Almost Linear Jordan Derivations on $C^*$–Algebras

Ick-Soon Chang
Department of Mathematics, Mokwon University
Mokwon Gil 21, Seo-gu, Daejeon, 302-318, Korea
ischang@mokwon.ac.kr

Madjid Eshaghi Gordji
Department of Mathematics, Semnan University
P.O. Box 35195-363, Semnan, Iran
madjid.eshaghi@gmail.com

Abbas Javadian
Department of Physics, Semnan University
P.O. Box 35195-363, Semnan, Iran
abasjavadian@gmail.com

Hark-Mahn Kim
Department of Mathematics
Chungnam National University
79 Daehangno, Yuseong-gu
Daejeon 305-764, Korea
hmkim@cnu.ac.kr

This work was supported by Basic Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (No. 2012R1A1A2008139 and No. 2012-0002410)
Abstract. In this paper, we investigate almost linear Jordan derivations on $C^*$–algebras associated to the generalized Cauchy–Jensen functional inequality. As results, we apply the almost linear Jordan derivations on $C^*$–algebras to contractive, weakly amenable or amenable Banach modules.

Mathematics Subject Classification: 39B72, 46H25, 46B06

Keywords: Cauchy-Jensen functional inequality, $C^*$-algebras, Jordan derivations, Banach $A$-module, weakly amenable Banach modules

1. Introduction

Let $A$ be a Banach algebra and let $X$ be a Banach $A$–module. A linear mapping $d : A \to X$ is a Jordan derivation if $d(ab + ba) = d(a)b + ad(b) + bd(a) + d(b)a$ for all $a, b \in A$. A linear mapping $d : A \to X$ is a derivation if $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$.

The dual space $X^*$ is a Banach $A$-module if for every $a \in A$, $x \in X$ and $x^* \in X^*$ we define

$$
\langle x, ax^* \rangle = \langle xa, x^* \rangle, \quad \langle x, x^* a \rangle = \langle ax, x^* \rangle.
$$

For $x \in X$, we define $\delta_x : A \to X$ by $\delta_x(a) = a \cdot x - x \cdot a := [a, x]$, $(a \in A)$, then $\delta_x$ is a derivation from $A$ into $X$. Derivations of this form are called inner derivations. We note that a Banach algebra $A$ is contractible if every continuous derivation from $A$ into $X$ is inner for all Banach $A$–module $X$. A Banach algebra $A$ is amenable if every continuous derivation from $A$ into $X^*$ is inner for all Banach $A$–module $X$. A Banach algebra $A$ is weakly amenable if every continuous derivation from $A$ into $A^*$ is inner (see [5]).

The stability problem of functional equations originated from a question of Ulam [26] concerning the stability of group homomorphisms. We are given a group $G_1$ and a metric group $G_2$ with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$? In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it.

First of all, Hyers [11] considered the case of approximately additive mappings between Banach spaces. The method which was provided by Hyers, and which produces the additive mapping $h$, was called a direct method. This method is the most important and most powerful tool for studying the stability of various functional equations. Hyers’ Theorem was generalized by Aoki [1] and Bourgin [3] for additive mappings by considering an unbounded Cauchy
Almost linear Jordan derivations on $C^*$–algebras

In 1978, Rassias [21] also provided a generalization of Hyers Theorem for linear mappings which allows the Cauchy difference to be unbounded like this $\|x\|^p + \|y\|^p$. A generalized result of Rassias’ theorem was obtained by Gavruta in [9] and Jung in [14]. In 1990, Rassias [22] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [8] following the same approach as in [21], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [8], as well as by Rassias and Šemrl [23], that one cannot prove a Rassias type theorem when $p = 1$. The counterexamples of Gajda [8], as well as of Rassias and Šemrl [23], have stimulated several mathematicians to invent new approximately additive or approximately linear mappings.

We generalize the functional inequality to the following generalized Cauchy–Jensen functional inequality

$$\left\| \sum_{i=1}^{l} f(x_i) + m \sum_{j=1}^{n} f(y_j) \right\| \leq \left\| mf\left(\frac{\sum_{i=1}^{l} x_i}{m} + \sum_{i=1}^{n} y_j\right)\right\|,$$

where $l \geq 2, m \geq 1, n \geq 0$ are integers and $\sum_{i=1}^{n} E(i) := 0$ by notational convenience. Now, it is easy to see that if a mapping $f$ with $f(0) = 0$ satisfies the generalized Cauchy–Jensen inequality (1.1), then $f$ is additive.

