A Numerical Resolution of a European Option Value
with a Stochastic Volatility

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Abstract

The aim of this work is to evaluate a European option with a stochastic volatility. For, we have a system of two stochastic differential equations (SDEs), where the first one describes the price of the underlying while the second one modelises the stochastic volatility. First we set the inconvenience of Black and Scholes model [1] and its limits, then we propose a model with a stochastic volatility. For this purpose, we use Garman partial differential equation (GPDE) to evaluate the option price where solution is approached by a finite difference method.

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1 Introduction

1.1 Black-Scholes Model

We consider a financial asset where the price is given by the stochastic differential equation

\[ dX_t = \mu X_t dt + \sigma X_t dW_t; \quad X_0 = x, \]

with

- \((X_t)\) is the price of the asset at time \(t\).
- \(\mu\) is the return that is to be supposed constant.
- \(W\) is a standard Brownian motion.
The underlying asset \((X_t)\) is a share or a stock index. In this model \(\sigma\) is the volatility supposed to be a strictly positive constant. We consider a riskless asset \(X^0\), with a value \(X^0_t = e^{rt}\) at time \(t\). We also suppose that the short term interest rate is constant and equal to \(r\).

Equation (1) has important consequences, mainly we can cite:

(i) The process \(X\) is a geometric Brownian motion and we have an explicit expression for \(X_t\)

\[
X_t = x \exp \left( \sigma W_t + \left( \mu - \frac{\sigma^2}{2} \right) t \right). \tag{2}
\]

(ii) The market is viable and complete, this is to say that there is a unique probability \(P^*\) for which the process of the updated prices \((e^{-rt}X_t)_{t \geq 0}\) of the risky asset is a \(P^*\) martingale, this probability is called neutral risk.

Any European option of a payoff \(H \in L^2(P^*, \mathcal{F}_T)\) i.e any option defined by a random variable \(H \in \mathcal{F}_T\) that is measurable and square integrable under the neutral risk probability \(P^*\), can be simulated.

(iii) There is a unique acceptable portfolio that is self-financed and under-estimated, containing both the riskless and the risky assets, with a value \(H\) at time \(T\).

For the particular case where \(H = h(X_t)\), with \(h\) a continuous and positive function, the price of the put can be given by

\[
P(t, X_t) = E^* \left[ e^{-r(T-t)} h \exp(\sigma(W^*_T - W^*_t) + \left( r - \frac{\sigma^2}{2} \right)(T-t)) \right], \tag{3}
\]

with \(P(t, x)\) a solution of the following partial differential equation (PDE) (see [5])

\[
\mathcal{L}_{BS}(\sigma)P = 0 \quad \forall x > 0 \quad P(t, x) = h(x), \tag{4}
\]

where

\[
\mathcal{L}_{BS}(\sigma) = \frac{\partial}{\partial t} + \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2} + \left( r \frac{\partial}{\partial x} - \cdot \right). \tag{5}
\]

The Black and Scholes model serves as a reference to all those who practice the financial market:

(i) It is a simple model, withere the prices \(X_t\) continuous.

(ii) It gives a closed formula for calls, puts and corresponding Greeks.

However, all the statistical tests do not validate the log-normal hypothesis for the underlying asset, therefore it is necessary to refine the model [8]. There are many ways to do it, for example we can authorize the price \(X_t\) to have jumps,
or depend on $t$ and $x$, the only source of noise remaining for the Brownian motion $W$; this is what was proposed by B. Dupire [3]. A natural way to expand Black and Scholes model is to authorize the volatility to be a stochastic process governed by a second noise, modelled by a second Brownian motion $\hat{Z}$ with the possibility to be correlated with $W$, but not perfectly correlated, contrary to the case of the model of B. Dupire [3]. In the same way we write

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t,$$

(7)

where $(\sigma_t)$ is a stochastic process. A naturel question how to choose it?.

In the following we try to answer this question. First, we require the volatility to be a quantity $F_t$- that is measurable and strictly positive; so we propose to write it in the following form:

$$\sigma_t = f(Y_t),$$

where $f : \mathbb{R} \to \mathbb{R}_+^*$ is a deterministic function and $Y$ is a real valued random process $F_t$-adapted. We limit ourselves to a Markovian process $Y$ satisfying the following stochastic differential equation

$$dY_t = \mu Y(t,Y_t) dt + \sigma Y(t,Y_t)d\hat{Z}_t.$$

(8)

1.2 The limits of Black and Scholes model

Empirical studies on the price return of the underlying asset have shown that the volatility has a stochastic behavior. A constant volatility doesn’t explain all the market phenomena, specially smile curves. It is a powerful modification which allows us to describe a more complex market than a Black and Scholes’s [1] and this is for the following reasons:

- We can reproduce more realistic laws for the return, in particular the tails of these distributions are more thicker than those of the lognormal laws.

