[
\mathcal{L}_u (t) := \{(x_{mn}) \in w^2 : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{mn}| < \infty \},
\]
\[
\mathcal{E}_{bp} (t) := C_p (t) \cap M_u (t) \text{ and } C_{0bp} (t) = C_{0p} (t) \cap M_u (t);
\]
where \(t = (t_{mn})\) is the sequence of strictly positive reals \(t_{mn}\) for all \(m,n \in \mathbb{N}\) and \(p - \lim_{m,n \to \infty}\) denotes the limit in the Pringsheim’s sense. In the case \(t_{mn} = 1\) for all \(m,n \in \mathbb{N}\); \(M_u (t)\), \(C_p (t)\), \(C_{0p} (t)\), \(\mathcal{L}_u (t)\), \(C_{0bp} (t)\) and \(C_{bp} (t)\) reduce to the sets \(M_u\), \(C_p\), \(C_{0p}\), \(\mathcal{L}_u\), \(C_{0bp}\) and \(C_{bp}\), respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [21,22] have proved that \(M_u (t)\) and \(C_p (t)\), \(C_{bp} (t)\) are complete paranormed spaces of double sequences and gave the \(\alpha-, \beta-, \gamma-\) duals of the spaces \(M_u (t)\) and \(C_{bp} (t)\). Quite recently, in her PhD thesis, Zelter [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the \(M-\) core for double sequences and determined those four dimensional matrices transforming every bounded double sequences \(x = (x_{jk})\) into one whose core is a subset of the \(M-\) core of \(x\). More recently, Altay and Basar [27] have defined the spaces \(\mathcal{BS}, \mathcal{BS}_p, \mathcal{CS}_p, \mathcal{CS}_bp, \mathcal{CS}_r, \mathcal{BV}\) of double sequences consisting of all double series whose sequence of partial sums are in the spaces \(M_u, M_u (t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r\) and \(\mathcal{L}_u\), respectively, and also examined some properties of those sequence spaces and determined the \(\alpha-\) duals of the spaces \(\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_p, \mathcal{CS}_bp\) and the \(\beta (\vartheta) -\) duals of the spaces \(\mathcal{CS}_bp\) and \(\mathcal{CS}_r\) of double series. Quite recently Basar and Sever [28] have introduced the Banach space \(\mathcal{L}_q\) of double sequences corresponding to the well-known space \(\ell_q\) of single sequences and examined some properties of the space \(\mathcal{L}_q\). Quite recently Subramanian and Misra [29] have studied the space \(\chi^2_M (p, q, u)\) of double sequences and gave some inclusion relations.

Spaces are strongly summable sequences were discussed by Kuttner [31], Maddox [32], and others. The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [8] as an extension of the definition of strongly Cesàro summable sequences. Connor
[33] further extended this definition to a definition of strong $A$- summability with respect to a modulus where $A = (a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A$- summability, strong $A$-statistical convergence. In [34] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [35]-[38], and [39] the four dimensional matrix transformation $(Ax)_{k,\ell} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}^{m,n} x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$ (a + b)^p \leq a^p + b^p $$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence $(s_{mn})$ is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n \in \mathbb{N})$ (see[1]).

A sequence $x = (x_{mn})$ is said to be Pringsheim’s double analytic if $\sup_{m,n} |x_{mn}|^{1/m+n} < \infty$. The vector space of all Pringsheim’s double analytic sequences will be denoted by $A^2$. A sequence $x = (x_{mn})$ is called Pringsheim’s double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The Pringsheim’s double entire sequences will be denoted by $\Gamma^2$. Let $\phi = \{all\ finite\ sequences\}.$

Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} z_{ij}$ for all $m, n \in \mathbb{N}$; where $z_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i, j)^{th}$ place for each $i, j \in \mathbb{N}$.

