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Cost Allocation for Outside Sources Replacing Service Departments

Franklin Lowenthal and Massoud Malek

California State University, East Bay
Hayward, CA 94542, USA
franklin.lowenthal@csueastbay.edu
massoud.malek@csueastbay.edu

Abstract

The validity of the linear algebra model to solve the reciprocal service department cost allocation problem has been widely recognized since Kaplan's seminal paper in The Accounting Review in 1973 [2].
In a manufacturing company, certain departments can be characterized as production departments and others as service departments. Examples of service departments are purchasing, computing services, repair and maintenance, security, food services etc. The costs of such service departments must be allocated to the production departments, which in turn will allocate them to the product. We shall determine the exact price that should be paid to external suppliers of the same services currently supplied by the service departments. We shall also prove that once the costs of services allocated to each production department is obtained, then they will stay the same even if some of the service departments are replaced by outside sources.

Keywords: Cost Allocation; Outsourcing; External Suppliers; Service Allocation; Reciprocal Service; Service Departments; Production Departments; Direct Cost; Traceable Cost; Full Cost; Unit Cost; Indifference Point; Reducible and Irreducible Matrices; Permutation Matrix; Spectral Radius; Row Norm

I. Introduction

In accounting certain costs, called *traceable costs* can be directly traced to a particular product. For example, in a manufacturing firm, direct materials and direct labor refer to material and labor costs that can be directly attributed to the product (e.g., wood for a chair). However, nails, sandpaper and plant security personnel all involve costs that cannot be directly traced to a particular product. Nevertheless, it is important in accounting that these costs be allocated to the products that the company manufactures. The firm needs to know the "full" cost of the product so that it can determine what an appropriate selling price should be.

In a manufacturing company, certain departments can be characterized as production departments and others as service departments. Examples of service departments are purchasing, computing services, repair and maintenance, security, food services etc. The costs of such service departments must be allocated to the production departments, which in turn will allocate them to the product. At first sight this appears to permit a straightforward methodology. For example, allocate the costs of food services to the production departments based on the number of personnel employed in the respective production departments. Thus if production department A has 10% of the production personnel, allocate 10% of the food services department cost to it. However, this ignores the phenomenon of reciprocal service. The personnel in computing services also eat in the cafeteria; conversely, computing services in all likelihood is responsible for the computing needs of the food services department. If a portion of the cost of the food services department were allocated to computing services, then it would seem appropriate that computing services allocate a portion of its costs right back to food services. Thus we seem to have conjured up the horror of allocating increasing costs back and forth forever among the service departments without ever being able to allocate all the costs to the production departments! Cost accountants refer to this as the *reciprocal service cost allocation problem* and they have devised several methods for dealing with it.

The one that we examine in this paper simply attacks the problem by finding a linear system of m equations in m unknowns to find the *true* or *full cost* of each service department and then allocating these costs directly to the production departments.

II. Notation and Example

Throughout this paper we shall assume that there are m service departments and n production departments, where some service departments may exclusively serve service departments. However, if a block of r $(1 \leq r \leq m)$ service departments serves among itself, then at least one member of this block must service some production departments or at least one service department outside of the block. We also make the following assumptions:

1. All service department costs are strictly variable and there is no fixed cost involved (i.e., $c(x) = u\,x$, where u represents the unit cost and x, the number of units).

2. The amount of the total cost originally allocated to the service departments must remain the same.

3. The external supplier will absorb any self-service requirements.

Definitions and Notations. *We define the following matrices and vectors:*

(D_1) The $m \times m$ matrix $S = (s_{ij})$ where s_{ij} is the number of units provided by the service department j to the service department i.

(D_2) The $n \times m$ matrix $P = (p_{kj})$, where p_{kj} is the number of units provided by the service department j to the production department k.

(D_3) The diagonal matrix $Z = diag(z_1, z_2, \ldots, z_m)$, where $z_j > 0$ is the number of units provided by the jth service department.

(D_4) The $m \times m$ matrix $B = (b_{ij}) = SZ^{-1}$ represents the proportion of the jth service department's output provided to the ith service department.

(D_5) The components of $z = (z_1, z_2, \ldots, z_m)^t$ are the diagonal entries of Z.

(D_6) The $n \times m$ matrix $C = (c_{kj}) = PZ^{-1}$, where the c_{kj} entry is the proportion of the jth service department's output that is provided to the kth production department.