- We can make these distributions asymmetric by correlating the noises $W$ et $\hat{Z}$.

- We can have the smile.

Nothing is free, especially in the market financial world, it is necessary to pay the price of these improvements. Among others, we cite:

- We cannot observe directly the volatility, so we estimate the model parameters and thus the current level of the volatility becomes a difficult problem.
The market so modelled is incomplete, this to say that when we treat an option, we cannot eliminate the risk by managing a portfolio containing some cash and the underlying. Indeed, the infinitesimal variation of the value of such a portfolio contains terms like $dW_t$ and $d\hat{Z}_t$ which vanish simultaneously.

Many studies showed that the volatility of the return is not constant and depends on the maturity $T$ and the strike $E$. The combination of the smile effect and the structure by the volatility term lead to a configuration in the form of a 2-dimensional matrix which is going to become the real input in the options evaluation formula. It has to be noted that the hypothesis of the normality of the return is not verified by the market. Hence the fact of considering a stochastic volatility is going to add major difficulties and complicate the model.

2 The PDE satisfying the price of the option

In this section, we propose a new model that takes in account a stochastic volatility. The main objective of these models is to reduce the biased estimation and to give the real value of the price option. We consider some specific criteria related to the dynamics of the state variables so that we set

$$\begin{cases}
    dS_t = \mu S_t dt + \sigma_t S_t dW_{t,S} \\
    \sigma_t = f(Z_t) \\
    dZ_t = p(t, S, Z_t) dt + q(t, S, Z_t) dW_{t,Z} \\
    dW_{t,S} dW_{t,Z} = \rho dt
\end{cases}$$

(9)

Hence we consider the partial differential equation, that models the value $V$ of a European option as

$$\frac{\partial V}{\partial t} + \frac{1}{2} Z^2 S^2 \frac{\partial^2 V}{\partial S^2} + \rho q Z S \frac{\partial^2 V}{\partial S \partial Z} + \frac{1}{2} q^2 Z^2 \frac{\partial^2 V}{\partial Z^2} + r S \frac{\partial V}{\partial S} + (p - q \lambda) \frac{\partial V}{\partial Z} - r V = 0.$$  

(10)

Where $\lambda$ is called the market price of the volatility risk and it is supposed to be constant. Equation (10) is the partial differential equation of an European put $V(S; Z; t)$ on an underlying where the instantaneous return has a stochastic volatility [7]. It is a linear equation, which is a particular case of the general equation developed by Garman et Al (1976)[6] et [4], Ross et Al (1985) [2]. Furthermore we can say that it is the starting point of all the models with stochastic volatility.

$$\left( \frac{\partial V}{\partial t} + \frac{1}{2} Z^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - r V \right) + \frac{\partial}{\partial Z} \left( \frac{1}{2} q^2 Z^2 \frac{\partial V}{\partial Z} + \rho q Z S \frac{\partial V}{\partial S} + (p - q \lambda) \frac{\partial V}{\partial Z} \right) = 0.$$  

(11)
We notice that, when the volatility is constant, the second term between the brackets takes a zero value and we find the equation of Black-Scholes.

Our article treats the case where the underlying follows the Heston model, with the volatility equal to $\sqrt{\sigma}$ and the option $V$ is given by

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + q \sigma S \frac{\partial^2 V}{\partial S \partial \sigma} + \frac{1}{2} q^2 \sigma^2 \frac{\partial^2 V}{\partial \sigma^2} + r S \frac{\partial V}{\partial S} + (p - q \lambda) \frac{\partial V}{\partial \sigma} - r V = 0 \quad (12)$$

**Proposition 2.1**
We suppose that $\mu S_t$ and $\sqrt{\sigma} S_t$ are locally Lipschitzian, $p$ and $q$ are constant therefore, the system

$$\begin{cases}
    dS_t = \mu S_t dt + \sqrt{\sigma} S_t dW_{t,S} \\
    d\sigma_t = p dt + q dW_{t,Z} \\
    dW_{t,S}.dW_{t,Z} = \rho dt
\end{cases}$$

has a unique solution (for details the reader is referred to [5] and [10]).

To eliminate the term of the mixed derivative equation (12), two cases appear, that is either to set the value of the correlation coefficient to zero ($\rho = 0$), which does not reflect the reality, or to make an adequate change of variables.

