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Some Ideas on Nonlinear Musical Analysis

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Abstract

In this article we use time series analysis methods such as recurrence quantification analysis, estimation of the correlation dimension and Liapunov exponents to study three different musical compositions. The results suggest a different level of complexity and chaoticity with respect to the degree of freedom of the composition style.

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Keywords: Nonlinear Time Series Analysis, Dynamical Systems, Musical Analysis

Introduction

In the last three decades many natural phenomena have been investigated by using the methods of non linear time series analysis (see for example [6], [20] and [15]). The general idea is to measure a variable of a complex system and use Takens' theorem (see [21]) to reconstruct the whole phase space by considering the data in a proper m-dimensional euclidian space and to study the attractor set, which is the set that contains all the information about the long term dynamics. The following step consists in studying the dynamics on the reconstructed phase space in order to understand the chaotic behavior of the phenomenon under consideration (see [19]). The estimate of the correlation dimension of the attractor and of the Liapunov exponents represent classical tools to detect chaotic behavior of the system. The final objective of the the

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theory is to make predictions of the behavior of the system by the knowledge of the data set. In the present paper we make an identification of a solo musical composition with a time series and apply some time series analysis techniques. We discuss some technical difficulties in applying these techniques to the context of musical analysis. There is a large literature about the relation between music and mathematics (see for example the interesting series of books [13] [11]and [12]) and in particular in finding hidden geometrical structures in musical compositions (see for example [22], [1], [2] and [8]). In this article, we suggest that Takens' theorem represents a natural way to search for hidden structure, i.e. attractors in a musical composition and although nonlinear analysis cannot substitute classical musical analysis such as Thematic, Motivic or Shaenkerian analysis, we expect that the study of the geometry of the attractor of a music composition should give some information about composition technics, styles, genres and peculiar features of the compositor, maybe revealing a new insight in human being's artistic production. The rest of the paper is organized as follows. In section 1 we present some preliminary notions while in section 2 we present some results on the search for the embedding dimension. In section 3 we apply a recurrence quantification analysis to the data. Section 4 is devoted to investigate the correlation dimension of the data. Section 5 consists on estimating the maximum Liapunov exponent. Finally, in section 6 we give some final remarks and conclusions.

1 Preliminaries

A musical composition, like many natural phenomena, is a complex products of the interaction of many variables: tones, rhythm, melody, harmony, form, tempo and dynamics. Since this is a first approach to musical analysis through the study of nonlinear time series analysis, in order to keep the presentation as simple as possible, in the present work we will consider only solo pieces in which we disregard all these variables but tones. We associate to each tone from the lowest to the highest a natural number in a chromatic way, that is, if $C4$ is the lowest tone then we make the identification $1 \to C4$, $2 \to C\#4$, $3 \to D4$ and so on. Consequently, to each solo music piece we associate a finite sequence of natural numbers called time series:

$$y_1, \dots, y_N, \qquad y_t \in \mathbb{N}, \qquad \forall t \in \{1, \dots, N\}, \tag{1}$$

where N is the total number of tones of the composition. We note that in the context of musical analysis there are two important features that are absent in the common fields of investigations:

- the time series consists only of natural numbers;

- since the series is taken from the compositor manuscript, it does not present any noise or errors or problems of interpretation and collection of the data;

- while in the current research the data is selected by a time delay (see [9]), in musical analysis the time delay is set as $\tau = 1$ and each element of the series is considered.

The main feature of the theory is Takens' Theorem, the idea is to construct an object in a Euclidean pseudo phase space whose dynamical properties are equivalent to that of the attractor set. Since this is a theoretical result, any application requires a careful interpretation, moreover some hypothesis are needed to be verified. In particular it is needed that:

1. the system can be represented by a nonlinear difference equation or O.D.E,

2. the measure function M is smooth,

3. the time series $\{y_t\}_{t=1}^{N}$ refers to the dynamic on the attractor,

4. the system is deterministic.

We consider that the previous hypothesis are actually satisfied if the musical piece is composed in a deterministic way and satisfying ideas, tastes and techniques that are typical of a musical style. Moreover the whole piece should have been written by the same composition method in a homogeneous way.
In the present work we consider three musical composition:

(S1) Prelude of Suite n. 1 for cello solo (1720-1721) by J.S. Bach (1685-1750) (see [23]);

(S2) Syrinx (1913) by C. Debussy (1862-1918) (see [4]);

(S3) Tenor Saxophone Solo from Acknowledgement (from the Album A Love Supreme, 1964) by J. Coltrane (1926-1967) (see [3]).

