Homotopy Analysis Method to Water Quality

Model in a Uniform Channel

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Abstract

In this paper, the Homotopy analysis method (HAM) for solving one-dimensional convection-diffusion equation with variable coefficients arising in the mathematical modeling of dispersion of pollutants in water is proposed. The Homotopy analysis method (HAM) for assessment of the chemical oxygen demand (COD) in a river is considered. In this proposed work, the HAM is used to compute the concentration of the pollutant for variable inputs. An illustrative
example is included to demonstrate the accuracy, efficiency and simplicity of the method.

**Keywords**: Water quality; uniform channel; convection-diffusion equation; homotopy analysis method

1 Introduction

This paper devotes a mathematical model for solving the dispersion of pollutants in a river. The Homotopy analysis method for assessment of the chemical oxygen demand (COD) concentration in a river is considered. Pochai et al. [1] addressed a mathematical model of water pollution using the finite element method. The same author(s) [2-4] implemented the finite difference method (FDM) to the hydrodynamic model with constant coefficients in the uniform reservoir and stream. This model requires the calculation of the substance dispersion given water velocity in the channel. In 1992, Liao [10] developed the basic ideas of the homotopy in topology to propose a general analytical method for nonlinear problems, namely homotopy analysis method. We show that the analytical solutions are in excellent agreement with numerical solutions obtained with MATLAB.

Non-linear differential equations describing the dynamics are known to be harder to solve then linear ODEs. One often has to resort to asymptotic techniques or classical perturbation theory to obtain analytical approximations [5]. Classical perturbation theory strongly depends on small/large physical parameters. Homotopy Analysis Method (HAM) is a quite new approach to explore highly non-linear systems. The method composes the non-linear system by linear parts and approximates the ‘real’ solution by an iterative process.

The convergence speed is governed by a tuning parameter $q$. The approximate solution then can be found as a linear combination of base functions. The main advantages of HAM are:

- Independence of small/large physical parameters
- Flexibility on the choice of the base functions
- Generality

The HAM is a nonperturbative analytical method for obtaining series solutions to nonlinear equations and has been successfully applied to numerous problems in
science and engineering [7-17]. In comparison with other perturbative and non-perturbative an analytical method, HAM offers the ability to adjust and control the convergence of a solution via the so-called convergence-control parameter. Because of this, the HAM has proved to be the most effective method for obtaining analytical solutions to highly nonlinear differential equations.

In the present paper, we introduce a Homotopy Analysis Method (HAM) is used to compute the concentration of the pollutant for variable inputs.

2 Mathematical model of dispersion in a uniform channel

The dispersion of COD is described by Convection-diffusion equation (CDE) [3] in the domain \([a,b]\),

\[
-D_x \frac{d^2 C}{dx^2} + U \frac{dC}{dx} + RC - Q = 0,
\]

where \(C(x)\) is the concentration of COD at the point \(x \in [a,b] \) \((kg/m^3)\), \(U\) is the flow velocity in \(x\) directions \((m/s)\), \(D_x\) is the diffusivity \((m^2/s)\), \(R\) is the substrate decay rate \((s^{-1})\), \(Q\) is an increasing rate substrate concentration due to a source \((kg/m^3s)\).

The boundary conditions are \(C = C_0\) at \(x = a\) and \(\frac{dC}{dx} = T_0\) at \(x = b\).

3 Basic idea of Homotopy Analysis Method (HAM)

In this section we present the basic ideas of the Homotopy analysis method (HAM). Here a description of the method is given to handle the general nonlinear problem.

\[
N\mathbf{u}_0(t) = 0, \ t > 0,
\]

Where \(N\) is a nonlinear operator and \(\mathbf{u}_0(t)\) is unknown function of the independent variable \(t\).