### 2.1 The new Form of the differential equation

Here we propose to make a change of variables in the following way:

We set: $X = X(S, \sigma, t)$ and $Y = Y(S, \sigma, t)$. To eliminate the term of the mixed derivatives in equation [12], we write:

$$X = \rho \ln S - \int \frac{\sqrt{\sigma}}{q} d\sigma, \quad Y = \sqrt{1 - \rho^2} \ln S.$$

If we consider the volatility of the volatility to be in the form $q = \eta \sigma^{\gamma - \frac{1}{2}}$, the variable $X$, will be:

$$X = \rho \ln S - \int \frac{\sqrt{\sigma}}{q} d\sigma = \rho \ln S - \frac{1}{\eta} \int \sigma^{-\gamma} d\sigma. \quad (13)$$

For cases appear; **Case 1:** $\gamma = \frac{3}{2}$, **Case 2:** $\gamma = \frac{5}{2}$, **Case 3:** $\gamma \neq \frac{3}{2}, \gamma \neq \frac{5}{2}$ and **Case 4:** $\gamma = \frac{1}{2}$. The fourth case is a particular case of the third, where the volatility of the volatility is constant; it has to be noted that the expressions of $S$ and $Y$, are the same whatever is the scenario.

$$S = S(X, Y) = \exp \left( \frac{Y}{\sqrt{1 - \rho^2}} \right), \quad (14)$$
\[ Y = Y(S, \sigma) = \sqrt{1 - \rho^2} \ln S. \quad (15) \]

The expressions of \( \sigma \) and \( X \) vary according to the relation between the volatility of the volatility and the volatility. The following table gives, the expressions for \( X(S, \sigma) \) and \( \sigma(X, Y) \) for the four above cases:

<table>
<thead>
<tr>
<th>Case</th>
<th>( q )</th>
<th>( X = X(S, \sigma) )</th>
<th>( \sigma = \sigma(X, Y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( q = \eta \sigma )</td>
<td>( X = \rho \ln S + \frac{2}{\eta \sqrt{\sigma}} )</td>
<td>( \sigma = \frac{2}{\eta} \left( X - \frac{\rho}{\sqrt{1 - \rho^2}} Y \right)^{-2} )</td>
</tr>
<tr>
<td>2</td>
<td>( q = \eta \sigma^2 )</td>
<td>( X = \rho \ln S + \frac{2 \rho}{3 \sigma \sqrt{\sigma}} )</td>
<td>( \sigma = \left( \frac{2}{3 \eta} \right)^{\frac{2}{3}} \left( X - \frac{\rho}{\sqrt{1 - \rho^2}} Y \right)^{-\frac{2}{3}} )</td>
</tr>
<tr>
<td>3</td>
<td>( q = \eta \sigma^{\gamma - \frac{1}{2}} ) ( \gamma \neq \frac{3}{2} ) et ( \gamma \neq \frac{5}{2} )</td>
<td>( X = \rho \ln S - \frac{1}{\eta} \left( \frac{\rho}{\sqrt{1 - \rho^2}} Y - X \right)^{\frac{1}{\gamma - \gamma}} )</td>
<td>( \sigma = \left( \frac{\eta}{\gamma - \gamma} \right)^{\frac{\gamma}{\gamma - \gamma}} \left( \frac{\rho}{\sqrt{1 - \rho^2}} Y - X \right)^{\frac{1}{\gamma - \gamma}} )</td>
</tr>
<tr>
<td>4</td>
<td>( q = \eta )</td>
<td>( X = \rho \ln S - \frac{2 \sqrt{\sigma}}{\eta} )</td>
<td>( \sigma = \frac{\eta^2}{4} \left( \frac{\rho}{\sqrt{1 - \rho^2}} Y - X \right)^{\frac{1}{4}} )</td>
</tr>
</tbody>
</table>

The term of the mixed derivative does not exist any more, even in the case when the correlation coefficient is useless. However the domain of \( V(X; Y) \) changes and this is a difficult task. Indeed, by making the change of variables for \( (S; \sigma) \) to be \( (X; Y) \), the domain is transformed.

### 3 The resolution of the differential equation

The Partial differential equation (12), with appropriate boundary conditions and where the market is supposed to be complete admits a unique analytical solution. However, the solution cannot be found explicitly, so we make use of a numerical scheme based on a finite difference method.