An FK-space(or a metric space)$X$ is said to have AK property if $(z_{mn})$ is a Schauder basis for $X$. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz[13] used the idea of Orlicz function to construct the space $(L^M)$. Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p (1 \leq p < \infty)$. subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektas and Altin [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].
Recalling [13] and [6], an Orlicz function is a function \( M : [0, \infty) \to [0, \infty) \) which is continuous, non-decreasing, and convex with \( M (0) = 0, M (x) > 0 \), for \( x > 0 \) and \( M (x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function \( M \) is replaced by subadditivity of \( M \), then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An modulus function \( M \) is said to satisfy the \( \Delta_2 \)-condition for small \( u \) or at 0 if for each \( k \in \mathbb{N} \), there exist \( R_k > 0 \) and \( u_k > 0 \) such that
\[
M (ku) \leq R_k M (u)
\]
for all \( u \in (0, u_k] \). Moreover, an modulus function \( M \) is said to satisfy the \( \Delta_2 \)-condition if and only if
\[
\lim_{u \to 0+} \sup \frac{M(2u)}{M(u)} < \infty
\]
Two Modulus functions \( M_1 \) and \( M_2 \) are said to be equivalent if there are positive constants \( \alpha, \beta \) and \( b \) such that
\[
M_1 (\alpha u) \leq M_2 (u) \leq M_1 (\beta u)
\]
for all \( u \in [0, b] \).

An modulus function \( M \) can always be represented in the following integral form
\[
M (u) = \int_0^u \eta (t) \, dt,
\]
where \( \eta \), the kernel of \( M \), is right differentiable for \( t \geq 0 \), \( \eta (0) = 0 \), \( \eta (t) > 0 \) for \( t > 0 \), \( \eta \) is non-decreasing and \( \eta (t) \to \infty \) as \( t \to \infty \) whenever \( \frac{M(u)}{u} \to \infty \) as \( u \to \infty \).

Consider the kernel \( \eta \) associated with the modulus function \( M \) and let
\[
\mu (s) = \sup \left\{ t : \eta (t) \leq s \right\}.
\]
Then \( \mu \) possesses the same properties as the function \( \eta \). Suppose now
\[
\Phi = \int_0^\Phi \mu (s) \, ds.
\]
Then, \( \Phi \) is an modulus function. The functions \( M \) and \( \Phi \) are called mutually complementary Orlicz functions.

Now, we give the following well-known results.

Let \( M \) and \( \Phi \) are mutually complementary modulus functions. Then, we have:
(i) For all \( u, y \geq 0 \),
\[
uy \leq M (u) + \Phi (y), \quad \text{(Young's inequality)}
\]
(ii) For all \( u \geq 0 \),
\[
u \eta (u) = M (u) + \Phi (\eta (u)).
\]
(iii) For all \( u \geq 0 \), and \( 0 < \lambda < 1 \),
\[
M (\lambda u) \leq \lambda M (u)
\]
Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

\[ \ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\}, \]

The space \( \ell_M \) with the norm

\[ \|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}, \]

becomes a Banach space which is called an Orlicz sequence space. For \( M(t) = t^p \) \( (1 \leq p < \infty) \), the spaces \( \ell_M \) coincide with the classical sequence space \( \ell_p \).

Any Orlicz function \( M_{mn} \) always has the integral representation

\[ M_{mn}(x) = \int_0^x p_{mn}(t) \, dt, \]

where \( p_{mn} \), known as the kernel of \( M_{mn} \) is non-decreasing, is right continuous for \( t > 0 \), \( p_{mn}(0) = 0 \), \( p_{mn}(t) > 0 \) for \( t > 0 \) and \( p_{mn}(t) \to \infty \), as \( t \to \infty \).

Given an Orlicz function \( M_{mn} \) with kernel \( p_{mn}(t) \), define

\[ q_{mn}(s) = \sup \{ t : p_{mn}(t) \leq s, s \geq 0 \} \]

Then \( q_{mn}(s) \) possesses the same properties as \( p_{mn}(t) \) and the function \( N_{mn} \) defined as

\[ N_{mn}(x) = \int_0^x q_{mn}(s) \, ds \]

is an Orlicz function. The functions \( M_{mn} \) and \( N_{mn} \) are called mutually complementary Orlicz functions.

For a sequence \( M = (M_{mn}) \) of Orlicz functions, the modular sequence class \( \tilde{\ell}_M \) is defined by

\[ \tilde{\ell}_M = \left\{ x = (x_{mn}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|) < \infty \right\}. \]

Using the sequence \( N = (N_{mn}) \) of Orlicz functions, similarly we define \( \tilde{\ell}_N \). The class \( \ell_M \) is defined by

\[
\ell_M = \left\{ x = (x_{mn}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{mn}y_{mn} \text{ converges for all } y \in \tilde{\ell}_N \right\}.
\]

For a sequence \( M = (M_{mn}) \) of Orlicz functions, the modular sequence class \( \ell_M \) is also defined as

\[ \ell_M = \left\{ x = (x_{mn}) : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|) < \infty \right\}. \]

The space \( \ell_M \) is a Banach space with respect to the norm \( \|x\|_M \) defined as

\[ \|x\|_M = \inf \left\{ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}|) \leq 1 \right\}. \]

The single sequence spaces were introduced by Woo [49] around the year 1973, and generalized the Orlicz sequence \( \ell_M \) and the modular sequence space considered earlier by Nakano in [12].