The choice of the material to analyze is of course questionable however we remark that the reasons for choosing the above compositions are:

- All three compositions are written for one instrument (in the last case it is an improvisation). We think that at this stage of the research it is convenient not to face chamber or orchestral music since the geometric information is distributed among each part of the composition and a higher dimensional analysis is required.

- The pieces correspond to different musical styles and historical periods. They are written with different compositional techniques (baroque, modern, and jazz improvisation).
- The composers of the pieces are among the most representative in their style and historical period.

The three series consist of 656, 259 and 818 data points respectively and are represented in figure 1 below which show the nonlinear profiles.

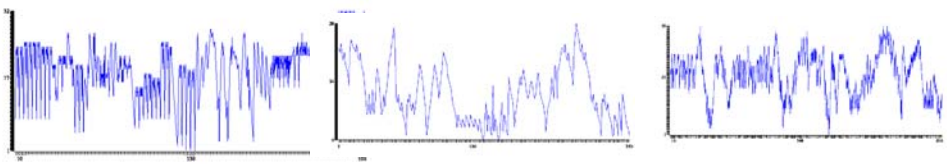

Figure 1: The three time series.

In the next sections we will start to analyze the above musical compositions with the techniques of non linear time series analysis.

2 Embedding dimension

In order to recover information about the dynamics on the attractor we consider the so called $m-$histories:

$$\begin{aligned} & y_N^m = (y_N, \ldots, y_{N-m+1}), \\ & y_{N-1}^m = (y_{N-1}, \ldots, y_{N-m}), \\ & \vdots \\ & y_m^m = (y_m, \ldots, y_1). \end{aligned}$$

From the Takens' theorem we have that there exists a natural number m, called embedding dimension such that the dynamics on the attractor set is diffeomorphic to the dynamics on the set of $m-$ histories:

$$\mathcal{B}_m = \{y_t^m\}_{t=m}^N \approx \mathcal{A}.$$

In particular they share the same Liapunov exponents and the same correlation dimension. Figures 2 and 3 below suggest that choosing $m = 2$ or $m = 3$ is not appropriate since the trajectories intersect. This contradicts the existence and uniqueness theorem for solutions of a smooth dynamical system. However both dimension m=2 and m=3 present a symmetry of the data with respect to the lines $x = y$ and $x = y = z$ respectively.

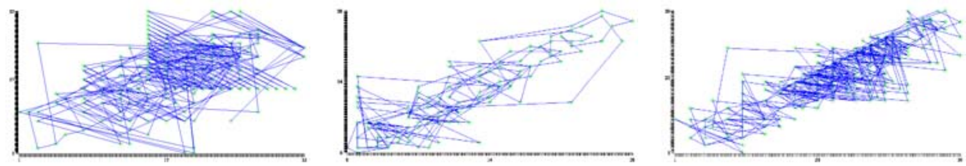

Figure 2: The time series represented in dimension $m = 2$.

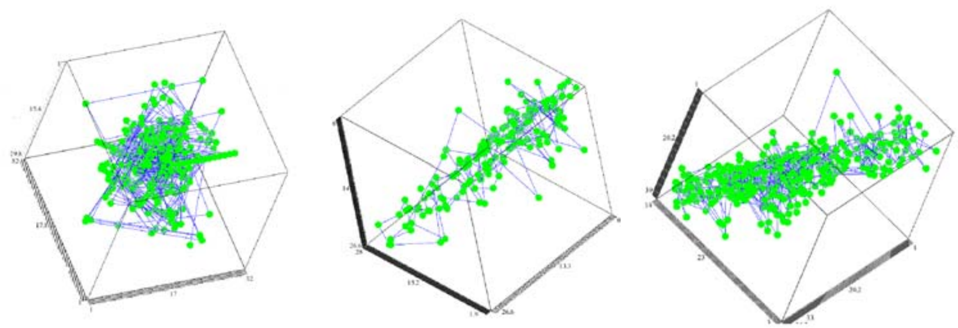

Figure 3: The time series represented in dimension $m = 3$.

A first indication of the geometric structure immerse in the above compositions is suggested by figure 4 below in which the number of repeated points is represented as a function of the dimension. The three composition share an half-gaussian distribution-like picture.

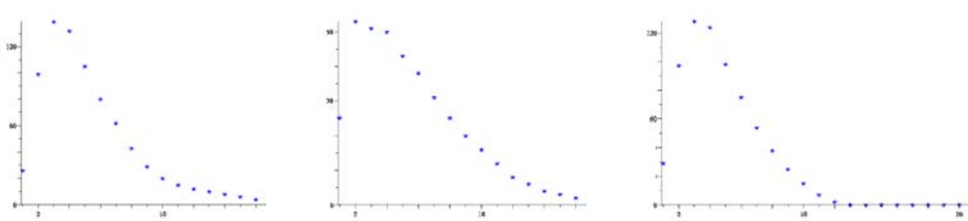

Figure 4: The number of repeated points of the three series in function of dimension.