(D_7) The $m \times m$ matrix $G = (g_{ij}) = S^tZ^{-1}$, where the jth component of the vector Gz is the total number of units provided by the service department j to all service departments including itself.

(D_8) The vector $p = z - Gz = (I - G)z$ represents the number of units allocated to the production departments.

(D_9) The ith component b_i, of the $m \times 1$ vector b, represents the traceable costs of the service department i.

(D_{10}) The traceable cost per unit vector $Z^{-1}b$ is denoted by c.

Example. *Algebra Inc. has 4 service departments $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, and $\mathcal{S}_4$ and three production departments $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$. Direct costs of \$ 78,000 for $\mathcal{S}_1$, \$ 200,000 for $\mathcal{S}_2$, \$ 100,000 for $\mathcal{S}_3$ and \$ 150,000 for $\mathcal{S}_4$ are to be allocated*

to $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ by the linear algebra reciprocal service method in accordance with the figures given in the chart below.

	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$
$\mathcal{S}_1$	40	200	500	100
$\mathcal{S}_2$	160	100	200	100
$\mathcal{S}_3$	400	100	300	100
$\mathcal{S}_4$	200	0	1000	100
$\mathcal{P}_1$	1800	400	2000	1600
$\mathcal{P}_2$	400	600	2000	2000
$\mathcal{P}_3$	1000	600	4000	1000
Total Units	4000 employees	2000 hours	10000 sq ft	5000 calls
Traceable Cost	\$ 78,000	\$ 200,000	\$ 100,000	\$ 150,000
Unit Cost	\$ 19.50	\$ 100.00	\$ 10.00	\$ 30.00

The above table produces the matrices dealing with the number of units used by the various departments:

$$S=\begin{bmatrix} 40 & 200 & 500 & 100 \\ 160 & 100 & 200 & 100 \\ 400 & 100 & 300 & 100 \\ 200 & 0 & 1000 & 100 \end{bmatrix},\ P=\begin{bmatrix} 1800 & 400 & 2000 & 1600 \\ 400 & 600 & 2000 & 2000 \\ 1000 & 600 & 4000 & 1000 \end{bmatrix},\ Z=\begin{bmatrix} 4000 & 0 & 0 & 0 \\ 0 & 2000 & 0 & 0 \\ 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 5000 \end{bmatrix};$$

the matrices of proportions:

$$B = SZ^{-1} = \begin{bmatrix} 0.01 & 0.10 & 0.05 & 0.02 \\ 0.04 & 0.05 & 0.02 & 0.02 \\ 0.10 & 0.05 & 0.03 & 0.02 \\ 0.05 & 0.00 & 0.10 & 0.02 \end{bmatrix},\quad C = PZ^{-1}=\begin{bmatrix} 0.45 & 0.20 & 0.20 & 0.32 \\ 0.10 & 0.30 & 0.20 & 0.40 \\ 0.25 & 0.30 & 0.40 & 0.20 \end{bmatrix},$$

$$G = S^tZ^{-1} = \begin{bmatrix} 0.010 & 0.080 & 0.04 & 0.040 \\ 0.050 & 0.050 & 0.020 & 0.000 \\ 0.125 & 0.100 & 0.030 & 0.200 \\ 0.025 & 0.050 & 0.010 & 0.020 \end{bmatrix};$$

and the vectors:

$$z = \begin{bmatrix} 4000 \\ 2000 \\ 10000 \\ 5000 \end{bmatrix},\ p = \begin{bmatrix} 3200 \\ 1600 \\ 8000 \\ 4600 \end{bmatrix},\ b = \begin{bmatrix} \$78,000 \\ \$200,000 \\ \$100,000 \\ \$150,000 \end{bmatrix},\ \text{and}\ \ c = \begin{bmatrix} \$19.50 \\ \$100.00 \\ \$10.00 \\ \$30.00 \end{bmatrix}.$$

III. Preliminary Results

In all that follows, the $n \times n$ identity matrix is denoted by I_n or simply I; the jth column of the identity matrix is denoted by e_j; and the $1 \times n$ vector whose entries are all one is denoted by δ_n.

A *permutation matrix* is obtain from the identity matrix I by permuting some of its rows or columns. A matrix A is said to be *reducible*, if there exists a permutation matrix Q such that

$$A = Q \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} Q^t,$$

where A_{11} and A_{22} are both square; otherwise, we call the matrix *irreducible.* All 1×1 matrices are considered irreducible.