An important subspace of \( \ell_M \), which is an AK-space is the space \( h_M \) defined as

\[ h_M = \left\{ x \in \ell_M : \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} M_{mn}(|x_{mn}| < \infty) \right\}. \]
A sequence \((M_{mn})\) of Orlicz functions is said to satisfy uniform \(\Delta_2\)-condition at 0' if there exist \(p > 1\) and \(k_0 \in \mathbb{N}\) such that \(x \in (0, 1)\) and \(k > k_0\), we have 
\[
\frac{\| M_{mn}(x) \|}{M_{mn}(2x)} \leq p,
\]
or equivalently, there exists a constant \(K > 1\) and \(k_0 \in \mathbb{N}\) such that 
\[
\frac{M_{mn}(2x)}{M_{mn}(x)} \leq K
\]
for all \(k > k_0\) and \(x \in (0, \frac{1}{2}]\). If the sequence \((M_{mn})\) satisfies uniform \(\Delta_2\)-condition, then \(h_M = \ell_M\) and vice versa [49].

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [30] as follows
\[
Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}
\]
for \(Z = c, c_0\) and \(\ell_\infty\), where \(\Delta x_k = x_k - x_{k+1}\) for all \(k \in \mathbb{N}\).
Here \(c, c_0\) and \(\ell_\infty\) denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference space \(bv_p\) of the classical space \(\ell_p\) is introduced and studied in the case \(1 \leq p \leq \infty\) by BaŞar and Altay in [42] and in the case \(0 < p < 1\) by Altay and BaŞar in [43]. The spaces \(c(\Delta), c_0(\Delta), \ell_\infty(\Delta)\) and \(bv_p\) are Banach spaces normed by
\[
\| x \| = | x_1 | + sup_{k \geq 1} | \Delta x_k | \quad \text{and} \quad \| x \|_{bv_p} = (\sum_{k=1}^{\infty} | x_k |^p)^{1/p}, \quad (1 \leq p < \infty).
\]
Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by
\[
Z(\Delta) = \{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \}
\]
where \(Z = \Lambda^2, \chi^2\) and \(\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}\) for all \(m, n \in \mathbb{N}\).

2. DEFINITION AND PRELIMINARIES

Throughout a double sequence is denoted by \(\langle X_{mn} \rangle\), a double infinite array of fuzzy real numbers.

Let \(D\) denote the set of all closed and bounded intervals \(X = [a_1, a_2]\) on the real line \(\mathbb{R}\). For \(X = [a_1, a_2] \in D\) and \(Y = [b_1, b_2] \in D\), define
\[
d(X, Y) = max(\{|a_1 - b_1|, |a_2 - b_2|\})
\]
It is known that \((D, d)\) is a complete metric space.

A fuzzy real number \(X\) is a fuzzy set on \(\mathbb{R}\), that is, a mapping \(X : \mathbb{R} \to I (= [0, 1])\) associating each real number \(t\) with its grade of membership \(X(t)\).

The \(\alpha\)-level set \([X]^{\alpha}\), of the fuzzy real number \(X\), for \(0 < \alpha \leq 1\); is defined by
\[
[X]^{\alpha} = \{ t \in \mathbb{R} : X(t) \geq \alpha \}.
\]
The 0-level set is the closure of the strong 0-cut that is, \(cl \{ t \in \mathbb{R} : X(t) > 0 \}\).

A fuzzy real number \(X\) is called convex if \(X(t) \geq X(s) \land X(r) = min \{ X(s), X(r) \}\), where \(s < t < r\). If there exists \(t_0 \in \mathbb{R}\) such that \(X(t_0) = 1\) then, the fuzzy real number \(X\) is called normal.

A fuzzy real number \(X\) is said to be upper-semi continuous if, for each \(\epsilon > 0\), \(X^{-1}((0, a + \epsilon))\) is open in the usual topology of \(\mathbb{R}\) for all \(a \in I\).

The set of all upper-semi continuous, normal, convex fuzzy real numbers is
denoted by $L(R)$.