The most used method to find the embedding dimension m is the so-called false nearest neighbor method. It is based on the idea that if we consider the data represented on a lower dimensional space, then points that are near in this space could be far in the actual phase space. This points are called false neighbors. We fix an $m \in \mathbb{N}$ and associate to each m-histories y_t^m its nearest neighborhood y_s^m and check if the distance increase more than a tolerance radium R when the dimension is increased. The function that counts the number of false neighborhoods is defined as follows (see [18]):

$$F(R,m) = \frac{\sum_{n=1}^{N-m-1} \theta\left(\frac{\|y_t^{m+1}-y_s^{m+1}\|}{\|y_t^m - y_s^m\|} - R\right) \Theta\left(\frac{\sigma}{R} - \|y_t^m - y_s^m\|\right)}{\sum_{n=1}^{N-m-1} \Theta\left(\frac{\sigma}{R} - \|y_t^m - y_s^m\|\right)},$$

where Θ is the heaviside function and σ is the standard deviation of the data (see table 1 below), Θ is the Heaviside function and $\|x-y\| = \sup_k |x_k - y_k|$.

Time Series	$\approx \sigma$
Bach	5.6320
Debussy	6.6640
Coltrane	7.2883

Table 1: The table represents the standard deviation of data of the three series

The factor $\Theta\left(\frac{\sigma}{R} - \|y_t^m - y_s^m\|\right)$ is used to select only data whose initial distance was not already larger than $\frac{\sigma}{R}$. The embedding dimension should correspond to the value of m for which the number of false neighbors is zero, i.e. $F(R,m) = 0$. In general there are several problems in applying this algorithm and in the case of musical analysis more problems arise. The main points are:

1. There is not a general method to choose the appropriate value of R.
2. The natural deviation of trajectories of a dynamical systems increases distances between points in the evolution. This makes hard to reach zero false neighborhoods.
3. In the case of musical analysis there are several parts of the data that repeat (depending on the form of the musical composition).

The choice of R can be made in the following manner:

$$R \approx 1 + \frac{q}{\sigma}, \tag{2}$$

where σ is the standard deviation of the data. The number q depends on the phenomenon under analysis, in particular in this case it is a natural number that estimates when two tones are far. We chose $q = 3$ that represents a musical interval of a minor third. The good choice of the parameter R can be supported by the analysis of the graphs that represent the embedding dimension. In figures 5 and 6 below, the number of false neighborhood for the three series are represented, from the previous discussion we expect that the embedding dimension should be found among (or near) the first local minima.

The rightness of the choice of the embedding dimension will also be supported by the results of the next section in which the consistency of results will be proved (in particular we will compare the correlation plot for different values of m).

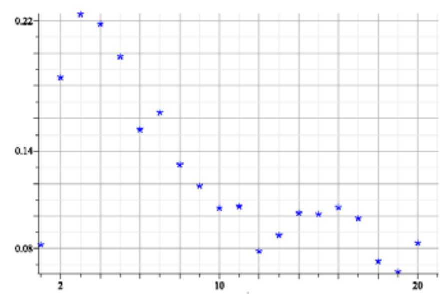

Figure 5: The number of false neighborhood, for the Bach series, as a function of dimension. We set $R = 1.45$, the local minima are $m = 6, 10, 12, 15$ while the global minimum is attained at $m = 19$.

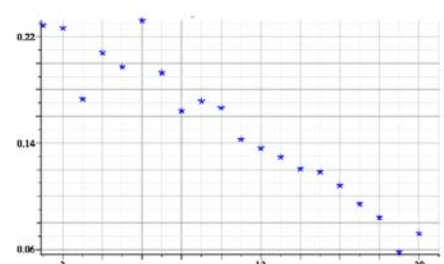

Figure 6: The number of false neighborhood, for the Debussy series, as a function of dimension. We set $R = 1.45$, the local minima are $m = 3, 5, 8, 14$ while the global minimum is attained at $m = 19$.

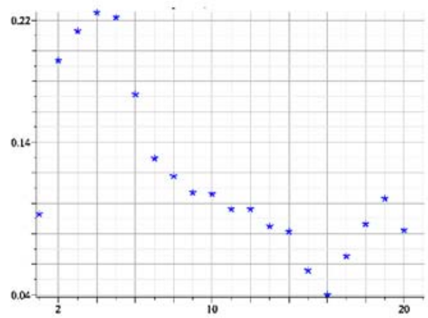

Figure 7: The number of false neighborhood, for the Coltrane series, as a function of dimension. We set $R = 1.68$, the local minima are $m = 9, 11, 13, 20$ while the global minimum is attained at $m = 16$.