Zero-order deformation equation

Let \(\mathbf{u}_0(t)\) denote the initial guess of the exact solution of Eq. (1),
h \neq 0\) an auxiliary parameter, \(H(t) \neq 0\) an auxiliary function and L is an auxiliary linear operator with the property.

\[
L(f(t)) = 0, \quad f(t) = 0. \tag{3}
\]

The auxiliary parameter \(h\), the auxiliary function \(H(t)\), and the auxiliary linear operator \(L\) play an important role within the HAM to adjust and control the convergence region of solution series. Using \(q \in [0,1]\) as an embedding parameter, the so-called zero-order deformation equation.

\[
(1 - q)L[(\varnothing(t;q) - u_0(t)] = qhH(t)N[(\varnothing(t;q)], \tag{4}
\]

where \(\varnothing(t;q)\) is the solution which depends on \(h, H(t), L, u_0(t)\) and \(q\). When \(q=0\), the zero-order deformation Eq. (4) becomes

\[
\varnothing(t;0) = u_0(t). \tag{5}
\]

And when \(q=1\), since \(h \neq 0\) and \(H(t) \neq 0\), the zero-order deformation Eq.(1) reduces to,

\[
N[\varnothing(t;1)] = 0, \tag{6}
\]

So, \(\varnothing(t;1)\) is exactly the solution of the nonlinear Eq. (2). Define the so-called \(m^{th}\) order deformation derivatives.

\[
u_m(t) = \frac{1}{m!} \frac{\partial^m \varnothing(t;q)}{\partial q^m} \tag{7}\]

If the power series (7) of \(\varnothing(t;q)\) converges at \(q=1\), then we gets the following series solution:

\[
u(t) = u_0(t) + \sum_{m=1}^{\infty} u_m(t). \tag{8}\]

where the terms \(u_m(t)\) can be determined by the so-called high order deformation described below.

**High-order deformation equation**

Define the vector,
Homotopy analysis method to water quality model

\( \bar{u}_n = \{u_0(t), u_1(t), u_2(t), \ldots, u_n(t)\} \) \hspace{1cm} (9)

Differentiating Eq. (4) \( m \) times with respect to embedding parameter \( q \), the setting \( q=0 \) and dividing them by \( m! \), we have the so-called \( m^{th} \) order deformation equation.

\[
L[u_m(t) - \mathcal{N}_m u_{m-1}(t)] = hH(t)R_m(\bar{u}_m, t),
\]

where

\[
\mathcal{N}_m = \begin{cases} 0, & m \leq 1 \\ 1, & \text{otherwise} \end{cases}
\]

and

\[
R_m(\bar{u}_m, t) = \frac{1}{(m-1)!} \frac{\partial^{m-1} N[\phi(t; q)]}{\partial q^{m-1}}
\]

For any given nonlinear operator \( N \), the term \( R_m(\bar{u}_m, t) \) can be easily expressed by (12). Thus, we can gain \( u_1(t), u_2(t), \ldots \) by means of solving the linear high-order deformation Eq. (12) one after the other order in order. The \( m^{th} \) order approximation of \( u(t) \) is given by

\[
u(t) = \sum_{k=0}^{m} u_k(t)
\]

ADM, VIM, and HPM are special cases of HAM when we set \( h = -1 \) and \( H(r, t) = 1 \) in Eq. (10).

We will get the same solutions for all the problems by above methods when we set \( h = -1 \) and \( H(r, t) = 1 \). When the base functions are introduced the \( H(r, t) = 1 \) is properly chosen using the rule of solution expression, rule of coefficient of ergodicity and rule of solution existence.

4 Applications and Results

In this section, we will use the homotopy analysis method (HAM) to solve Convection-diffusion equations.
Consider the convection-diffusion equation in the form,
\[ c'' = p(x)c' + q(x)c + r(x). \]  
(14)

We apply the HAM to solve Eq. (14) as follows:

Since \( m \geq 1 \), \( \gamma_m = 1 \). Set \( h = -1 \) and \( H(r, t) = 1 \)
\[ c_m(x) = c_{m-1}(x) - L^{-1}\left( R_m\left( c_{m-1}\right) \right) \]  
(15)