In this paper, we investigate the almost linear Jordan derivations on $C^*$–algebras. Moreover, we apply the main results of paper to investigate the Jordan derivations on nuclear $C^*$–algebras and Jordan derivations from $C^*$–algebras into its dual spaces.

2. Main results

From now on, we suppose that $A$ is a $C^*$–algebra and $X$ is a Banach $A$–module. Moreover, we assume that $n_0 \in \mathbb{N}$ is a positive integer and suppose that $\mathbb{T}_{\frac{1}{n_0}} := \{e^{i\theta} : 0 \leq \theta \leq \frac{2\pi}{n_0}\}$.

In this section, we establish linear derivations from $C^*$–algebras to Banach $A$–modules. We start our work with the lemma 2.1 and the next lemma [6]. Quite recently, Peralta and Russo [20] proved the following lemma that generalized the result of Johnson (see for example, [12, 13]).

**Lemma 2.1.** Suppose $A$ is a $C^*$–algebra and $X$ is a Banach $A$–module. Then each Jordan derivation $d : A \to X$ is a derivation.

**Lemma 2.2.** Assume that a mapping $f : A \to B$ is additive and for each fixed $x \in A$, $f(tx) = tf(x)$ for all $t \in \mathbb{T}_{\frac{1}{n_0}}$. Then $f$ is $C$-linear.

We first introduce our main theorem on linear derivations from $C^*$–algebras to Banach $A$–modules.
\textbf{Theorem 2.3.} Let $f : A \to X$ be a mapping with $f(0) = 0$ and
\begin{equation}
\|f(ab + ba) - f(a)b - af(b) - f(b)a - bf(a)\| \leq \varphi(a, b, 0, 0, \ldots, 0) \tag{2.1}
\end{equation}
for all $a, b \in A$. Let $f$ satisfies the functional inequality
\begin{equation}
\left\| \sum_{i=1}^{l} f(x_i) + m \sum_{j=1}^{n} f(y_j) + f(tx) - tf(x) \right\| \leq \left\| mf \left( \sum_{i=1}^{l} x_i/m + \sum_{j=1}^{n} y_j \right) \right\| \tag{2.2}
\end{equation}
for all $x_1, \ldots, x_l, y_1, \ldots, y_n, x \in A$ and all $t \in \mathbb{T}^{n \to 2}$, and there exists a constant $L$ with $0 < L < 1$ for which the perturbing function $\varphi : A^{l+n+1} \to \mathbb{R}^+$ satisfies
\begin{equation}
\varphi \left( \frac{l}{mn} (x_1, \ldots, x_l, y_1, \ldots, y_n, x) \right) \leq L \cdot \frac{l}{mn} \varphi(x_1, \ldots, x_l, y_1, \ldots, y_n, x) \tag{2.3}
\end{equation}
for all $x_1, \ldots, x_l, y_1, \ldots, y_n, x \in A$. Then there exists a unique linear derivation $d_1 : A \to X$, defined as $d_1(x) = \lim_{k \to \infty} (\frac{mn}{l})^k f((\frac{l}{mn})^k x), (x \in A)$ such that
\begin{equation}
\|f(x) - d_1(x)\| \leq \frac{1}{1-L} \Psi(x) \tag{2.4}
\end{equation}
for all $x \in A$, where
\begin{align*}
\Psi(x) & := \frac{1}{K} \varphi \left( \frac{x-x, \ldots, x-x, 0, \ldots, 0, x-x, \ldots, x-x, 0, \ldots, 0}{2^{\left[\frac{l}{2}\right]}} \right) \\
& \quad + \frac{1}{l} \varphi \left( \frac{-x, \ldots, -x, \frac{l}{mn}x, \ldots, \frac{l}{mn}x, 0}{n} \right), (x \in A),
\end{align*}
and $K := \left[\frac{l}{2}\right] + m \left[\frac{n}{2}\right]$ and $[\cdot]$ denotes Gaussian notation.