3 Recurrence Quantification Analysis

In this section we use the method of recurrence quantification analysis to compare the different local minima and to choose the correct embedding dimension. This method was introduced by Zbilut and Webber in [24] and is based on the observation that patterns of recurrence in nature necessarily have mathematical underprintings [5]. Recurrence quantification analysis can be seen as a quantification of this observation. It is based on the analysis of the so-called recurrence plots.

The recurrence plot analysis was introduced by Eckmann, Kamphorst and Ruelle in [5]. In is a graphical method that allows to identify chaotic or deterministic behavior for a given time series. It is a very useful method when the dimension of the phase space is known to be greater that 3 and consequently, it cannot be graphed. The recurrence plot analysis enables to study the phase space through a two-dimensional graph.

Here, we will use recurrence and quantification plot analysis in order to determine the correlation dimension among the different local minima found in the previous section. The method consists of constructing a symmetric matrix consisting of the Euclidean distances between all pairs of m-histories:

$$M_{i,j}(m) = \|y_i^m - y_j^m\|. \tag{3}$$

For a given $\epsilon > 0$, the percentage of recurrence $\mathrm{REC}(M, \epsilon)$ of a matrix M is defined as the percentage of entries of the M that are less that ϵ. The entries of M having a value less than ϵ will be called *recurrence points*. This variable measures how sparse the matrix is.

Then each entry (i, j) of the matrix will be plotted as a black dot if the entry is less than a fixed value of ε (In our case we took $\varepsilon = 4$) . Figures 8-10 show the recurrence plots for each local minima obtained for each time series.

In general terms, we see that the global geometry of the plots does not abruptly change for the same time series and different values of m. Also, we see that for high values of m information is lost probably due to the fact that the number of data considerably reduces. Consequently we could infer that the analysis of the dynamical behavior of the time series will not be severely affected by the change of the dimension embedding among the firsts local minima found by the method of nearest neighbors.

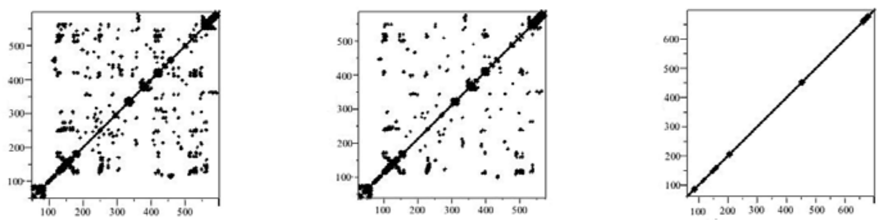

Figure 8: The matrix $\tilde{M}$ for the Bach series for the values of $m = 6, 7, 19$.

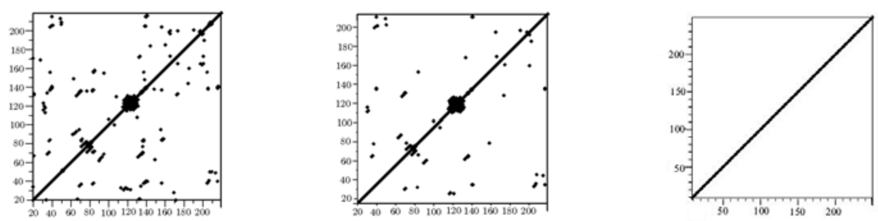

Figure 9: The matrix $\tilde{M}$ for the Debussy series for the values of $m = 6, 7, 19$.

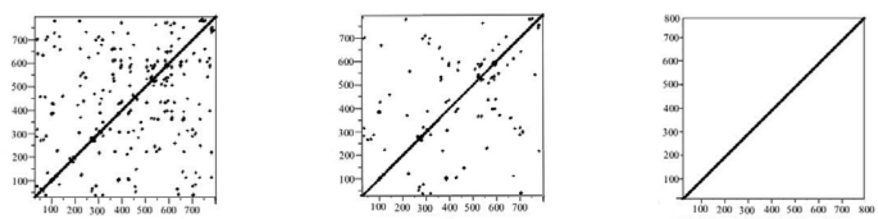

Figure 10: The matrix $\tilde{M}$ for the Coltrane series for the values of $m = 8, 9, 16$.

Although recurrence plot analysis is widely used, the conclusions we may reach by just looking at the graph are mostly empirical and sensitive to the bias one observer may have. In order to prevent that problem to happen, Zbilut and Webber devised in [24] a quantification method in which certain recurrence features can be extracted.

The second recurrence variable we calculate is the percentage of determinism $\mathrm{DET}(M, \epsilon)$, which is defined as the proportion of recurrent points in M that belong to a diagonal of length at least 2.