The absolute value, $|X|$ of $X \in L(R)$ is defined by

$$|X|(t) = \begin{cases} \max \{X(t), X(-t)\}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Let $\bar{d}: L(R) \times L(R) \to \mathbb{R}$ be defined by

$$\bar{d}(X,Y) = \sup_{0 \leq \alpha \leq 1} d([X]^{\alpha}, [Y]^{\alpha}).$$

Then, $\bar{d}$ defines a metric on $L(R)$ and it is well-known that $(L(R), \bar{d})$ is a complete metric space.

A sequence $(X_{mn}) \subset L(R)$ is said to be null if $\bar{d}(X_{mn}, 0) = 0$.

A double sequence $(X_{mn})$ of fuzzy real numbers is said to be entire in Pringsheim’s sense to a fuzzy number $0$ if $\lim_{m,n \to \infty} (X_{mn})^{1/m+n} = 0$.

A double sequence $(X_{mn})$ is said to regularly if it converges in the Pringsheim’s sense and the following limits zero:

$$\lim_{m \to \infty} (X_{mn})^{1/m+n} = 0 \text{ for each } n \in \mathbb{N},$$

and

$$\lim_{n \to \infty} (X_{mn})^{1/m+n} = 0 \text{ for each } m \in \mathbb{N}.$$ 

By the null of double we mean the null on the Pringsheim sense that is, a double sequence $x = (x_{mn})$ has Pringsheim limit $0$ (denoted by $P - \lim x = 0$).

We shall write more briefly as "$P - \text{null}"$.

We denote $w^2(F)$ the set of all sequences $X = (X_{mn})$ of fuzzy numbers.

A sequence $X = (X_{mn})$ of fuzzy numbers is said to be Pringsheim’s double analytic if the set $\{X_{mn} : (m, n) \in \mathbb{N}\}$ of fuzzy numbers is Pringsheim’s double analytic.

A $K-$ space of sequences for which the coordinate linear functionals are continuous.

Let $\sigma$ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^{mn}(\eta) = \sigma(\sigma^{m-1n-1}(\eta)), m,n = 1,2,3,\ldots$

2.1. Definition. Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn}), Y = (Y_{mn})$ be two sequences of fuzzy numbers are said to be $\sigma-$ asymptotically equivalent if

$$X^{12F}_{f,\sigma,\mu,\lambda} \approx Y = f\left[\bar{d}\left(\frac{1}{X_{mn}}(X_{\sigma^{mn}(\eta)})^{1/m+n} \bar{0}, \bar{0}\right)\right] \to 0 as \ m,n \to \infty, \text{ uniformly in } \eta.$$

2.2. Definition. Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn}), Y = (Y_{mn})$ be two sequences of fuzzy numbers are said to be $S_{f,\sigma,\mu,\lambda}^{12F}$ if for every $\epsilon > 0$

$$S_{f,\sigma,\mu,\lambda}^{12F} = \frac{1}{\mu_{rs}} \left\{ m,n \in I_{rs} : f\left[\bar{d}\left(\frac{1}{X_{mn}}(X_{\sigma^{mn}(\eta)})^{1/m+n} \bar{0}, \bar{0}\right)\right] \geq \epsilon \right\} \to 0 as r,s \to \infty, \text{ uniformly in } \eta.$$ 

In this case, we write $X_{mn} \to \bar{0}\left(S_{f,\sigma,\mu,\lambda}^{12F}\right)$. 

2.3. **Definition.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn}), Y = (Y_{mn})$ be two sequences of fuzzy numbers are said to be asymptotically $\mu-$ invariant statistical equivalent of multiple $\overline{0}$ provided that for every $\epsilon > 0$,

$$X_{f,\sigma,\mu,\lambda}^{S^{2F}} \approx Y = \frac{1}{\mu_{rs}} \left\{ m, n \in I_{rs} : f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(\eta)}{Y_{mn}(\eta)} \right)^{1/m+n}, 0 \right) \right] \geq \epsilon \right\} \rightarrow 0 \text{ as } r, s \rightarrow \infty, \text{ uniformly in } \eta.$$

