Pseudo-Valuation Maps and Pseudo-Valuation Domains

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Abstract. Corresponding to a valuation map there exist a valuation domain and vice versa. In this article we established a pseudo-valuation map \( \omega \) from \( K^* \) to \( G^* \) (a partially ordered group equipped with \(*\) property), then constructed a pseudo-valuation domain. We confirmed our map \( \omega \) by using it while generating the few characteristics of Pseudo-valuation domains.

1. Introduction and Preliminaries

Let \( R \) be an integral domain with quotient field \( K \). A prime ideal \( P \) of \( R \) is called strongly prime if \( x, y \in K \) and \( xy \in P \) imply that \( x \in P \) or \( y \in P \) (alternatively \( P \) is strongly prime if and only if \( x^{-1}P \subset P \) whenever \( x \in K \setminus R \) \cite[Definition, page2]{5}. A domain \( R \) is called a pseudo-valuation domain if every prime ideal of \( R \) is a strongly prime \cite[Definition, page2]{5}. It was shown in Hedstrom and Houston \cite[Theorem 1.5(3)]{5} that an integral domain \( R \) is a pseudo-valuation domain if and only if for every nonzero \( x \in K \), either \( x \in R \) or \( ax^{-1} \in R \) for every nonunit \( a \in R \). Every valuation domain is a pseudo-valuation domain \cite[Proposition. 1.1]{5} but converse is not true for example the valuation domain \( V \) of the form \( K + M \), where \( K \) is a field and \( M \) is the maximal ideal of \( V \). If \( F \) is a proper subfield of \( K \), then \( R = F + M \) is a pseudo-valuation domain which is not a valuation domain. Whereas \( R \) and \( V \) have the same quotient field \( L \) and that \( M \) is the maximal ideal of \( R \) \cite[Theorem A, page 560]{8}. A quasi-local domain \((R, M)\) is a pseudo-valuation domain if and only
if \( x^{-1}M \subset M \) whenever \( x \in K \setminus R \) (cf. [5, Theorem 1.4]). Also a Noetherian pseudo-valuation domains have discussed in [5]. A Noetherian domain \( R \) with quotient field \( K \) is a pseudo-valuation domain if and only if \( x^{-1} \in R' \) whenever \( x \in K \setminus R \), where \( R' \) is the integral closure of \( R \) in \( K \) [5, Theorem 3.1]. In [5, Example 3.6] \( \mathbb{Z}[\sqrt{5}]_{(2,1+\sqrt{5})} \) is a Noetherian pseudo-valuation domain which is not a valuation domain and is not in the form of \( D + M \).

There are number of studies on pseudo-valuation domains through different point of views. In [4] the group of divisibility of pseudo-valuation domains has discussed on the basis of semivaluation map. Further [4] deals with the group of divisibility of quasi-local domains, for example see [4, Cor 3.5] and [4, Proposition 3.6].

As every pseudo-valuation domain is necessarily a quasi-local [5, Cor 1.3] and a quasi-local domain is pseudo-valuation domain if and only if its maximal ideal is strongly prime [5, Theorem 1.4]. We give much importance to the local property and strongly prime ideal of the pseudo-valuation domain while constructing the pseudo-value map. Ohm’s [4, Cor 3.5] bring us a clue about group of divisibility of pseudo-valuation domain.

Valuation maps and corresponding valuation domains have been studied in the literature. We established pseudo-value map \( \omega \) and through that map we found pseudo-value group, which is the group of divisibility of pseudo-valuation domains. We proved the characteristics of PVD through that map, while establishing \( \omega \) we ignored the \( GCD \)-behavior of valuation map. During construction of pseudo-value map we observed that \( \omega \) is the map lying between semi-value map and valuation map and consequently the pseudo-value group is lying between semi-value group and value group.

2. PSEUDO-VALUE MAP AND GROUP OF DIVISIBILITY OF PSEUDO-VALUE DOMAIN

2.1. Partially ordered group and \( \ast \) property. Let \( K \) be a field and \( D \) be a subring of \( K \) with identity, \( K^* = K \setminus \{0\} \), the multiplicative group be the group of units of \( K \) and \( U(D) \) represent the units of \( D \), which is subgroup of \( K^* \). \( G = K^*/U(D) \) the factor group with operation addition defined as

\[
xU + yU = xyU,
\]

we define for each \( xU, yU \in G \),

\[
xU \leq yU \text{ if and only if } y/x \in D.
\]

We note that the relation \( \leq \) is partially ordered relation. The positive subset with respect to relation \( \leq \) is

\[
G_+ = \{xU : xU \geq U\} = \{xU : x \in D\}.
\]

The set \( G_+ \subset G \) is a cone i.e a subset of \( G \) containing 0 and closed under addition.
Here we give few terminology from [7] which shall be helpful further.