We remark that in this calculations it is enough to take into account just the upper (or lower) triangle of the matrix M since the matrix is symmetric at has zeroes over the diagonal.

The idea of studying this variables is that each diagonal in the matrix corresponds to a diagonal in the recurrence plot. If there is a diagonal line in the plot, this corresponds to a repeating or deterministic pattern in the dynamics. Deterministic time series will result in long diagonal lines. On the other hand, chaotic time series will give very few short diagonal lines (See [24]).

In figures 11 to 13 we see a graph of ϵ vs. $\mathrm{REC}(M, \epsilon)$ for each time series and for each value of the candidates for the embedding dimension found in previous sections. As we can see in any case the percentage of recurrence is very low even for high values of ϵ and it does not significantly change by modifying the value of the embedding dimension.

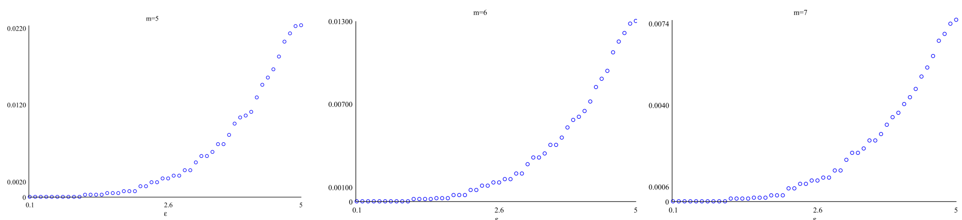

Figure 11: ϵ vs. $\mathrm{REC}(M, \epsilon)$ of the Bach time series with $m = 5, 6, 7$.

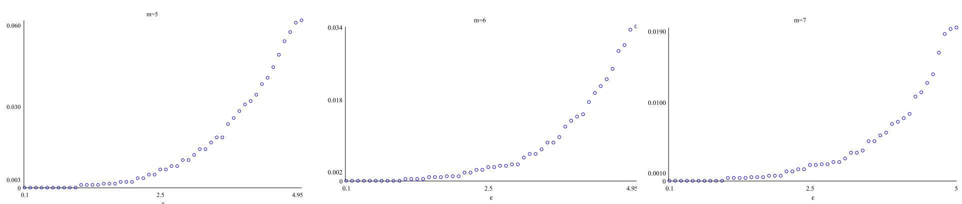

Figure 12: ϵ vs. $\mathrm{REC}(M, \epsilon)$ of the Debussy time series with $m = 5, 6, 7$.

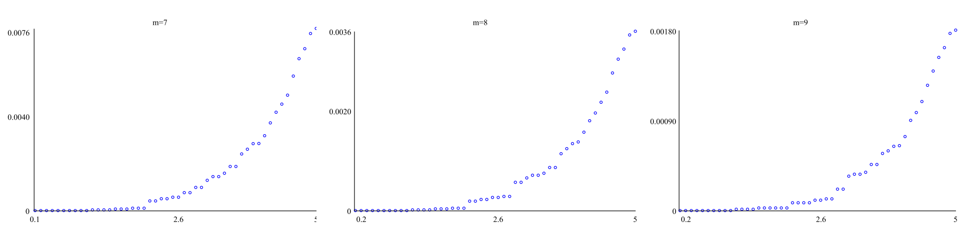

Figure 13: ϵ vs. $\mathrm{REC}(M, \epsilon)$ of the Coltrane time series with $m = 7, 8, 9$.

In tables 2, 3 and 4 it is shown the calculation of the parameter $\mathrm{DET}(M, \varepsilon)$ for $\epsilon = 4$ for each one of the three time series. We can see that for all of them, the determinism percentage is very low and the values are very close to each other when comparing each time series with the same value of m. However, we can still observe for a fixed value of m Bach's series has the highest percentage of determinism among all three and Debussy's has the lowest one for the choice of m as the embedding dimension (see also section 5 for the comparison between the maximum Liapunov exponents). This result is somewhat expected due to the style of each musical composition.

Serie	**Embedding Dimension**	$\epsilon = 4$
Bach		
	5	0.464746
	6	0.615468
	7	0.601973

Table 2: Estimate of DET(M, ε) for Bach's Series.

Serie	**Embedding Dimension**	$\epsilon = 4$
Debussy		
	5	0.454545
	6	0.475177
	7	0.475609

Table 3: Estimate of DET(M, ε) for Debussy's Series.

Serie	**Embedding Dimension**	$\epsilon = 4$
Coltrane		
	7	0.428211
	8	0.436274
	9	0.500000

Table 4: Estimate of DET(M, ε) for Coltrane's Series.

