An Identity of the Generalized $q$-Euler Polynomials with Weak Weight $\alpha$ Associated with $p$-Adic Invariant $q$-Integral on $\mathbb{Z}_p$

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Abstract

In this paper, we study the symmetry for generalized $q$-Euler numbers $E_{n,\chi,q}^{(\alpha)}$ and polynomials $E_{n,\chi,q}^{(\alpha)}(x)$ with weak weight $\alpha$. We obtain some interesting identities of the power sums and generalized $q$-Euler polynomials $E_{n,\chi,q}^{(\alpha)}(x)$ using the symmetric properties for the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$.

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1 Introduction

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of $p$-adic rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, $\mathbb{C}$ denotes the complex number field, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or $p$-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable function on $\mathbb{Z}_p$. For $g \in UD(\mathbb{Z}_p)$ the fermionic $p$-adic invariant $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-q}(x) = \lim_{N \to \infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \text{ see } [1, 2]. \quad (1.1)$$
If we take $g_n(x) = g(x + n)$ in (1.1), then we see that

$$q^n I_q(g_n) + (-1)^{n-1} I_q(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \text{ see [1, 2, 3, 4].} \quad (1.2)$$

Let a fixed positive integer $d$ with $(p, d) = 1$, set

$$X = X_d = \lim_{N \to \infty} (\mathbb{Z}/dp^N\mathbb{Z}), \quad X_1 = \mathbb{Z}_p,$$

$$X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} a + dp\mathbb{Z}_p,$$

$$a + dp^n\mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^n} \},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$(cf. [1, 2, 3, 4, 5, 6]).

It is easy to see that

$$I_{-q}(g) = \int_X g(x) d\mu_{-q}(x) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x). \quad (1.3)$$

Ryoo [7] introduced generalized $q$-Euler numbers $E^{(\alpha)}_{n, \chi, q}$ and polynomials $E^{(\alpha)}_{n, \chi, q}(x)$ attached to $\chi$.

Let $\chi$ be the primitive Dirichlet character with conductor $d \in \mathbb{N}$ with $d \equiv 1 \pmod{2}$. We assume that $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|q - 1|_p < 1$. Let $g(y) = \chi(y)e^{(y+x)t}$. By (1.1), we derive

$$\int_X \chi(y)e^{(y+x)t} d\mu_{-q^\alpha}(y) = \frac{[2]_q^\alpha \sum_{a=0}^{d-1} \chi(a)(-1)^a q^{\alpha a} e^{at}}{q^{\alpha d} e^{dt} + 1}e^{xt}$$

$$= \sum_{n=0}^{\infty} E^{(\alpha)}_{n, \chi, q}(x) \frac{t^n}{n!}. \quad (1.4)$$

By using Taylor series of $e^{(y+x)t}$ in the above equation (1.4), we obtain

$$\sum_{n=0}^{\infty} \left( \int_X \chi(y)(y + x)^n d\mu_{-q^\alpha}(y) \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} E^{(\alpha)}_{n, \chi, q}(x) \frac{t^n}{n!}.$$

By comparing coefficients of $\frac{t^n}{n!}$ in the above equation, we have the Witt formula for the generalized $q$- Euler polynomials attached to $\chi$ as follows:

**Theorem 1.1** For positive integers $n$, we have

$$E^{(\alpha)}_{n, \chi, q}(x) = \int_X \chi(y)(y + x)^n d\mu_{-q^\alpha}(y). \quad (1.5)$$
Observe that for $x = 0$, the equation (1.5) reduces to (1.6).

**Corollary 1.2** For positive integers $n$, we have

$$E_{n,\chi,q}^{(\alpha)} = \int_X \chi(y)y^n d\mu_{-q^\alpha}(y). \quad (1.6)$$

By (1.5) and (1.6), we have the following theorem.

**Theorem 1.3** For positive integers $n$, we have

$$E_{n,\chi,q}^{(\alpha)}(x) = \sum_{l=0}^{n} \binom{n}{l} E_{l,\chi,q}^{(\alpha)} x^{n-l}. \quad (1.7)$$