The *spectrum* of a square matrix is the set of all its eigenvalues. An eigenvalue with the largest modulus is called a *maximal eigenvalue.* The *spectral radius* of A denoted by $\rho(A)$ is the modulus of a maximal eigenvalue. The *row norm* of an $n \times n$ matrix $A = (a_{ij})$ is defined as follows:

$$||A||_\infty = \max_{1 \le i \le n} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}.$$

It is well known that $\rho(A) \le ||A||_\infty$ and if $\rho(A) < 1$, then the matrix $I - A$ is invertible [1] with

$$(I - A)^{-1} = I + A + A^2 + A^3 + \cdots$$

In every accounting paper related to the cost allocation problem, including Kaplan's paper, it is assumed that each service department serves at least one production department; this clearly implies that $||B^t|| < 1$ which makes the matrix $I_m - B$ invertible. We shall prove the invertibility of $I_m - B$ in a more general case mentioned at the beginning of the previous section. But first we need the following result [1, page 363].

Lemma 1. *Let A be an $n \times n$ irreducible matrix and suppose for at least one value of i,*

$$\sum_{j=1}^{n} |a_{ij}| < ||A||_\infty,$$

then $\rho(A) < ||A||_\infty$.

Now we may prove the invertibility of the matrix $I_m - B$.

Lemma 2. *Let B be the matrix defined in (D_4), then the matrix $I_m - B$ is invertible.*

Proof. For every $j \in \{1, 2, \ldots, m\}$, the sum of entries in the jth column of B plus the sum of entries in the jth column of C equals one. If B is irreducible, then the preceding lemma (for B^t) along with our assumption that at least one of the service departments must serve a production department, allow us to say that $\rho(B^t) < 1$; this clearly makes $I_m - B$ invertible. Now suppose B is reducible, then for some permutation matrix Q,

$$B = Q \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ B_{h1} & B_{h2} & \ldots & B_{hh} \end{bmatrix} Q^t,$$

where B_{kk}'s are irreducible. According to our assumption, at least one column of each B_{kk} has sum strictly less than 1 which makes $\rho(B_{kk}) = \rho(B_{kk}^t) < 1$; hence $I_m - B$ is invertible. ∎

The $m \times 1$ *full cost* vector:

$$v = (I_m - B)^{-1} b = (I_m + B + B^2 + B^3 + \cdots)\, b = b + B\, b + B^2 b + B^3 b + \cdots \tag{1}$$

represents the redistributed service department costs after accounting for the interactions among departments [2]. Since B and b are both non-negative, it follows from (1) that no component of b may exceed the corresponding component of v (i.e. $v \geq b$). The only time $v_i = b_i$ is when the ith row of B is zero. This is the case when the service department i is either an external supplier or acts as one.

We denote by T, the *total traceable cost* (i.e., $T = c^t z$). The vector $u = Z^{-1} v$ is the *full cost per unit* vector. The service department costs allocated to each production department is then given by the $n \times 1$ vector w determined by the following equation:

$$w = Cv = C(I_m - B)^{-1} b = (PZ^{-1})v = P(Z^{-1}v) = Pu. \tag{2}$$

Lemma 3. *The sum of the allocated costs to production departments is equal to the total service department costs (i.e., the sum of the components of w equals the sum of the components of b).*

Proof. We need to show that $\delta_n w = \delta_m b$. The fact that $\delta_m B + \delta_n C = \delta_m$ implies that

$$\begin{aligned} \delta_n w &= \delta_n C v = \delta_n C (I_m - B)^{-1} b = [\delta_n C][(I_m - B)^{-1}] b = [\delta_m - \delta_m B][(I_m - B)^{-1}] b \\ &= [\delta_m (I_m - B)][(I_m - B)^{-1}] b = \delta_m (I_m - B)(I_m - B)^{-1} b = \delta_m I_m b = \delta_m b = T. \end{aligned}$$ ∎