4 Correlation Dimension

For the class of dissipative systems the dynamics takes place in a lower dimensional set called attractor. Estimating the dimension of the attractor will give an indication of the chaoticity or regularity of the system. An integer dimension it is the indication of the regularity of motion while a fractal dimension may suggest that the dynamics is chaotic. However to support this hypothesis further analysis is needed, for example the estimate of the Liapunov exponents (see section 6 below). The most used tool is the so-called Correlation Dimension method proposed by Grassberger and Procaccia (see [7]). This is a simple algorithm that does not require of too many calculations. Let $\tilde{N}$ denote the number of distinct couples of $m-$histories and define the correlation integral (see [9]) as

$$C_m(\varepsilon) = \frac{2}{\tilde{N}(\tilde{N}-1)} \sum_{i=1}^{\tilde{N}} \sum_{j=i+1}^{\tilde{N}} \Theta\left(\varepsilon - \|y_i^m - y_j^m\|\right),$$

where ε is a positive parameter to be chosen later. If ε is properly chosen we have that in the case of fractal dimension $C_m(\varepsilon) \approx \varepsilon^D$, where D is the so-called correlation dimension

$$D \approx \frac{\log C_m(\varepsilon)}{\log \varepsilon}.$$

We will estimate the correlation dimension D by the so-called GP-plot method. It consists on representing $\log C_m(\varepsilon)$ as a function of $\log \varepsilon$ and then compute the slope of the curves with a linear regression method. The proper choice of ε is in correspondence to the values of ε for which the curves appear to be close enough to a line. In figures 14 and 15 the function $\log C_m(\varepsilon)$ is represented as a function of $\log \varepsilon$, for ε in the closed interval $[3, 7]$. We chose the values of m in correspondence of the first local minima and the global minima of the function that counts the false neighbors.

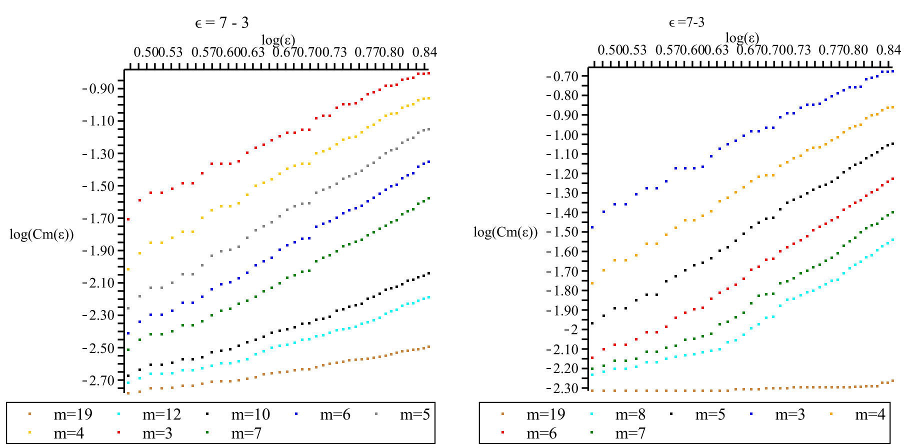

Figure 14: $\log C_m(\varepsilon)$ as a function of $\log \varepsilon$; with $\varepsilon \in [3, 7]$ for the Bach and Debussy time series for different values of m

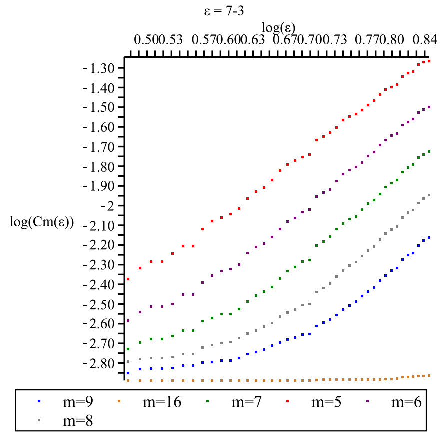

Figure 15: $\log C_m(\varepsilon)$ as a function of $\log \varepsilon$; $\varepsilon \in [3, 7]$ for the Coltrane time series for different values of m

This computation is another indirect estimation of the embedding dimension found in the previous section, in fact we have that $m \geq 2D + 1$. The values of m that satisfy the previous inequality correspond to local minima of the function that counts the false nearest neighbors. This together with an empiric analysis of the correlation plot, leads us to conclude that the embedding dimension for the three series should be 7 for the Bach and Debussy series and 9 for the Coltrane series.