**Definition 1.** If \( G \) is a partially ordered group, then a subset \( X \) of \( G \) is convex if \( X \) contains, for each pair of elements \( x, y \) of \( X \) with \( x \leq y \), each element of \( G \) between \( x \) and \( y \) [7, Page 197].

**Definition 2.** Let \( H \) be a subgroup of a partially ordered abelian group \( G \). \( H \) is an ordered ideal in \( G \) if for every \( x, y \in H \) and \( z \in G \) such that \( x \leq z \leq y \), it follows that \( z \in H \). The family of ordered ideals in a partially ordered abelian group is a complete lattice under the relation of inclusion [6, Cor 1.10].

**Definition 3.** [7, Page 233] If \( G \) is partially ordered abelian group, then an element \( b \geq 0 \) of \( G \) is bounded if there is an element \( g \) of \( G \) such that \( nb < g \) for each positive integer \( n \). Thus bounded subset of \( G \) form subsemigroup of \( G \) [7, Proposition 19.10].

By adding * property with a partially ordered group let we define below.

2.1.1. * Property: A partially ordered group \( G \) in which each \( g \geq 0 \), either \( g \geq 0 \) or \( g < h \) for all \( h \in G \) with \( h > 0 \).

A partially ordered group \( G \) with * property will be denoted by \( G^* \).

**Definition 4.** A partially ordered set \( X \) is said to be directed if every two elements have both an upper bound and lower bound. A partially ordered group \( G \) whose partial order is directed is called directed group [1, Page 2].

If a directed partially ordered group \( G \) having * property then we will see later that it become the group of divisibility of pseudo-valuation domain. The above * property has been discussed in [2, Proposition 5.1(b)] to prove that an integral domain a pseudo-valuation domain, but we organized here as a special case of group of divisibility while constructing pseudo-value map. Although the group of divisibility of pseudo-valuation domain has been discussed in the literature but we just considered its only one important characteristic and define pseudo-valuation map.

2.2. **Pseudo-valuation map.** Let \( G^* \) be a partially ordered group as defined above and \( K \) be a field then we have the mapping

**Definition 5.** Let \( \omega \) be a map from \( K^* \to G^* \), which has the following properties:

(a) \( \omega(xy) = \omega(x) + \omega(y) \).

(b) \( \omega(x - y) \geq \omega(t) \) for each "\( t \)" in \( K^* \), with such that \( \omega(t) \leq \omega(x) \) and \( \omega(t) \leq \omega(y) \),

(c) \( \omega(x) = g \geq 0 \) or \( \omega(x) < \omega(y) = h \), where \( g, h \in G \) and \( h > 0 \), where \( x, y \in K^* \).

The map \( \omega \) is an additive map if it satisfy

(d) \( \omega(x) < \omega(y) \) implies that \( \omega(x + y) = \omega(x) \) for all \( x, y \in K \).
In the above map each $\omega(x) \geq 0$ for all $x \in K \setminus \{0\}$ the map above is closely related to semi-valuation map. In the above definition condition (c) plays an important role, in which the $*$ property of $G^*$ has been observed. We also call $\omega$ the pseudo-valuation map.

No doubt (d) implies that it is a quasilocal domain. We shall prove that $D_\omega = \{x \in K : \omega(x) \geq 0\}$ a pseudo-valuation domain and $G^*$ is corresponding pseudo-value group of $D_\omega$. Through the map as in definition 5 a gcd property is exempted, we definitely say that it is non GCD-domain and fulfil quasilocal domains properties too.

**Example 1.** Let $K$ be any field and $(G^*, +) \cup \{1\}$ be any partially ordered group, define map

$$\omega : K^* \rightarrow (G^*, +) \cup \{1\}$$

$$\omega(x) = \begin{cases} 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Then it can easily be prove that $\omega$ is pseudo valuation.

**Remark 1.** Let $G$ be a partially ordered group and $X$ be a set of bounded elements of $G$; $X$ is convex subsemigroup of $G$. The subgroup $B(G)$ of $G$ generated by $X$ is a convex subgroup of $G$; if $G$ is lattice ordered, then $B(G)$ is a sublattice subgroup of $G$ [7, Proposition 19.10].