## 2 Identity for generalized $q$-Euler polynomials

In this section, we assume that $q \in \mathbb{C}_p$. In [2], Kim investigated interesting properties of symmetry $p$-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli polynomials and Euler polynomials. By using same method of [2], expect for obvious modifications, we obtain some interesting identities of the power sums and generalized $q$-Euler polynomials $E_{n,\chi,q}^{(\alpha)}(x)$ using the symmetric properties for the $p$-adic invariant $q$-integral on $\mathbb{Z}_p$. If $n$ is odd from (1.2), we obtain

$$q^n I_q(g_n) + I_q(g) = [2] q^\sum_{l=0}^{n-1} (-1)^l q^l g(l). \quad (2.1)$$

It will be more convenient to write (2.1) as the equivalent integral form

$$q^n \int_{\mathbb{Z}_p} g(x+n) d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-q^\alpha}(x) = [2] q^\sum_{k=0}^{n-1} (-1)^k q^{\alpha k} g(k). \quad (2.2)$$

Substituting $g(x) = \chi(x)e^{xt}$ into the above, we obtain

$$q^n \int_X \chi(x+nd)e^{(x+nd)t} d\mu_{-q^\alpha}(x) + \int_X \chi(x)e^{xt} d\mu_{-q^\alpha}(x) = [2] q^\sum_{j=0}^{nd-1} (-1)^j \chi(j) q^{\alpha j}e^{jt}. \quad (2.3)$$

For $k \in \mathbb{Z}_+$, let us define the $p$-adic functional $T_{k,\chi,q}^{(\alpha)}(n)$ as follows:

$$T_{k,\chi,q}^{(\alpha)}(n) = \sum_{l=0}^{n} (-1)^l \chi(l) q^{\alpha l} l^k. \quad (2.4)$$
After some elementary calculations, we have
\[
\int_X \chi(x)e^{xt}d\mu_{-q^\alpha}(x) = \frac{[2]q^\alpha \sum_{a=0}^{d-1} \chi(a)(-1)^aq^{\alpha a}e^{at}}{q^{\alpha d}e^{dt} + 1},
\]
\[
\int_X \chi(x)e^{(x+n)t}d\mu_{-q^\alpha}(x) = e^{nt}\frac{[2]q^\alpha \sum_{a=0}^{d-1} \chi(a)(-1)^aq^{\alpha a}e^{at}}{q^{\alpha d}e^{dt} + 1}.
\]

By using (2.5), we have
\[
q^{\alpha nd} \int_X \chi(x)e^{(x+nd)t}d\mu_{-q^\alpha}(x) + \int_X \chi(x)e^{xt}d\mu_{-q^\alpha}(x) = \frac{[2]q^\alpha \sum_{a=0}^{d-1} \chi(a)(-1)^aq^{\alpha a}e^{at}}{q^{\alpha d}e^{dt} + 1}.
\]

From the above, we get
\[
q^{\alpha nd} \int_X \chi(x)e^{(x+nd)t}d\mu_{-q^\alpha}(x) + \int_X \chi(x)e^{xt}d\mu_{-q^\alpha}(x) = \frac{[2]q^\alpha \sum_{a=0}^{d-1} \chi(a)(-1)^aq^{\alpha a}e^{at}}{q^{\alpha d}e^{dt} + 1}.
\]

By substituting Taylor series of \(e^{xt}\) into (2.3), we obtain
\[
\sum_{m=0}^{\infty} \left( q^{\alpha nd} \int_X \chi(x)(x+nd)^m d\mu_{-q^\alpha}(x) + \int_X \chi(x)x^md\mu_{-q^\alpha}(x) \right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left( [2]q^\alpha \sum_{j=0}^{nd-1} (-1)^j \chi(j)q^{\alpha j}j^m \right) \frac{t^m}{m!}
\]

By comparing coefficients \(\frac{t^m}{m!}\) in the above equation, we obtain
\[
q^{\alpha nd} \sum_{k=0}^{m} \binom{m}{k} (nd)^{m-k} \int_X \chi(x)x^kd\mu_{-q^\alpha}(x) + \int_X \chi(x)x^md\mu_{-q^\alpha}(x) = \frac{[2]q^\alpha \sum_{j=0}^{nd-1} (-1)^j \chi(j)q^{\alpha j}j^m}{t^m/m!}.
\]

By using (2.4), we have
\[
q^{\alpha nd} \sum_{k=0}^{m} \binom{m}{k} (nd)^{m-k} \int_X \chi(x)x^kd\mu_{-q^\alpha}(x) + \int_X \chi(x)x^md\mu_{-q^\alpha}(x) = [2]q^\alpha T_{m,\chi,q}^{(\alpha)}(nd - 1).
\]

By using (2.6) and (2.7), we arrive at the following theorem:
Theorem 2.1 Let \( n \) be odd positive integer. Then we obtain
\[
\frac{\int_X \chi(x)e^{xt}d\mu_{-q^\alpha}(x)}{\int_X q^{a(nd-1)x}e^{ndtx}d\mu_{-q^\alpha}(x)} = \sum_{m=0}^{\infty} \frac{(T_{m,\chi,q}(nd-1))^{\frac{t^m}{m!}}}{m!}.
\]

Let \( w_1 \) and \( w_2 \) be odd positive integers. By (2.5), Theorem 2.1, and after some elementary calculations, we obtain the following theorem.