Going back to our example,

$$v = \begin{bmatrix} \$111,182.30 \\ \$\,221,554.43 \\ \$\,129,520.58 \\ \$\,171,950.18 \end{bmatrix} \geq \begin{bmatrix} \$\,78,000 \\ \$\,200,000 \\ \$\,100,000 \\ \$\,150,000 \end{bmatrix} = b, \quad u = \begin{bmatrix} \$\,27.80 \\ \$\,110.77 \\ \$\,12.95 \\ \$\,34.39 \end{bmatrix} \text{ and } w = \begin{bmatrix} \$\,175,271.09 \\ \$\,172,268.74 \\ \$\,180,460.17 \end{bmatrix}.$$

$$\begin{aligned} T &= \$\,78,000 + \$\,200,000 + \$\,100,000 + \$\,150,000 \\ &= \$\,175,271.09 + \$\,172,268.74 + \$\,180,460.17 = \$\,528,000\,. \end{aligned}$$

IV. Proof of Theorem on "Full Cost" as Cost to Pay External Supplier

Through an elementary example [2], Kaplan finds a fair value of service provided by an outside supplier for a single service department; then he states that "it is possible to confirm this conclusion by another calculation" [2 pp 744 and 748]. Kaplan's assertion was that were the jth service department eliminated and an external supplier contracted to perform this identical service, then paying this external supplier exactly u_j per unit of service will lead to the same total cost to the firm as it presently incurs. Thus this cost per unit represents the *indifference point* as far as accepting the actual bid of an external supplier. At first this may seem to be too high a price to pay ($u_j \geq c_j$). But further reflection shows that elimination of the service department leads to cost savings in all the other service departments that no longer need to supply their services to the department that is eliminated. This elimination leads to even further savings since the reduced level of operations of the other service departments in turn reduces the level of service that they require from the external supplier. Finally, the external supplier will absorb any self-service requirements.

Through the use of some elementary matrix manipulations, we rigorously prove that the indifference point is fixed and is independent of the number of service departments replaced by external suppliers.

By using $I_m = ZZ^{-1}$ and $E = (Z - S^t)^{-1}$, we define or redefine the following matrices:

$$I_m - B = ZZ^{-1} - SZ^{-1} = (Z - S)Z^{-1} \tag{3}$$

$$I_m - G = ZZ^{-1} - S^tZ^{-1} = (Z - S^t)Z^{-1} \tag{4}$$

$$(I_m - B)^{-1} = Z(Z - S)^{-1} = ZE^t. \tag{5}$$

Since $I_m - B = (Z - S)Z^{-1}$ is invertible and $I_m - G = (Z - S^t)Z^{-1}$, we conclude that $(I_m - G)^{-1}$ exists; hence

$$H = (h_{ij}) = [H_1, H_2, ...H_n] = (I_m - G)^{-1} = Z(Z - S^t)^{-1} = ZE. \tag{6}$$

The full cost per unit vector u is then obtained as follows:

$$\begin{aligned} u^t &= (u_1, u_2, ..., u_n) = (Z^{-1}v)^t = [Z^{-1}(I_m - B)^{-1}b]^t \\ &= b^t EZZ^{-1} = b^t E = (c^t Z)E = c^t(ZE) \\ &= c^t H = c^t[H_1, H_2, \dots, H_m] = [c^t H_1, c^t H_2, \dots, c^t H_m]\,. \end{aligned} \tag{7}$$

Thus $u_j = c^t H_j$. Suppose now that r $(1 \le r \le m)$ service departments are replaced by outside suppliers. By relabeling some of the service departments, if necessary, we may assume without loss of generality that the first r service departments are replaced by outside suppliers:

$$\mathcal{O}_1\,,\ \mathcal{O}_2\,,\ \dots\,,\ \mathcal{O}_r.$$

We shall prove that $\mathcal{O}_k$ $(k = 1, 2, \dots, r)$ should be paid u_k dollars per unit for its services; but first we need to define the matrix $\widetilde{G}$ by replacing the first r columns of G by zero columns. We have

$$G = [G_1, G_2, \dots, G_r, G_{r+1}, \dots, G_n] \text{ and } \widetilde{G} = [0, 0, \dots, 0, G_{r+1}, \dots, G_n].$$

Hence the invertible matrix

$$(I_m - \widetilde{G}) = [\ e_1, e_2, \dots, e_r, (e_{r+1} - G_{r+1}), \dots, (e_n - G_n)\ ] = \begin{pmatrix} I_r & M \\ 0 & N \end{pmatrix} \tag{8}$$

The vector $\widetilde{z}$ represents the total units required for each supplier or each remaining service department and is a solution to the system of linear equations:

$$(I_m - \widetilde{G})\,\widetilde{z} = p.$$

Note that, originally we had $\qquad (I_m - G)\,z = p.$

The vectors c, v, u, and w are now replaced by $\widetilde{c}$, $\widetilde{v}$, $\widetilde{u}$, and $\widetilde{w}$ respectively.