2.4. **Example.** Let $\mu_{rs} = rs$ and $\sigma^{mn}(\eta) = \eta + 1$ for all $\eta, r, s \in \mathbb{N}$. Consider the sequences of fuzzy numbers $X = (X_{mn}), Y = (Y_{mn})$ defined by $X_{rs} = (rs)^{-2}$ and $Y_{rs} = (rs)^{-1}$ for all $r, s \in \mathbb{N}$. Then

$$\lim_{rs} \frac{1}{\mu_{rs}} \left\{ m, n \in I_{rs} : f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(\eta)}{Y_{mn}(\eta)} \right)^{1/m+n}, 0 \right) \right] \geq \epsilon \right\} =$$

$$\lim_{rs} \frac{1}{\mu_{rs}} \left\{ m, n \in I_{rs} : f \left[ d \left( \frac{1}{(rs)^{-1}}, 0 \right) \right] \geq \epsilon \right\} = 0.$$

If we take $\mu_{rs} = rs$ for all $r, s \in \mathbb{N}$, the above definition reduces to following definition:

2.5. **Definition.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn}), Y = (Y_{mn})$ be two sequences of fuzzy numbers are said to be asymptotically invariant statistical equivalent of multiple $\overline{0}$ provided that for every $\epsilon > 0$,

$$X_{f,\sigma,\mu,\lambda}^{S^{2F}} \approx Y = \frac{1}{rs} \left\{ m, n \leq (rs) : f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(\eta)}{Y_{mn}(\eta)} \right)^{1/m+n}, 0 \right) \right] \geq \epsilon \right\} \rightarrow 0 \text{ as } r, s \rightarrow \infty, \text{ uniformly in } \eta.$$

2.6. **Definition.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn}), Y = (Y_{mn})$ be two sequences of fuzzy numbers are said to be strong $V_{f,\sigma,\mu,\lambda}^{S^{2F}}$ asymptotically equivalent of multiple $\overline{0}$ provided that

$$X_{f,\sigma,\mu,\lambda}^{S^{2F}} \approx Y = \frac{1}{\lambda_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(\eta)}{Y_{mn}(\eta)} \right)^{1/m+n}, 0 \right) \right] \rightarrow 0 \text{ as } r, s \rightarrow \infty, \text{ uniformly in } \eta.$$

2.7. **Example.** Let $\mu_{rs} = rs$ and $\sigma^{mn}(\eta) = \eta + 1$ for all $\eta, r, s \in \mathbb{N}$. Consider the sequences of fuzzy numbers $X = (X_{mn}), Y = (Y_{mn})$ defined by $X_{rs} = (rs)^{-2}$ and $Y_{rs} = (rs)^{-1}$ for all $r, s \in \mathbb{N}$. Then

$$\lim_{rs} \frac{1}{\mu_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(\eta)}{Y_{mn}(\eta)} \right)^{1/m+n}, 0 \right) \right] =$$

$$\lim_{rs} \frac{1}{\mu_{rs}} \sum_{m=1}^{r} \sum_{n=1}^{s} f \left[ d \left( \frac{(rs)^{-2(m+n)}}{(rs)^{-(m+n)}}, 0 \right) \right] = \lim_{rs} \frac{1}{\mu_{rs}} \sum_{m=1}^{r} \sum_{n=1}^{s} \frac{1}{(rs)^{m+n}} < \infty.$$

If we take $\lambda_{rs} = rs$ for all $r, s \in \mathbb{N}$, the above definition reduces to the following definition:
2.8. **Definition.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn}), Y = (Y_{mn})$ be two sequences of fuzzy numbers are said to be strong $C^{2\Phi}_{f,\sigma,\mu,\lambda}$ asymptotically equivalent of multiple $\overline{0}$ provided that

$$X^{C^{2\Phi}}_{f,\sigma,\lambda} \approx Y = \frac{1}{\lambda_{rs}} \sum_{m=1}^{r} \sum_{n=1}^{s} f \left[ \overline{d} \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{m+n,\eta}^{\mu,\lambda,\sigma}}{Y_{m+n,\eta}^{\mu,\lambda,\sigma}} \right)^{1/m+n}, \overline{0} \right) \right] \to 0 \text{ as } r, s \to \infty, \text{ uniformly in } \eta.$$

If we take $\sigma_{mn}(\eta) = \eta + 1$, the above definitions reduce the following definitions:

2.9. **Definition.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn}), Y = (Y_{mn})$ be two sequences of fuzzy numbers are said to be asymptotically almost equivalent if $X^{C^{2\Phi}}_{f,\sigma,\lambda} \approx Y = f \left[ \overline{d} \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{m+n,\eta}^{\mu,\lambda,\sigma}}{Y_{m+n,\eta}^{\mu,\lambda,\sigma}} \right)^{1/m+n}, \overline{0} \right) \right] \to 0 \text{ as } m, n \to \infty, \text{ uniformly in } \eta.$