Theorem 2.2 Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have
\[
\frac{\int_X \chi(x)e^{w_2xt}d\mu_{-q^{w_2^{\alpha}}}(x)}{\int_X q^{a(w_1w_2d-1)x}e^{w_1w_2tx}d\mu_{-q^\alpha}(x)} = \frac{[2]_{q^{w_2^{\alpha}}}}{[2]_{q^\alpha}} \sum_{m=0}^{\infty} \frac{(T_{m,\chi,q^{w_2}}(w_1d-1)w_2^{m})^{\frac{t^m}{m!}}}{m!}.
\]

Then we set
\[
S(w_1, w_2) = \frac{\int_X \int_X \chi(x_1)\chi(x_2)e^{(w_1x_1+w_2x_2+w_1x_2)t}d\mu_{-q^{w_1^{\alpha}}}(x_1)d\mu_{-q^{w_2^{\alpha}}}(x_2)}{\int_X q^{a(w_1w_2d-1)x}e^{w_1w_2tx}d\mu_{-q^\alpha}(x)}.
\]

By Theorem 2.2 and (2.8), after elementary calculations, we obtain
\[
S(w_1, w_2) = \left( \int_X \chi(x_1)e^{(w_1x_1+w_1x_2)t}d\mu_{-q^{w_1^{\alpha}}}(x_1) \right) \left( \int_X \chi(x_2)e^{w_2xt}d\mu_{-q^{w_2^{\alpha}}}(x_2) \right) \left( \sum_{m=0}^{\infty} \frac{[2]_{q^{w_2^{\alpha}}}}{[2]_{q^\alpha}} \sum_{m=0}^{\infty} \frac{T_{m,\chi,q^{w_2}}(w_1d-1)w_2^{m}}{m!} \right)
\]

By using Cauchy product in the above, we have
\[
S(w_1, w_2) = \sum_{m=0}^{\infty} \frac{[2]_{q^{w_2^{\alpha}}}}{[2]_{q^\alpha}} \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} \frac{E^{(\alpha)}_{m,\chi,q^{w_1}}(w_1x)w_1^{j}T^{(\alpha)}_{m-j,\chi,q^{w_1}}(w_2d-1)w_2^{m-j}}{m!}
\]

From the symmetry of \( S(w_1, w_2) \) in \( w_1 \) and \( w_2 \), we also see that
\[
S(w_1, w_2) = \left( \int_X \chi(x_1)e^{w_1x_1t}d\mu_{-q^{w_1^{\alpha}}}(x_1) \right) \left( \int_X \chi(x_2)e^{w_2x_2t}d\mu_{-q^{w_2^{\alpha}}}(x_2) \right) \left( \sum_{m=0}^{\infty} \frac{[2]_{q^{w_1^{\alpha}}}}{[2]_{q^\alpha}} \sum_{m=0}^{\infty} \frac{T_{m,\chi,q^{w_1}}(w_2d-1)w_2^{m}}{m!} \right)
\]

Thus we have
\[
S(w_1, w_2) = \sum_{m=0}^{\infty} \frac{[2]_{q^{w_1^{\alpha}}}}{[2]_{q^\alpha}} \sum_{j=0}^{m} \frac{m!}{j!(m-j)!} \frac{E^{(\alpha)}_{m,\chi,q^{w_2}}(w_2x)w_2^{j}T^{(\alpha)}_{m-j,\chi,q^{w_2}}(w_1d-1)w_1^{m-j}}{m!}
\]

By comparing coefficients \( \frac{t^m}{m!} \) in the both sides of (2.10) and (2.11), we arrive at the following theorem:
Theorem 2.3 Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain

\[
[2]_{q^{w_1}} \sum_{j=0}^{m} \binom{m}{j} w_1^{m-j} w_2^{j} E_{j,\chi,q^{w_2}}^{(\alpha)}(w_1 x) T_{m-j,\chi,q^{w_1}}^{(\alpha)}(w_2 d - 1)
\]

\[
= [2]_{q^{w_2}} \sum_{j=0}^{m} \binom{m}{j} w_1^{j} w_2^{m-j} E_{j,\chi,q^{w_1}}^{(\alpha)}(w_2 x) T_{m-j,\chi,q^{w_2}}^{(\alpha)}(w_1 d - 1),
\]

where \( E_{j,\chi,q^{w}}^{(\alpha)}(x) \) and \( T_{m,\chi,q^{w}}^{(\alpha)}(k) \) denote generalized q-Euler polynomials with weak weight \( \alpha \) and p-adic functional, respectively.