\textbf{Proof.} Putting
\begin{align*}
(x_1, \ldots, x_l) & := \left( \frac{x-x, \ldots, x-x, 0, \ldots, 0}{2^{\left[\frac{l}{2}\right]}} \right) \\
\text{and} \\
(y_1, \ldots, y_n, x) & := \left( \frac{x-x, \ldots, x-x, 0, \ldots, 0}{2^{\left[\frac{n}{2}\right]}} \right)
\end{align*}
in (2.2), we have an approximate odd condition
\begin{equation}
\|f(x) + f(-x)\| \leq \frac{1}{K} \varphi \left( \frac{x-x, \ldots, x-x, 0, \ldots, 0, x-x, \ldots, x-x, 0, \ldots, 0}{2^{\left[\frac{l}{2}\right]}} \right) \tag{2.5}
\end{equation}
for all $x \in A$, where $K := [\frac{l}{2}] + m[\frac{n}{2}]$ and $[\cdot]$ denotes Gaussian notation. Replacing
\[(x_1, \ldots, x_l, y_1, \ldots, y_n, 0) := (-x, \ldots, -x, \underbrace{\frac{l}{mn} x, \ldots, \frac{l}{mn} x}_l, 0)\]
in (2.2), we lead to
\[\|lf(-x) + mnf\left(\frac{l}{mn} x\right)\| \leq \varphi(-x, \ldots, -x, \underbrace{\frac{l}{mn} x, \ldots, \frac{l}{mn} x}_l, 0)\]
for all $x \in A$. Associating (2.5) with (2.6) yields
\[\|f(x) - \frac{mn}{l} f\left(\frac{l}{mn} x\right)\| \leq \Psi(x)\] (2.7)
for all $x \in A$. Thus, it follows from (2.7) that for all nonnegative integers $k$ and $j$ with $j > k \geq 0$ and $x \in A$
\[\left\| \left(\frac{mn}{l}\right)^k f\left(\frac{l}{mn} x\right) - \left(\frac{mn}{l}\right)^{k+j} f\left(\frac{l}{mn} x\right) \right\| \leq \sum_{i=k}^{k+j-1} \left\| \left(\frac{mn}{l}\right)^i f\left(\frac{l}{mn} x\right) - \left(\frac{mn}{l}\right)^{i+1} f\left(\frac{l}{mn} x\right) \right\| \leq \sum_{i=k}^{k+j-1} \left(\frac{mn}{l}\right)^i \left(\frac{l}{mn}\right)^i \Psi\left(\frac{l}{mn} x\right) \leq \sum_{i=k}^{k+j-1} L^i \Psi(x),\]
which tends to zero as $k \to \infty$. Hence the sequence $\left\{ \left(\frac{mn}{l}\right)^k f\left(\frac{l}{mn} x\right) \right\}$ is Cauchy for all $x \in A$, and so we can define a function $d_1 : A \to X$ by
\[d_1(x) = \lim_{k \to \infty} \left(\frac{mn}{l}\right)^k f\left(\frac{l}{mn} x\right), \quad x \in A.\]
Moreover, letting $k = 0$ and $j \to \infty$ in the last inequality yields
\[\|f(x) - d_1(x)\| \leq \frac{1}{1-L} \Psi(x)\] (2.8)
for all $x \in A$, which yields the estimation (2.4).

Next, let $d_1' : A \to X$ be another additive mapping satisfying the inequality (2.8). Then it is obvious that $d_1'(\left(\frac{l}{mn}\right)^k x) = (\frac{mn}{l})^k d_1'(x)$ and $d_1\left(\left(\frac{l}{mn}\right)^k x\right) = (\frac{mn}{l})^k d_1(x)$ for all $k \in \mathbb{N}$ and all $x \in A$. Thus, we have
\[\|d_1(x) - d_1'(x)\| = (\frac{mn}{l})^k \|d_1\left(\left(\frac{l}{mn}\right)^k x\right) - d_1'\left(\left(\frac{l}{mn}\right)^k x\right)\| \leq (\frac{mn}{l})^k \left\{ \|d_1\left(\left(\frac{l}{mn}\right)^k x\right) - f\left(\left(\frac{l}{mn}\right)^k x\right)\| + \|f\left(\left(\frac{l}{mn}\right)^k x\right) - d_1'\left(\left(\frac{l}{mn}\right)^k x\right)\| \right\} \leq \frac{2}{1-L} (\frac{mn}{l})^k \Psi\left(\left(\frac{l}{mn}\right)^k x\right) \leq \frac{2}{1-L} L^k \Psi(x)\]
for all \( k \in \mathbb{N} \) and all \( x \in A \). Taking the limit as \( k \to \infty \), we lead to the uniqueness of the mapping \( d_1 \) near \( f \) satisfying the inequality (2.8).