Theorem. *Let $\widetilde{z} = (\widetilde{z}_1, \widetilde{z}_2, \dots, \widetilde{z}_r, \widetilde{z}_{r+1}, \dots, \widetilde{z}_n)^t$ be a solution to $(I_m - \widetilde{G})\,\widetilde{z} = p$, then*
(i) $\widetilde{c}^{\,t} = (u_1, u_2, \dots, u_r, c_{r+1}, \dots, c_n)^t$;
(ii) $\widetilde{u} = u$;
(iii) $\widetilde{w} = w$.

Proof. (i): Since $p = (I_m - G)\,z = (I_m - \widetilde{G})\,\widetilde{z}$, then by using the matrix $H = (I_m - G)^{-1}$, we obtain

$$z = H\,p = H(I_m - \widetilde{G})\widetilde{z}\,.$$

The total cost $T = c^t z$ is fixed; so we need to find a vector $\widetilde{c}$, where $T = \widetilde{c}^{\,t}\widetilde{z}$. From

$$H(I_m - \widetilde{G}) = [H_1, H_2, \dots, H_r, H_{r+1}, \dots, H_n](I_m - \widetilde{G}) = [H_1, H_2, \dots, H_r, e_{r+1}, \dots, e_n],$$

and from (7), we have $u_j = c_j H_j$. Hence

$$\begin{aligned} T = c^t z \quad &= c^t H(I_m - \widetilde{G})\widetilde{z} = c^t[H_1, H_2, \dots, H_r, e_{r+1}, \dots, e_n]\widetilde{z} \\ &= (u_1, u_2, \dots, u_r, c_{r+1}, \dots, c_n)\widetilde{z}\,. \end{aligned}$$

Thus $\widetilde{c}^{\,t} = (u_1, u_2, \dots, u_r, c_{r+1}, \dots, c_n)^t$ may be chosen to be the new traceable cost.

(ii): According to (7), $u^t = c^t H = c^t(I_m - G)^{-1}$, or $c^t = u^t(I_m - G)$.
Similarly, we have $\widetilde{c}^{\,t} = \widetilde{u}^{\,t}(I_m - \widetilde{G})$.
For $k = 1, 2, \dots, r$,

$$u_k = \widetilde{c}_k = \widetilde{u}^{\,t} e_k = \widetilde{u}_k \quad \Longrightarrow \quad \widetilde{u}_k = u_k.$$

From (8), we have

$$u^{\,t}\begin{pmatrix} M \\ N \end{pmatrix} = (c_{r+1}, c_{r+2}, \dots, c_m) = (\widetilde{c}_{r+1}, \widetilde{c}_{r+2}, \dots, \widetilde{c}_m) = \widetilde{u}^{\,t}\begin{pmatrix} M \\ N \end{pmatrix}.$$

Now we use the invertibility of the matrix N to deal with the remaining components of $\widetilde{u}$ and u. We already showed that for $k = 1, 2, \dots, r$, $\widetilde{u}_k = u_k$. Therefore we only need to use the equation

$$(\widetilde{u}_{r+1}, \widetilde{u}_{r+2}, \dots, \widetilde{u}_m)N = (u_{r+1}, u_{r+2}, \dots, u_m)N.$$

The fact that the matrix N is invertible completes the proof of this part.

(iii): From (2), we have $w = Pu$ and from part (ii) $\widetilde{u} = u$. Thus

$$\widetilde{w} = P\widetilde{u} = Pu = w. \blacksquare$$

According to this theorem, if $\mathcal{O}_k$ $(k = 1, 2, \dots, r)$ is paid exactly u_k dollars per unit for its service, then the original amount of the total cost, the full cost per unit and the costs of service departments allocated to each production department will remain intact. This proves that Kaplan's assertion concerning the indifference point is also true for multiple replacements.