Tables 5-7 below show, for each series, the estimated value of the correlation dimension for $m = 5, 6, 7$ in the case of Bach and Debussy's time series and for $m = 7, 8, 9$ in the case of Coltrane's time series using linear regression. We perform the calculation for the whole interval of ε, that is $0.48 \leq \log \varepsilon \leq 0.84$ and for subinterval where the curves appear to be closer to a straight line.

Serie	m	D
Bach		$\log \varepsilon \in [0.70, 0.84]$
	5	3.05312268
	6	3.05866535
	7	2.86418984

Table 5: The estimate of the correlation dimension of the Bach's series.

Serie	m	D
Debussy		$\log \varepsilon \in [0.70, 0.84]$
	5	2.48703915
	6	2.76516076
	7	2.80476919

Table 6: The estimate of the correlation dimension of the Debussy's series.

Serie	m	D
Coltrane		$\log \varepsilon \in [0.70, 0.84]$
	7	3.64054770
	8	3.69778462
	9	3.36536608

Table 7: The estimate of the correlation dimension of the Coltrane's series.

In table 8 below, we represent the mean value of the correlation dimension for the values of m considered before (that is values of m such that $m \geq 2D+1$).

Serie	**D**
Bach	2.985325956 ± 0.09
Debussy	2.685656366 ± 0.15
Coltrane	3.567899466 ± 0.14

Table 8: The mean values of the correlation dimension for the three series.

5 Liapunov Exponents

We observed that the fractal dimension of the attractor does not always imply chaoticity of the system. As a consequence a fractal correlation dimension is not enough to conclude about the chaoticity of the series, further analysis is needed. An important tool to detect chaos are the so called Liapunov exponents which describe the rate of divergence of the trajectories. A positive Liapunov exponent may be considered as an indication of chaotic behavior. The idea is to consider an $m-$history $y_{t_0}^m$ and a neighborhood $B(y_{t_0}^m, \varepsilon)$ of radius $\varepsilon > 0$. Then we let $B(y_{t_0}^m, \varepsilon)$ evolve for a small time Δt and the mean value of the evolution of the points of the neighborhood approximate the evolution of $y_{t_0}^m$. The logarithm of this mean value represents a good approximation of the exponential deviation of the trajectory of $y_{t_0}^m$ from infinitesimally close trajectories. In order to obtain the estimate we consider the mean value among any $m-$ history $y_{t_0}^m$. The algorithm that estimates the greatest Liapunov Exponent is the following:

$$L(\Delta t) = \frac{1}{N_m} \sum_{t_0=1}^{N_m} \ln \left(\frac{1}{\left| B(y_{t_0}^m, \varepsilon) \right|} \sum_{y_t^m \in B(y_{t_0}^m, \varepsilon)} \| y_{t_0+\Delta t}^m - y_{t+\Delta t}^m \| \right),$$

where N_m is the number of $m-$histories and $\left| B(y_{t_0}^m, \varepsilon) \right|$ is the number of elements of $B(y_{t_0}^m, \varepsilon)$. In order to obtain correct estimates, the neighborhood $B(y_{t_0}^m, \varepsilon)$ should contain a sufficiently large number of points. In the present paper we chose the radius ε in such a fashion that $\left| B(y_{t_0}^m, \varepsilon) \right| \geq 10$ for any $y_{t_0}^m$ considered. We note that the number of m-histories considered in the above calculation should correspond to a number of data of magnitude of order $\frac{N}{2}$. In fact, a small quantity of data could not be representative of the dynamics of the phenomenon under analysis. When ε is properly chosen we consider different values for the elapsed time $\Delta t \in \{1, \ldots, H\}$ for a proper choice of $H \in \mathbb{N}$, and graph the curves of $\exp(L(\Delta t))$ as a function of Δt for different choices of the embedding dimension. Then by linear regression we estimate the maximum Liapunov exponent in the intervals where the graphs are stable. In figure 16 and 17 below we represent $\exp(L(\Delta t))$ as a function of Δt for $m = 3, 4, 5, 6, 7$ and for each time series. We note that if we increase the value of m we obtain a translation of the graphs without significative changes in the slope.