2.10. **Definition.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn})$ be a sequences of fuzzy numbers are said to be statistically almost convergent $\overline{0}$

$$\frac{1}{\lambda_{rs}} \left\{ (mn) \in I_{rs} : f \left[ \overline{d} \left( \frac{1}{\lambda_{mn}} \left( X_{m+n,\eta}^{\mu,\lambda,\sigma} \right)^{1/m+n}, \overline{0} \right) \right] \geq \epsilon \right\} \to 0 \text{ as } r, s \to \infty, \text{ uniformly in } \eta.$$

In this we write $X_{mn} \to \overline{0} \left( S^{2\Phi}_{f,\lambda} \right)$.

2.11. **Definition.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn})$ be a sequences of fuzzy numbers are said to be asymptotically almost $\mu_{rs}$ statistical equivalent of multiple $\overline{0}$ provided that for every $\epsilon > 0$

$$X^{S^{2\Phi}}_{f,\mu,\lambda} \approx Y = \frac{1}{\mu_{rs}} \left\{ m, n \in I_{rs} : f \left[ \overline{d} \left( \frac{1}{\lambda_{mn}} \left( X_{m+n,\eta}^{\mu,\lambda,\sigma} \right)^{1/m+n}, \overline{0} \right) \right] \geq \epsilon \right\} \to 0 \text{ as } r, s \to \infty, \text{ uniformly in } \eta.$$

If we take $\mu_{rs} = rs$ for all $r, s \in \mathbb{N}$, the above definition reduces to the following definition:

2.12. **Definition.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn})$ be a sequences of fuzzy numbers are said to be asymptotically almost statistical equivalent of multiple $\overline{0}$ provided that for every $\epsilon > 0$

$$X^{S^{2\Phi}}_{f,\mu,\lambda} \approx Y = \frac{1}{\mu_{rs}} \left\{ m, n \in I_{rs} : f \left[ \overline{d} \left( \frac{1}{\lambda_{mn}} \left( X_{m+n,\eta}^{\mu,\lambda,\sigma} \right)^{1/m+n}, \overline{0} \right) \right] \geq \epsilon \right\} \to 0 \text{ as } r, s \to \infty, \text{ uniformly in } \eta.$$

2.13. **Definition.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn})$ be a sequences of fuzzy numbers are said to be strong asymptotically almost $\mu_{rs}$ statistical equivalent of multiple $\overline{0}$ provided that
Then, we can assume that

\[ X^{f_{r,s},\lambda} \approx Y = \frac{1}{\mu_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn(q)}}{Y_{mn(q)}} \right)^{1/m+n}, \overline{0} \right) \right] \rightarrow 0 \text{ as } r, s \rightarrow \infty, \text{ uniformly in } \eta. \]

If we take \( \mu_{rs} = rs \) for all \( r, s \in \mathbb{N} \), the above definition reduces to following definition.

2.14. **Definition.** Let \( f_{mn} \) and \( g_{mn} \) are two complementary functions and \( X = (X_{mn}) \) be a sequences of fuzzy numbers are said to be strong asymptotically almost equivalent of multiple \( \overline{0} \) provided that

\[ X^{f_{r,s},\lambda} \approx Y = \frac{1}{rs} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn(q)}}{Y_{mn(q)}} \right)^{1/m+n}, \overline{0} \right) \right] \rightarrow 0 \text{ as } r, s \rightarrow \infty, \text{ uniformly in } \eta. \]

3. **Main Results**

3.1. **Theorem.** Let \( f_{mn} \) and \( g_{mn} \) are two complementary functions and \( X = (X_{mn}) \) and \( Y = (Y_{mn}) \) be two real valued sequences of fuzzy numbers. Then, the following conditions are satisfied:

(i) If \( X^{f_{r,s},\lambda} \approx Y \), then \( X^{f_{r,s},\lambda} \approx Y \)

(ii) If \( X \in \Lambda^{2F} \) and \( X^{f_{r,s},\lambda} \approx Y \), then \( X^{f_{r,s},\lambda} \approx Y \); hence, \( X^{f_{r,s},\lambda} \approx Y \).