By Theorem 1.3, we have the following corollary.

Corollary 2.4 Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we obtain

\[
[2]_{q^{w_1}} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^{j-k} E_{j,\chi,q^{w_2}}^{(\alpha)} T_{m-j,\chi,q^{w_1}}^{(\alpha)}(w_2 d - 1)
\]

\[
= [2]_{q^{w_2}} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{j} w_2^{m-k} x^{j-k} E_{j,\chi,q^{w_1}}^{(\alpha)} T_{m-j,\chi,q^{w_2}}^{(\alpha)}(w_1 d - 1).
\]

Now we will derive another interesting identities for generalized q-Euler polynomials with weak weight \( \alpha \) using the symmetric property of \( S(w_1, w_2) \).

\[
S(w_1, w_2) = \left[ \frac{2}{2} \right]_{q^{\alpha}} \sum_{j=0}^{w_1 d - 1} (-1)^j \chi(j) q^{w_2 \alpha j} \int_X \chi(x_1) e^{x_1 + w_2 x + j \frac{w_2}{w_1}} (w_1 t) \, d\mu_{-q^{w_1 \alpha}}(x_1)
\]

\[
= \sum_{n=0}^{\infty} \left( \left[ \frac{2}{2} \right]_{q^{\alpha}} \sum_{j=0}^{w_1 d - 1} (-1)^j \chi(j) q^{w_2 \alpha j} E_{n,\chi,q^{w_1}}^{(\alpha)} \left( w_2 x + j \frac{w_2}{w_1} \right) w_1^n \right) \frac{t^n}{n!}
\]  \hspace{1cm} (2.12)

By using the symmetry property in (2.12), we also have

\[
S(w_1, w_2) = \left[ \frac{2}{2} \right]_{q^{\alpha}} \sum_{j=0}^{w_2 d - 1} (-1)^j \chi(j) q^{w_1 \alpha j} \int_X \chi(x_2) e^{x_2 + w_1 x + j \frac{w_1}{w_2}} (w_2 t) \, d\mu_{-q^{w_2 \alpha}}(x_2)
\]

\[
= \sum_{n=0}^{\infty} \left( \left[ \frac{2}{2} \right]_{q^{\alpha}} \sum_{j=0}^{w_2 d - 1} (-1)^j \chi(j) q^{w_1 \alpha j} E_{n,\chi,q^{w_2}}^{(\alpha)} \left( w_1 x + j \frac{w_1}{w_2} \right) w_2^n \right) \frac{t^n}{n!}
\]  \hspace{1cm} (2.13)

By comparing coefficients \( \frac{t^n}{n!} \) in the both sides of (2.12) and (2.13), we have the following theorem.
Theorem 2.5 Let $w_1$ and $w_2$ be odd positive integers. Then we obtain

\[ [2]_{q^{w_2}}^{w_1} \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) q^{w_2 \alpha j} E_{n, \chi, q^{w_1}}^{(\alpha)} \left( w_2 x + j \frac{w_2}{w_1} \right) w_1^n \]

\[ = [2]_{q^{w_1}}^{w_2} \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 \alpha j} E_{n, \chi, q^{w_2}}^{(\alpha)} \left( w_1 x + j \frac{w_1}{w_2} \right) w_2^n. \]

(2.14)

If we take $x = 0$ in Theorem 2.5, we also derive the interesting identity for generalized $q$-Euler numbers with weak weight $\alpha$ as follows:

Corollary 2.6 Let $w_1$ and $w_2$ be odd positive integers. Then we obtain

\[ [2]_{q^{w_2}}^{w_1} \sum_{j=0}^{w_1 d-1} (-1)^j \chi(j) q^{w_2 \alpha j} E_{n, \chi, q^{w_1}}^{(\alpha)} \left( j \frac{w_2}{w_1} \right) w_1^n \]

\[ = [2]_{q^{w_1}}^{w_2} \sum_{j=0}^{w_2 d-1} (-1)^j \chi(j) q^{w_1 \alpha j} E_{n, \chi, q^{w_2}}^{(\alpha)} \left( j \frac{w_1}{w_2} \right) w_2^n. \]

References


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