From remark1 corresponding to prime ideal in $D$ a convex set in $G^*$ which generate a convex subgroup of $G^*$, similarly for a strongly prime ideal $P$ of $D$.

**Remark 2.** Let $P$ be a strongly prime ideal in $D$ and $G^*$ be a group of divisibility of $D$, then there is one to one corresponding between strongly prime ideals and a subsets $X$ in $G^*$ which generate the convex subgroups $B(G)$ as in remark1. By the definition of strongly prime ideal i.e $P$ is strongly prime if and only if $x^{-1}P \subseteq P$ whenever $x \in K \setminus R$ we have convex set $C$, $x \in G \setminus C$ such that $-x + C \subseteq C$ and also generating a convex subgroup. We call such $C$ a strongly convex set.

**Remark 3.** We have from [7, Page 250] for a partially ordered group $H$, the ascending sequence

$$B_1(H) \subseteq B_2(H) \subseteq ...$$

of subgroups of $H$.

**Proposition 1.** Let $K$ be a field and $D$ be a subring of $K$ with identity also $G^*$ be the group of divisibility of $K$ with respect to $D$. Let $\omega$ be the map defined in (def 5) then $D_\omega$ is an integral domain.

**Proof.** Clearly $1 \in D_\omega$ and definition5(a) implies that $D_\omega$ is closed under multiplication, and if $x, y \in D_\omega$, then $\omega(x - y) \geq \omega(1) = 0$ since $\omega(x) \geq \omega(1)$ and $\omega(y) \geq \omega(1)$. Thus $D_\omega$ is a subring of $K$ with identity. The mapping $\omega$ is no doubt a group homomorphism and its kernel is $U = \{x \in K : \omega(x) = 0\}$, which shows $U$ is a group of units of $D_\omega$. So $D_\omega$ is an integral domain. \qed
If \( \omega \) satisfy additional property \((d)\) of the definition 5, then it is quasi-local domain.

After proving \( D_\omega \) is integral domain now we shall prove that it is pseudo-valuation domain. As we know that \( D_\omega \) is pseudo-valuation domain if each of its prime ideal is strongly prime, also \( D_\omega \) is pseudo-valuation domain if and only if for every nonzero \( x \in K \), either \( x \in D_\omega \) or \( ax^{-1} \in D_\omega \) for every nonunit \( a \in D_\omega \).

**Proposition 2.** \( D_\omega = \{ x \in K : \omega(x) \geq 0 \} \) is a pseudo-valuation domain.

**Proof.** Let us suppose that \( x \in K \) and \( x \notin D_\omega \) then \( \omega(x) < 0 \), so \( \omega(x^{-1}) = -\omega(x) > 0 \Rightarrow x^{-1} \in D_\omega \) and hence \( ax^{-1} \in D_\omega \) for every nonunit \( a \in D_\omega \), as \( D_\omega \) is an integral domain. On the other hand if \( x \in K \) then \( \omega(x) \geq 0 \) or \( \omega(x) < \omega(a) = h \), where \( g, h \in G \) and \( h > 0 \). If \( \omega(x) \geq 0 \), then we are done otherwise there exist \( \omega(a) > \omega(x) \) such that \( \omega(ax^{-1}) = \omega(a) - \omega(x) \geq 0 \Rightarrow ax^{-1} \in D_\omega \). Both the cases shows that \( D_\omega \) is pseudo-valuation domain. \( \square \)

Again let us try to prove \( D_\omega \) is pseudo-valuation domain by its characteristics as every pseudo-valuation domain is necessarily local.

**Proposition 3.** \( D_\omega \) is a local (non-noetherian) and a pseudo-valuation domain.

**Proof.** Through that map \( \omega \) we will first prove the necessarily condition of pseudo-valuation domain i.e local, and then prove the maximal ideal is strongly prime. Let us suppose that \( D_\omega \) has two maximal ideals \( M \) and \( N \) and corresponding to these ideal there are two convex subgroups \( C_1 \) and \( C_2 \) generated by the bounded sets \( X_1 \) and \( X_2 \), then choose \( x \in M \setminus N \) and \( y \in N \setminus M \) also \( x \in X_1 \setminus X_2 \) and \( y \in X_2 \setminus X_1 \). Since \( \omega \) is also a homomorphism hence \( X_1 \) correspond to \( M \) and \( X_2 \) correspond to \( N \). Let \( \omega(xy^{-1}) = g \) and \( \omega(y) = h > 0 \) then clearly \( g \notin 0 \) and \( g \notin h \), while \( h > 0 \) which contradict the \((e) \) of definition 5) i.e the * property, So \( D_\omega \) must be local. Let \( x \in K \setminus D_\omega \) and \( m \in M \subset D_\omega \) since \( \omega(x) \notin 0 \) because \( x \notin D_\omega \), then again \( \omega(x) < \omega(m) \) since \( \omega(m) > 0 \). Thus \( x^{-1}m \in M \) and so \( x^{-1}M \subset M \). Hence \( D_\omega \) is a pseudo-valuation domain. \( \square \)

So we may discuss the pseudo-valuation domain through map \( \omega \), as valuation domain has been studied through valuation map.