(iii) \( X^{f_{r,s},\lambda} \approx Y \cap \Lambda^{2F} = X^{f_{r,s},\lambda} \approx Y \cap \Lambda^{2F} \).

**Proof:** If \( \epsilon > 0 \) and \( X^{f_{r,s},\lambda} \approx Y \), then

\[
\sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn(q)}}{Y_{mn(q)}} \right)^{1/m+n}, \overline{0} \right) \right] \geq \sum_{m \in I_{rs}} \sum_{n \in I_{rs}, \epsilon} d \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn(q)}}{Y_{mn(q)}} \right)^{1/m+n}, \overline{0} \right) \right] \geq \epsilon \left\{ m, n \leq (rs) : f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn(q)}}{Y_{mn(q)}} \right)^{1/m+n}, \overline{0} \right) \right] \geq \epsilon \right\}.
\]

Therefore, \( X^{f_{r,s},\lambda} \approx Y \).

(ii) Suppose that \( X = (X_{mn}) \) and \( Y = (Y_{mn}) \) are in \( \Lambda^{2F} \) and \( X^{f_{r,s},\lambda} \approx Y \). Then, we can assume that

\[
f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn(q)}}{Y_{mn(q)}} \right)^{1/m+n}, \overline{0} \right) \right] \leq M, \text{ for all } m, n \text{ and } \eta.
\]

Given \( \epsilon > 0 \),

\[
\frac{1}{\mu_{rs}} \sum_{m \in I_{rs}} \sum_{n \in I_{rs}} f \left[ d \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn(q)}}{Y_{mn(q)}} \right)^{1/m+n}, \overline{0} \right) \right]
\]
For a given $X_h$. Hence,

\[ \left| \mu_{rs} \right| \leq 1 \}

Therefore, $X_{y}^{1/2F} \approx Y$. Further, we have

\[ \frac{1}{rs} \sum_{m=1}^{r} \sum_{n=1}^{s} f \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(q)}{Y_{mn}(q)} \right)^{1/m+n}, 0 \right) = \frac{1}{rs} \sum_{m=1}^{r} \sum_{n=1}^{s} \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(q)}{Y_{mn}(q)} \right)^{1/m+n}, 0 \right) \]

Hence, $X_{f,\sigma,\lambda}^{G} \approx Y$ since $X_{f,\sigma,\lambda}^{G} \approx Y$.

(iii) Follows from (i) and (ii).

**3.2. Theorem.** Let $f_{mn}$ and $g_{mn}$ are two complementary functions and $X = (X_{mn}), Y = (Y_{mn})$ be two real valued sequences of fuzzy numbers, $X_{f,\sigma,\lambda}^{G} \approx Y \Rightarrow X_{f,\sigma,\mu,\lambda}^{G} \approx Y$ if

\[ (3.1) \quad \liminf \left( \frac{\mu_{rs}}{rs} \right) \]

**Proof:** For a given $\epsilon > 0$, we have

\[ \left\{ m, n \leq (rs) : f \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(q)}{Y_{mn}(q)} \right)^{1/m+n}, 0 \right) \geq \epsilon \right\} \supset \left\{ m, n \in I_{rs} : f \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(q)}{Y_{mn}(q)} \right)^{1/m+n}, 0 \right) \geq \epsilon \right\} \]

Therefore,

\[ \frac{1}{rs} \left\{ m, n \leq (rs) : f \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(q)}{Y_{mn}(q)} \right)^{1/m+n}, 0 \right) \geq \epsilon \right\} \supset \frac{1}{rs} \left\{ m, n \in I_{rs} : f \left( \frac{1}{\lambda_{mn}} \left( \frac{X_{mn}(q)}{Y_{mn}(q)} \right)^{1/m+n}, 0 \right) \geq \epsilon \right\} \]
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$$\frac{\mu_{rs}}{\mu_{rs}} - \mu_{rs} \left\{ m, n \in I_{rs} : f \left[ \bar{d} \left( \frac{1}{X_{mn}} \left( \frac{X_{mn} Y_{mn}}{Y_{mn} Y_{mn}} \right)^{1/m+n}, 0 \right) \right] \geq \epsilon \right\}.$$ Taking the limit as $r, s \to \infty$ and using equation (3.1), we get the result.

3.3. **Conclusions.** The concept of asymptotic equivalence was first suggested by Marouf in 1993. After that, several authors introduced and studied some asymptotically equivalent sequences. The results are obtained modular space of double entire sequences with respect of fuzzy numbers.

**References**


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