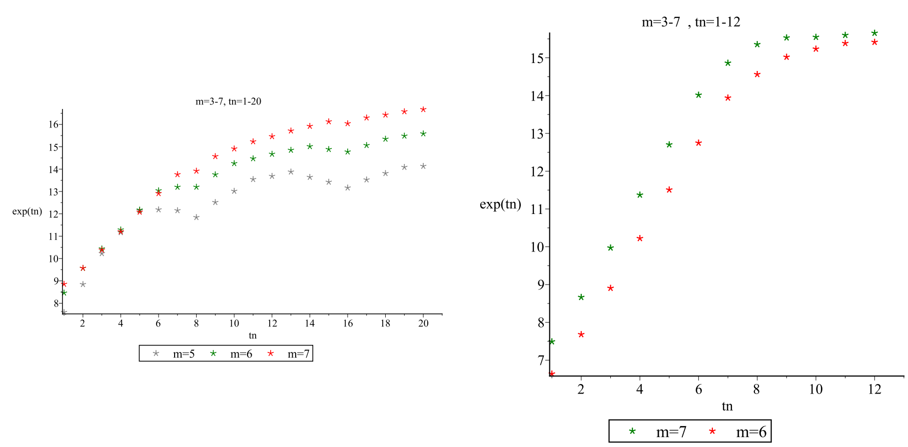

Figure 16: $L(\Delta t)$ as a function of Δt for the Bach and Debussy series for several values of m.

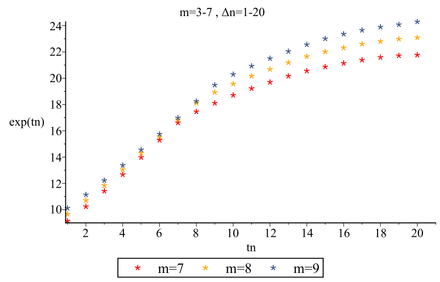

Figure 17: $L(\Delta t)$ as a function of Δt for the Coltrane series for several values of m.

Tables 9-11, present the maximum Liapunov exponent calculated for each value of m for each time series.

Serie	**Embedding Dimension**	**Maximum Liapunov Exponent**
Bach		
	5	0.283283784
	6	0.321671166
	7	0.393982544

Table 9: Estimate of the maximum Liapunov exponent for Bach's Series.

Serie	**Embedding Dimension**	**Maximum Liapunov Exponent**
Debussy		
	6	0.772038617
	7	0.854937776

Table 10: Estimate of the maximum Liapunov exponent for Debussy's Series.

Serie	**Embedding Dimension**	**Maximum Liapunov Exponent**
Coltrane		
	7	0.659067186
	8	0.716975219
	9	0.770629193

Table 11: Estimate of the maximum Liapunov exponent for Coltrane's Series.

We remark that a high number of the value of m will drastically reduce the number of m-histories available for the estimate, this will result in a bad estimate of the Liapunov Exponents. We consider the mean value of the maximum Liapunov Exponent for the values of m as indicated in the above tables and show the results in table 12.

Serie	Maximum Liapunov Exponent
Bach	0.332979164 ± 0.05
Debussy	0.813488196 ± 0.04
Coltrane	0.715557199 ± 0.06

Table 12: The mean value of the maximum Liapunov exponent for the three series.

6 Conclusion

We have thus presented and estimated the embedding dimension, correlation dimension and the maximun Liapunov exponent for three different musical compositions. The values obtained are not an exact calculation but they can be seen as an indicative of the features of the three musical compositions. From the view point of musical analysis the above results are not surprising: Bach's composition follows instrumental baroque counterpoint rules, Debussy's composition has a free style while Coltrane's composition is an improvisation constructed on many repeated patterns. Then the following inequalities concerning the embedding as well as correlation dimension

$$m_{\mathrm{Bach}} < m_{\mathrm{Debussy}} < m_{\mathrm{Coltrane}}, \tag{4}$$

$$D_{\mathrm{Bach}} < D_{\mathrm{Debussy}} < D_{\mathrm{Coltrane}}, \tag{5}$$

appear reasonable due to the high or low number of patterns that are present in the three series. The percentage of determinism and the maximum Liapunov exponent satisfy a different inequality:

$$L_{\mathrm{Bach}} < L_{\mathrm{Coltrane}} < L_{\mathrm{Debussy}}. \tag{6}$$

These results suggest an increasing level of complexity and chaoticity in this three musical compositions which could be explained by arguing that a more free composition style needs more parameters (higher embedding and correlation dimension) and it turns out to be less predictable (higher maximum Liapunov exponent and lower percentage of determinism). In particular, patterns-based improvisation such as Coltrane's (see [16]) could need a higher dimension embedding but could be less predictable than free composition of Debussy's style. There is a large musical production based on ideas such as fractals (see [14]) and random dynamics (see [17]). The methods of nonlinear time series analysis suggest the existence of some hidden geometric structure that could be of fractal type. In the present work we have associated to the three compositions the fractal dimension as in table 8. It could be possible to catalogue music by fractal dimension, analyzing the relation between dimension, style genre, and other variables. It could be interesting to develop these ideas, looking for patterns and hidden dimensions within authors, periods, music styles and genres. Moreover the analysis has been carried on starting with a one-dimensional time series made of tones, it could be possible to consider more variables (rhythm, dynamics,...) in order to face more complex musical compositions such as orchestral or chamber music.

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