It follows from (2.2) and (2.3) that

\[
(\frac{mn}{l})^k \left\| \frac{1}{l} \sum_{i=1}^{l} f \left( (\frac{l}{mn})^k x_i \right) + m \sum_{j=1}^{n} f \left( (\frac{l}{mn})^k y_j \right) \right\| \\
\leq (\frac{mn}{l})^k \left\| m f \left( \frac{l}{m} \sum_{i=1}^{l} (\frac{l}{mn})^k x_i + \sum_{j=1}^{n} (\frac{l}{mn})^k y_j \right) \right\| \\
+ L^k \varphi(x_1, \ldots, x_l, y_1, \ldots, y_n, 0)
\]

for all \( k \in \mathbb{N} \) and all \( x_1, \ldots, x_l, y_1, \ldots, y_n \in A \). Taking \( k \to \infty \) in the last relation, we see that

\[
\left\| \frac{1}{l} \sum_{i=1}^{l} d_1(x_i) + m \sum_{j=1}^{n} d_1(y_j) \right\| \leq \left\| m d_1 \left( \frac{\sum_{i=1}^{l} x_i}{m} + \sum_{j=1}^{n} y_j \right) \right\|
\]

for all \( x_1, \ldots, x_l, y_1, \ldots, y_n \in A \). This implies that the mapping \( d_1 \) is additive.

Putting \( x_1, \ldots, x_l, y_1, \ldots, y_n := 0 \) in (2.2), then we have

\[
\left\| f(tx) - tf(x) \right\| \leq 0(0, 0, \ldots, 0, x)
\]

for all \( x \in A \) and all \( t \in T_{\frac{1}{l0}}^{l} \). By definition of \( d_1 \) and (2.3), we have

\[
\left\| d_1(tx) - td_1(x) \right\| = \lim_{k \to \infty} \left\| (\frac{mn}{l})^k f \left( \left( \frac{l}{mn} \right)^k tx \right) - t(\frac{mn}{l})^k f \left( \left( \frac{l}{mn} \right)^k x \right) \right\|
\]

\[
\leq \lim_{k \to \infty} L^k \varphi(0, 0, \ldots, 0, x) = 0
\]

for all \( x \in A \) and all \( t \in T_{\frac{1}{l0}}^{l} \). This means that

\[
d_1(tx) = td_1(x)
\]

for all \( x \in A \) and all \( t \in T_{\frac{1}{l0}}^{l} \). Thus, the additive mapping \( d_1 \) is \( C \)-linear by Lemma 2.1.

On the other hand, we have observe by using (2.1) and (2.3) that

\[
\left\| d_1(ab + ba) - d_1(a)b - ad_1(b) - d_1(b)a - bd_1(a) \right\|
\]

\[
= \lim_{k \to \infty} \left\| (\frac{mn}{l})^k \left( f \left( \left( \frac{l}{mn} \right)^k (ab + ba) \right) - f \left( \left( \frac{l}{mn} \right)^k ab \right) - a f \left( \left( \frac{l}{mn} \right)^k b \right) - f \left( \left( \frac{l}{mn} \right)^k ba \right) - b f \left( \left( \frac{l}{mn} \right)^k a \right) \right) \right\|
\]

\[
\leq \lim_{k \to \infty} L^k \varphi(a, b, 0, 0, \ldots, 0) = 0
\]

for all \( a, b \in A \). Therefore, it follows that \( d_1 \) is a Jordan derivation. Then by Lemma 2.1, \( d_1 \) is a derivation. \qed
Remark 2.4. Suppose that a mapping \( f : A \to X \) with \( f(0) = 0 \) satisfies the functional inequalities (2.1), (2.2) for which the control function \( \varphi : A^{l+n} \to \mathbb{R}^+ \) satisfies
\[
\sum_{i=0}^{\infty} \left( \frac{mn}{l} \right)^i \varphi \left( \left( \frac{1}{mn} \right)^i (x_1, \ldots, x_l, y_1, \ldots, y_n, x) \right) < \infty
\]
for all \( x_1, \ldots, x_l, y_1, \ldots, y_n, x \in A \) instead of the condition (2.3). Then it follows from the similar argument to Theorem 2.3 that there exists a unique linear derivation \( d_1 : A \to X \), defined as \( d_1(x) = \lim_{k \to \infty} \left( \frac{1}{mn} \right)^k f((\frac{1}{mn})^k x), (x \in A) \) such that
\[
\| f(x) - d_1(x) \| \leq \sum_{i=0}^{\infty} \left( \frac{mn}{l} \right)^i \Psi \left( \left( \frac{1}{mn} \right)^i x \right)
\]
for all \( x \in A \), where \( \Psi \) is defined as in Theorem 2.3.