For, we consider the case where the volatility of the volatility is constant \( q = \eta \) with

\[ S = S(X, Y) = \exp \left( \frac{Y}{\sqrt{1 - \rho^2}} \right), \quad (16) \]

\[ \sigma = \frac{\eta^2}{4} \left( \frac{\rho}{\sqrt{1 - \rho^2}} Y - X \right)^{\frac{1}{2}} \quad (17) \]

\[ \frac{\partial V}{\partial t} + \frac{\rho}{2} (1 - \rho^2) \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) + \left( \rho r - (p - \lambda_\sigma q) \sqrt{\sigma} \right) \frac{\partial V}{\partial X} + r \sqrt{1 - \rho^2} \frac{\partial V}{\partial Y} - r V = 0, \quad (18) \]

and we put \( D = (p - \lambda_\sigma q) \) to get

\[ \frac{\partial V}{\partial t} + \frac{\rho}{2} (1 - \rho^2) \left( \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} \right) + \left( \rho r - D \sqrt{\sigma} \right) \frac{\partial V}{\partial X} + r \sqrt{1 - \rho^2} \frac{\partial V}{\partial Y} - r V = 0. \quad (19) \]
3.1 Numerical discretisation

We consider the following discretisation:

\[ V_{k+1}^{i,j} = a_{i,j}^k V_{i,j}^k + b_{i,j}^k V_{i,j+1}^k + c_{i,j}^k V_{i,j-1}^k + d_{i,j}^k V_{i,j}^{k+1} + e_{i,j}^k V_{i,j}^{k-1}, \]

where \( V_{k+1}^{i,j} = V(t + \Delta t, x, y) \), \( V_{i,j} = V(t, x \pm \Delta x, y) \), \( V_{i,j}^{k+1} = V(t, x, y \pm \Delta y) \); with \( \Delta t \) the time step and \( \Delta x, \Delta y \) the space steps. \( 1 \leq i \leq I, 1 \leq j \leq J, 0 \leq k \leq K, I, J, K \in \mathbb{N} \).

\[ a_{i,j}^k = 1 - \left[ r + 2\sigma_{i,j}(1 - \rho^2)(\frac{1}{(\Delta x)^2} + \frac{1}{(\Delta y)^2}) \right] \Delta t; \]
\[ b_{i,j}^k = \left[ \sigma_{i,j}(1 - \rho^2)(\frac{1}{(\Delta x)^2}) + (pr - D)\frac{\sigma_{i,j}}{q} \frac{1}{2\Delta x} \right] \Delta t; \]
\[ c_{i,j}^k = \left[ \sigma_{i,j}(1 - \rho^2)(\frac{1}{(\Delta y)^2}) + r\sqrt{1 - \rho^2} \frac{1}{2\Delta y} \right] \Delta t; \]
\[ d_{i,j}^k = \left[ \sigma_{i,j}(1 - \rho^2)(\frac{1}{(\Delta y)^2}) - r\sqrt{1 - \rho^2} \frac{1}{2\Delta y} \right] \Delta t; \]
\[ e_{i,j}^k = \left[ \sigma_{i,j}(1 - \rho^2) + (pr - D)\frac{\sigma_{i,j}}{q} \frac{1}{2\Delta x} \right] \Delta t; \]

and

\[ S_{i,j}(X_i, Y_j) = \exp \left( \frac{Y_j}{\sqrt{1 - \rho^2}} \right), \]
\[ \sigma_{i,j}(X_i, Y_j) = \frac{\eta^2}{4} \left( \frac{prY_j}{\sqrt{1 - \rho^2}} - X_i \right)^2, \]

with the following conditions:

\[ C1 : V_{0,j}^k = 0; \forall j, \forall k; \]
\[ C2 : V_{i,j}^k = S_{i,j}; \forall j, \forall k; \]
\[ C3 : V_{i,j}^0 = \text{Max}(0, S_{i,j}^0 - E); \forall i, \forall j; \]
\[ C4 : V_{i,0}^k = \text{Max}(0, S_{i,0}^k - Ee^{-k\rho\Delta t}); \forall i, \forall k; \]
\[ C5 : V_{i,j}^k = S_{i,j}; \forall i, \forall k; \]
\[ C6 : V_{i,1}^k = V_{i,0}^k; \forall i, \forall k. \]

4 Numerical Results and Comments

According to the numerical experimentations, it is clear that the price of the underlying behave exponentially with respect to the variable \( Y \) as shown in
figure 4.1 below. We also notice that we experimented no unstability behavior as long as we respect the CFL (the stability condition) that is deducted from the stability analysis of the finite difference scheme considered.

5 Conclusion

We have established a partial differential equation, as well as the boundary conditions on a bounded domain, when the underlying asset is modelled by a stochastic differential equation. The square of the volatility is expressed as a diffusion term and the volatility of the volatility is constant. By a change of variables, we get rid of the mixed derivative term in equation (12) and remain in the situation where the coefficient of the correlation is not equal to zero. The evaluation of a European option is approached by solving equation (19). For numerical approximations we make use of a semi implicit finite difference scheme to evaluate the price of the underlying as well as the price of a European call option as illustrated on the left and the right of figure 4.1 respectively. This scheme is chosen for numerical stabiltily and consitency reasons.

References


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