The matrices $\widetilde{S}$ and $\widetilde{B}$ are obtained as follows:

$$\widetilde{S} = (\widetilde{s}_{ij}) = \widetilde{Z}(\widetilde{G}^t) \quad \text{and} \quad \widetilde{B} = \widetilde{S}(\widetilde{Z}^{-1}), \text{ where } \widetilde{Z} = diag(\widetilde{z}_1, \widetilde{z}_2, \dots, \widetilde{z}_n).$$

If $1 \le j \le r$ (resp. $r < j \le m$), then the entry $\widetilde{s}_{ij}$ $(i = r+1, \dots, m)$ represents the number of units provided by the outside supplier (resp. the service department) j to the service department i. Note that the matrix P will remain the same but $\widetilde{C} = P(\widetilde{Z})^{-1}$. Finally, we point out the fact that since $\widetilde{z}_i \le z_i$ for $i = 1, 2, \dots, m$, no component of the new full cost vector

$\widetilde{v} = \widetilde{Z}u$ may exceed the corresponding component of the original full cost vector $v = Zu$. Therefore, each time a service department is replaced by an outside supplier, the full cost of every remaining service department will be reduced. Also note that the traceable cost vector $\widetilde{b} = \widetilde{Z}\,\widetilde{c}$.

In our example, if the first 2 service departments are replaced by outside suppliers $\mathcal{O}_1$ and $\mathcal{O}_2$, then the company must pay $u_1 = \$\,27.80$ per unit to the supplier $\mathcal{O}_1$ and $u_2 = \$\,110.77$ per unit to the supplier $\mathcal{O}_2$ but it will need only

$$\widetilde{z} = \begin{bmatrix} 3760.91 \\ 1692.35 \\ 9234.66 \\ 4788.11 \end{bmatrix} \quad \text{units of services instead of} \quad z = \begin{bmatrix} 4000 \\ 2000 \\ 10000 \\ 5000 \end{bmatrix} \quad \text{units.}$$

The new traceable cost vector will be

$$\widetilde{b} = \widetilde{Z}\,\widetilde{u} = \begin{bmatrix} 3760.91 & 0 & 0 & 0 & \$\,27.80 & \$\,104,536.68 \\ 0 & 1692.35 & 0 & 0 & \$\,110.77 & \$\,187,473.44 \\ 0 & 0 & 9234.66 & 0 & \$\,10.00 & \$\,92,346.62 \\ 0 & 0 & 0 & 4788.11 & \$\,30.00 & \$\,143,643.26 \end{bmatrix}$$

and the new full cost:

$$\widetilde{v} = \begin{bmatrix} \$104,536.68 \\ \$187,473.44 \\ \$119,607.87 \\ \$164,663.23 \end{bmatrix} \quad \text{will have smaller components than} \quad v = \begin{bmatrix} \$\,111,182.30 \\ \$\,221,554.43 \\ \$\,129,520.58 \\ \$\,171,950.18 \end{bmatrix};$$

but the full cost per unit vector and the costs of service departments allocated to each production department remain intact (i.e., $\widetilde{u} = \widetilde{Z}^{-1}\widetilde{v} = Z^{-1}v = u$ and $\widetilde{w} = \widetilde{C}\widetilde{v} = Cv = w$). The total cost $T = \$528,000$ will also remain the same, i.e.,

$$\begin{aligned} T = \widetilde{c}^{\,t}\,\widetilde{z} &= \$\,27.80(3760.91) + \$\,110.77(1692.35) + \$\,10.00(9234.66) + \$\,30.00(4788.11) \\ &= \$\,104,536.68 + \$\,187,473.44 + \$\,92,346.62 + \$\,143,643.26 \\ &= \$\,528,000.00 \end{aligned}$$

Here is the final table (we use integer values for the number of employees and calls):

	$\mathcal{O}_1$	$\mathcal{O}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$
$\mathcal{S}_3$	369	92.35	277.04	92
$\mathcal{S}_4$	192	0	957.62	96
$\mathcal{P}_1$	1800	400	2000	1600
$\mathcal{P}_2$	400	600	2000	2000
$\mathcal{P}_3$	1000	600	4000	1000
Total Units	3761 employees	1692.35 hours	9234.66 sq ft	4788 calls
Traceable Cost	\$ 104,536.68	\$ 187,473.44	\$ 92,346.62	\$ 143,643.26
Unit Cost	\$ 27.80	\$ 110.77	\$ 10.00	\$ 30.00

References

[1] Roger A. Horn and Charles A. Johnson, Matrix Analysis, Cambridge University Press, 1985.

[2] Robert S. Kaplan, Variable and Self-Service Costs in Reciprocal Allocation Models, The Accounting Review (October 1973), pp.738-748.

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