Theorem 2.5. Suppose that a mapping \( f : A \to X \) with \( f(0) = 0 \) satisfies the functional inequalities (2.1), (2.2) and there exists a constant \( L \) with \( 0 < L < 1 \) for which the perturbing function \( \varphi : A^{l+n} \to \mathbb{R}^+ \) satisfies
\[
\varphi \left( \frac{mn}{l} (x_1, \ldots, x_l, y_1, \ldots, y_n, x) \right) \leq L \cdot \frac{mn}{l} \varphi(x_1, \ldots, x_l, y_1, \ldots, y_n, x) \quad (2.9)
\]
for all \( x_1, \ldots, x_l, y_1, \ldots, y_n, x \in A \). Then there exists a unique linear derivation \( d_2 : A \to X \), defined as \( d_2(x) = \lim_{k \to \infty} \left( \frac{1}{mn} \right)^k f((\frac{1}{mn})^k x), (x \in A) \) such that
\[
\| f(x) - d_2(x) \| \leq \frac{L}{1 - L} \Psi(x) \quad (2.10)
\]
for all \( x \in A \), where \( \Psi \) is given as in Theorem 2.3.

Proof. It follows from the inequality (2.7) that
\[
\left\| \left( \frac{1}{mn} \right)^k f \left( \left( \frac{1}{mn} \right)^k x \right) - \left( \frac{1}{mn} \right)^{k+j} f \left( \left( \frac{1}{mn} \right)^{k+j} x \right) \right\|
\leq \sum_{i=k}^{k+j-1} \left( \frac{mn}{l} \right)^{i+1} \Psi \left( \left( \frac{mn}{l} \right)^{i+1} x \right) \leq \sum_{i=k}^{k+j-1} L^{i+1} \Psi(x),
\]
which tends to zero as \( k \to \infty \).

The remaining proof is similar to the corresponding proof of Theorem 2.3.

Remark 2.6. Suppose that a mapping \( f : A \to X \) with \( f(0) = 0 \) satisfies the functional inequalities (2.1), (2.2) for which the perturbing function \( \varphi : A^{l+n} \to \mathbb{R}^+ \) satisfies
\[
\sum_{i=0}^{\infty} \left( \frac{1}{mn} \right)^i \varphi \left( \left( \frac{mn}{l} \right)^i (x_1, \ldots, x_l, y_1, \ldots, y_n, x) \right) < \infty
\]
for all \(x_1, \ldots, x_k, y_1, \ldots, y_n, x \in A\) instead of the condition (2.9). Then it follows from the similar argument to Theorem 2.5 that there exists a unique linear derivation \(d_2 : A \to X\), defined as \(d_2(x) = \lim_{k \to \infty} \left( \frac{1}{mn} \right)^k f((\frac{mn}{l})^k x), (x \in A)\) such that

\[
\|f(x) - d_2(x)\| \leq \sum_{i=0}^{\infty} \left( \frac{l}{mn} \right)^{i+1} \Psi \left( \left( \frac{mn}{l} \right)^{i+1} x \right)
\]

for all \(x \in A\), where \(\Psi\) is defined as in Theorem 2.3.

**Corollary 2.7.** Let \(0 < p \neq 1, l \neq mn\) and \(\theta > 0\). If a mapping \(f : A \to X\) with \(f(0) = 0\) satisfies the following functional inequalities

\[
\left\| f(ab + ba) - af(b) - f(a)b - b(f(a) - f(b)a) \right\| \leq \theta \left( \|a\|^p + \|b\|^p \right),
\]

\[
\left\| \sum_{i=1}^{l} f(x_i) + m \sum_{j=1}^{n} f(y_j) + f(tx) - tf(x) \right\| \leq \left\| m f \left( \sum_{i=1}^{l} \frac{x_i}{m} \right) + \sum_{j=1}^{n} y_j \right\| + \theta \left( \sum_{i=1}^{l} \|x_i\|^p + \sum_{j=1}^{n} \|y_j\|^p + \|x\|^p \right)
\]