**Remark 4.** Clearly directed group \( G^* \) is the group of divisibility of pseudo-valuation domain \( D \) with quotient field \( K \). Hence every partially ordered group with * property is the group of divisibility of a \( PVD \) and \( G^* \) need not be a torsion free. In other words * property shows that \( A \) is a \( PVD \) if and only if its group of divisibility is a lexicographic extension of a trivially ordered group by a totally ordered group.

The above Remark is confirm by [2, Page 462].
If $D$ is $PVD$ then its maximal fractional ideals and maximal ideals are comparable i.e let $D$ be a local domain with maximal ideal $M$ and quotient field $K$ which is a $PVD$ and $M : M$ is a valuation overring of $D$ with maximal ideal $M$. Then we have lexicographically exact sequence which relates the group of divisibility of $D$ and group of divisibility of $V = M : M$ see [4, Page 577]. Now through map $\omega$ we describe the valuation overring of $PVD$'s.

**Proposition 4.** Let $D$ be a domain with quotient field $K$ with group of divisibility $G = K^*/U(D)$ and $\omega$ be the defined above, then the following are equivalent

(a) $D$ is local with $M : M$ is a valuation domain,
(b) $G$ is satisfying $*$ property.

**Proof.** (a) $\Rightarrow$ (b). Corresponding to maximal ideal $M$ in $D$ there is maximum convex set in $G$ which generate maximal convex subgroup. Let $\omega(x) = xU = g \neq 0 \in G$ and $h > 0$, if $g \not< h$, then for some $h \in G$ with $h > 0$, then there is $\omega(x^{-1}) = -xU \in G$ and $\omega(m) = c \in C$ (maximal convex subgroup) such that

$$\omega(x^{-1}m) = \omega(x^{-1}) + \omega(m) = -\omega(x) + \omega(m) \notin C.$$ 

$$\Rightarrow x^{-1}m \notin M \text{ thus } xM \subset M$$

Hence $g + c > 0$ for all $c \in G$ with $c > 0$, or from $g + c > 0$ we have $g > c$ for all $c \in G$ with $c < 0$.

(b) $\Rightarrow$ (a). $D$ is clearly local prove is similar to proposition3, the rest of proof if followed as (a) $\Rightarrow$ (b).

As it has been demonstrated in several ways that a valuation domain is a pseudo-valuation domain. In following we do it through newly established pseudo-valuation map but here we shall consider $G^*$ a totally ordered group instead of partially ordered.

**Remark 5.** It is quite clear that when we will talk about valuation domain then $G^*$ should be a totally ordered.

**Proposition 5.** Every valuation domain is a pseudo-valuation domain.

**Proof.** Let $V$ a valuation domain and $P$ be a prime ideal in $V$. Let us suppose that our defined group $G^*$ is extend to linear order then there is each convex subgroup of $G$ correspond to prime ideal in $V$. This reversible correspondence is such that if $H$ is convex in $G$ then $\omega^{-1}((G\uparrow H)U\{0\})$ is prime ideal in $V$ by [7, Page 198, 199]. Now as linearly ordered group enjoy the partial ordering properties thus we have to show that each prime ideal in $V$ is strongly prime. To prove $\omega^{-1}((G\uparrow H)U\{0\})$ is prime ideal is similar to that [3, Proposition 3.3]. To prove it is strongly prime let $P$ be a prime ideal in $V$. Suppose $xy \in P$ where $x, y \in K$ the quotient field of $V$. If both $x, y \in V$ then we are done by condition (1) of map $\omega$ in (definition 5). Suppose $y \notin V$, then there exist $y^{-1} \in V \subset K$, hence $x = yx.y^{-1} \in P \Rightarrow x \in P$. Now due to reverse correspondent and map $\omega$ from $K$ to group of divisibility and (definition 5) the result follows. $\square$
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References


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