Then
\[ c(x) = c_0(x) + c_1(x) + c_2(x) + \ldots \]  
(16)

Consider the convection-diffusion equation with variable coefficients
\[ c' = p(x)c' + q(x)c + r(x). \]  
(17)

We assume that there is a plant which discharge waste water into the channel at
the starting point \( 0.0 \) k.m. and that the COD concentrations of the waste water are
12 kg/m\(^3\). Let the physical parameter values are: diffusion coefficient \( 2 \) 
m\(^2\)/s, flow velocity \( u = 5 - x \) m/s, \( x \in [0, 4] \), substance decay rate \( 3s^{-1} \)
and rate of change of substance concentration due to the source \( 1Kg/m^3s \). The
space increment size 100m is used for the numerical simulation.

\[
p(x) = \frac{5 - x}{D_x} = \frac{5 - x}{12} \]

\[
q(x) = \frac{R}{D_x} = \frac{3}{12} \]  
(18)

\[
r(x) = \frac{-Q}{D_x} = \frac{-1}{12}. \]

Eq. (17) becomes
\[ c' = \left( \frac{5 - x}{12} \right) c' + \left( \frac{3}{12} \right) c - \left[ \frac{1}{12} \right] \]  
(19)

subject to the conditions \( c(0) = 12 \) and \( c'(0) = 0 \).

The approximate analytical solution of Eq. (19) using the HAM is given by
\[ c(x) = 12 - \frac{1}{4}\left[ \frac{x^2}{2!} + \frac{5x^3}{18} + \frac{49x^4}{432} + \frac{5x^5}{432} + \frac{1x^6}{432} + \ldots \right] \]  
(20)

From Eq. (20), we can obtain the numerical COD concentration.
5. Error estimation

The accuracy of the results be estimated by error function

\[ E = \left| C_{\text{exact}} - C_{\text{HAM}} \right| \]

The results are shown in Table 1.

<table>
<thead>
<tr>
<th>Distance (Km.)</th>
<th>COD concentration ( \left( \text{Kg} / \text{m}^3 \right) )</th>
<th>COD concentration ( \left( \text{Kg} / \text{m}^3 \right) )</th>
<th>Error ( E_f )</th>
<th>Error ( E_H )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>12.0000</td>
<td>12.0000</td>
<td>0.00000</td>
<td>0.00000</td>
</tr>
<tr>
<td>0.5</td>
<td>10.0833</td>
<td>11.95552</td>
<td>1.88393</td>
<td>0.01171</td>
</tr>
<tr>
<td>1.0</td>
<td>8.44260</td>
<td>11.84890</td>
<td>3.40792</td>
<td>0.001620</td>
</tr>
<tr>
<td>1.5</td>
<td>7.07910</td>
<td>11.65443</td>
<td>4.59459</td>
<td>0.01926</td>
</tr>
<tr>
<td>2.0</td>
<td>5.97830</td>
<td>11.38065</td>
<td>5.40948</td>
<td>0.00713</td>
</tr>
</tbody>
</table>

\( E_f \) - Error estimation of finite difference method (FDM)

\( E_H \) - Error estimation of homotopy analysis method (HAM)

Our results can be compared with Pochai’s results [7]. In order to assess the advantages, efficiency and the accuracy of the HAM for solving one-dimensional convection-diffusion equation with variable coefficients we use our method to solve another convection-diffusion equation, whose exact solutions are known. Results in the Table.1 show that the HAM agrees with the results obtained in [7]. In this paper, the HAM has been compared with the finite difference method (FDM) [7]. By the numerical solutions, it can be obtained that the COD concentration along a uniform channel will be decreasing.

6 Conclusion

In this work, the HAM has been applied to obtain the approximate/analytical solutions of the one-dimensional convection-diffusion equation with variable
coefficients. This scheme provides us a simple way to adjust and control the convergence of the series solution by choosing proper values of auxiliary and homotopy parameters. In conclusion, HAM gives accurate approximate solution for nonlinear problems in comparison with other methods.

References


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