for all \(a, b, x_1, \ldots, x_l, y_1, \ldots, y_n, x \in A\) and all \(t \in T_1\), then there exists a unique linear derivation \(D : A \to X\), defined as

\[
D(x) = \begin{cases} 
\lim_{k \to \infty} \left( \frac{mn}{l} \right)^k f((\frac{mn}{l})^k x), (x \in X), & \text{if } l < mn, p > 1, (or \ l > mn, 0 < p < 1); \\
\lim_{k \to \infty} \left( \frac{mn}{l} \right)^k f((\frac{mn}{l})^k x), (x \in X), & \text{if } l < mn, 0 < p < 1, (or \ l > mn, p > 1),
\end{cases}
\]

such that

\[
\|f(x) - D(x)\| \leq \begin{cases} 
\frac{(mn)^{p-1}}{(mn)^{p-1}} \left( \frac{2^{\frac{1}{p}} + 2^{\frac{1}{p}}}{\sqrt[2]{\frac{1}{p} + m l \frac{1}{2}}} \right) + 1 + \frac{\theta}{\left( \frac{mn}{l} \right)^{p-1}} \|x\|^p, & \text{if } l < mn, p > 1, \ (or \ l > mn, 0 < p < 1); \\
\frac{(mn)^{p-1}}{p-1} \left( \frac{2^{\frac{1}{p}} + 2^{\frac{1}{p}}}{\sqrt[2]{\frac{1}{p} + m l \frac{1}{2}}} \right) + 1 + \frac{\theta}{\left( \frac{mn}{l} \right)^{p-1}} \|x\|^p, & \text{if } l < mn, 0 < p < 1, \ (or \ l > mn, p > 1),
\end{cases}
\]

for all \(x \in A\).

**Corollary 2.8.** Let \(l \neq mn\) and \(\theta > 0\). If a mapping \(f : A \to X\) with \(f(0) = 0\) satisfies the following functional inequalities

\[
\left\| f(ab + ba) - af(b) - f(a)b - b(f(a) - f(b)a) \right\| \leq \theta,
\]

\[
\left\| \sum_{i=1}^{l} f(x_i) + m \sum_{j=1}^{n} f(y_j) + f(tx) - tf(x) \right\| \leq \left\| m f \left( \sum_{i=1}^{l} \frac{x_i}{m} \right) + \sum_{j=1}^{n} y_j \right\| + \theta
\]

...
Almost linear Jordan derivations on $C^\ast$–algebras

for all $a, b, x_1, \ldots, x_l, y_1, \ldots, y_n, x \in X$ and all $t \in \mathbb{T}^1_{\frac{1}{\infty}}$, then there exists a unique linear derivation $D : A \to X$, defined as

$$D(x) = \begin{cases} 
\lim_{k \to \infty} (\frac{mn}{l})^k f((\frac{1}{mn})^k x), (x \in X), & \text{if } l > mn; \\
\lim_{k \to \infty} (\frac{mn}{n})^k f((\frac{1}{mn})^k x), (x \in X), & \text{if } l < mn,
\end{cases}$$

such that

$$\|f(x) - D(x)\| \leq \begin{cases} 
\frac{1}{l \cdot mn} (\frac{1}{K} + \frac{1}{T})\theta, & \text{if } l > mn; \\
\frac{1}{mn} (\frac{1}{K} + \frac{1}{T})\theta, & \text{if } l < mn,
\end{cases}$$

for all $x \in A$, where $K := \lceil \frac{1}{2} \rceil + m\lceil \frac{a}{2} \rceil$.

3. Applications

Now, in this part we apply the main theorem to investigate almost linear Jordan derivations from $C^\ast$–algebras to contractive, weakly amenable or amenable Banach $A$–modules.

**Theorem 3.1.** Let $A$ be a finite dimensional $C^\ast$–algebra and let $f : A \to X$ and $\varphi : A^{l+n+1} \to \mathbb{R}^+$ be mappings satisfying conditions of Theorem 2.3. If

$$\sup\{\|f(x) + \Psi(x)\| : \|x\| \leq 1\} < \infty,$$  
(3.1)

then there exist an $x_0 \in X$ and an inner linear derivation defined as $a \to [a, x_0], a \in A$ such that

$$\|f(a) - [a, x_0]\| \leq \frac{1}{1 - L}\Psi(a)$$

for all $a \in A$, where $\Psi$ is defined as in Theorem 2.3.

**Proof.** By Theorem 2.3, there exists a linear derivation $d_1 : A \to X$ satisfying (2.4). Then by (3.1), $d_1$ is bounded, and hence $d_1$ is continuous. On the other hand, we know that every finite dimensional $C^\ast$–algebra is contractible (see [5]). Then every continuous derivation from $A$ into $X$ is inner. Thus, it follows that $d_1$ is inner derivation, and so there exists $x_0 \in X$ such that $d_1(a) = ax_0 - x_0a$ for all $a \in A$. \hfill \Box

Note that a Banach $A$–module $X$ is symmetric if $ax = xa$ for all $a \in A, x \in X$.

**Corollary 3.2.** Let $A$ be finite dimensional and $X$ be a symmetric $A$–module. Let $f : A \to X$ and $\varphi : A^{l+n+1} \to \mathbb{R}^+$ be mappings which satisfy (3.1) and conditions in Theorem 2.3. Then

$$\lim_{k \to \infty} (\frac{mn}{l})^k f((\frac{1}{mn})^k a) = 0$$

and

$$\|f(a)\| \leq \frac{1}{1 - L}\Psi(a)$$  
(3.2)
for all \( a \in A \), where \( \Psi \) is defined as in Theorem 2.3.

**Theorem 3.3.** Let \( f : A \to A^* \) and \( \varphi : A^{l+n+1} \to \mathbb{R}^+ \) be mappings which satisfy (3.1) and conditions in Theorem 2.3. Then there exist an \( a' \in A^* \) and an inner linear derivation defined as \( a \to [a, a'], a \in A \) such that

\[
\|f(a) - [a, a']\| \leq \frac{1}{1 - L} \Psi(a)
\]

for all \( a \in A \), where \( \Psi \) is defined as in Theorem 2.3.

**Proof.** By Theorem 2.3, there exists a linear derivation \( d_1 : A \to A^* \) satisfying (2.4). Then by (3.1), \( d_1 \) is bounded. Hence \( d_1 \) is continuous. On the other hand, we know that every \( C^* \)-algebra is weakly amenable (see [5]). Then every continuous derivation from \( A \) into \( A^* \) is inner. It follows that \( d_1 \) is inner derivation. Then there exists \( a' \in A^* \) such that \( d_1(a) = aa' - a'a \) for all \( a \in A \).

**Theorem 3.4.** Let \( A \) be a nuclear \( C^* \)-algebra. Let \( f : A \to X^* \) and \( \varphi : A^{l+n+1} \to \mathbb{R}^+ \) be mappings which satisfy (3.1) and conditions in Theorem 2.3. Then there exist an \( x' \in X^* \) and an inner linear derivation defined as \( a \to [a, x'], a \in A \) such that

\[
\|f(a) - [a, x']\| \leq \frac{1}{1 - L} \Psi(a)
\]

for all \( a \in A \), where \( \Psi \) is defined as in Theorem 2.3.

**Proof.** We know that a \( C^* \)-algebra \( A \) is nuclear if and only if it is amenable (see [5]). Then for each Banach \( A \)-module \( X \), every derivation from \( A \) into dual of \( X \) is inner. The rest of proof is similar to the proof of previous Theorem.

The following is an immediate consequence of Theorem 2.3 and Sakai’s result [25] for almost linear Jordan derivations.

**Theorem 3.5.** Let \( A \) be a \( C^* \)-algebra with unit. Let \( f : A \to A \) and \( \varphi : A^{l+n+1} \to \mathbb{R}^+ \) be mappings which satisfy (3.1) and conditions in Theorem 2.3. Then there exist an \( x \in A \) and an inner linear derivation defined as \( a \to [a, x], a \in A \) such that

\[
\|f(a) - [a, x]\| \leq \frac{1}{1 - L} \Psi(a)
\]

for all \( a \in A \), where \( \Psi \) is defined as in Theorem 2.3.

**ACKNOWLEDGMENT**

This work was supported by Basic Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (No. 2012R1A1A2008139 and No. 2012-0002410). Corresponding author: hmkim@cnu.ac.kr; madjid.eshaghi@gmail.com.
Almost linear Jordan derivations on $C^*$-algebras

References


Received: October, 2012