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Analytical Solution of the Frenet-Serret Systems of Circular Motion Bodies

Iyad A. Suwan, Methqal G. Daraghmeh and Abdelhalim M. Ziqan

Department of Mathematics and Statistics
Arab American University Jenin, Jenin, Palestine

Abstract

In this paper, the general Frenet-Serret system of circular motion body with constant velocity is analytically solved in three dimensional space. The tangent, normal, and binormal vectors are found by reducing the system into a high order ordinary differential equation. Solving this equation gives a closed form of those vectors. A special case of four dimensional Frenet-Serret system is also solved in this work.

Keywords: Frenet-Serret, high order ODE, Tangent, Normal, Binormal, Curvature, Torsion, circular orbits

1. Introduction

The Frenet-Serret frame is one of the most important tools that analyze and describe the properties of a particle along differentiable curves in Eucledian space [1,10].

Frenet and Serret [2] adapted the frame to curves by directly expressing the changes in derivatives of the tangent, normal and binormal vectors in terms of the frame. A few decades later, after the result of Frenet and Serret, their theory was extended to surfaces [3], also an n-dimensional vector calculus formulations of the system is developed [4]. Moreover, extensions to the frame have been proposed using quaternion-formulations [1]. In applications, studying the Frenet-Serret systems is of great importance in applied mathematics, physics, engineering and many fields of science [5-19].
One of the most important applications of the Frenet-Serret frames is understanding the kinematic properties of circular bodies, like the circular orbits in black hole space [10,11]. In this case, understanding the frame is useful in studying the properties of these orbits and provides interpretation of their geometry.
The general three dimensional Frenet-Serret system to be discussed in this paper is defined by:

$$\begin{bmatrix} T'(t) \\ U'(t) \\ V'(t) \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & \tau \\ -\kappa_2 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T(t) \\ U(t) \\ V(t) \end{bmatrix} \tag{1}$$

Where T, U, V are the tangent, normal and binormal vector fields respectively, t is the time, $\kappa_1 = T' \bullet U$, $\kappa_2 = T' \bullet V$, and τ is the torsion.
Studying of systems like (1) has been carried out in both analytical and numerical approaches as in [20-25]. This System will be analytically solved in this paper for bodies of circular motion with constant velocities.

2. Analysis and results

2.1 The Three Dimensional System

Considering a circular motion body with constant velocity leads to a constant curvature and torsion, hence, differentiating the third equation of (1) twice and differentiating the first and the second equations once with respect to time give

$$V''' = -\kappa_2 T'' - \tau U'' \tag{2}$$

$$T^{//} = \kappa_1 U^{/} + \kappa_2 V^{/} \tag{3}$$

and
$$U^{//} = -\kappa_1 T^{/} + \tau\, V^{/} . \tag{4}$$

Substituting (3) and (4) into (2) gives:

$$V^{///} = -\kappa_2 (\kappa_1 U^{/} + \kappa_2 V^{/}) - \tau(-\kappa_1 T^{/} + \tau\, V^{/}) \tag{5}$$

which is written as:

$$V^{///} = \kappa_1 \tau\ T^{/} - \kappa_1\ \kappa_2 U^{/} - (\kappa_2{}^2 + \tau^2)\ V^{/} \tag{6}$$

But

$$T^{/} = \kappa_1 U + \kappa_2 V \tag{7}$$

and

$$U^{/} = -\kappa_1 T + \tau\ V\ . \tag{8}$$

Therefore, substituting (7) and (8) in (6) yields to

$$V^{///} = \kappa_1{}^2 \kappa_2 T + \kappa_1{}^2 \tau\ U + (-\kappa_2{}^2 - \tau^2)\ V^{/} \tag{9}$$

hence,

$$V^{///} = \kappa_1{}^2 (\kappa_2 T + \tau\ U) - (\kappa_2{}^2 + \tau^2)\ V^{/} \tag{10}$$

but from (1)**,**

$$\kappa_2 T + \tau\ U = -V^{/}\quad . \tag{11}$$

Substituting (11) in (10) gives

$$V^{///} = -\kappa_1{}^2 V^{/} - (\kappa_2{}^2 + \tau^2)\ V^{/} \tag{12}$$

which is

$$V^{///} + (\kappa_1{}^2 + \kappa_2{}^2 + \tau^2)\ V^{/} = 0 \tag{13}$$

The characteristic equation of the homogeneous ordinary differential equation (13) is

$$r^3 + (\kappa_1{}^2 + \kappa_2{}^2 + \tau^2)\ r = 0\ . \tag{14}$$

In addition to the trivial solution, The solution of (14) is $r = \pm i\sqrt{\kappa_1{}^2 + \kappa_2{}^2 + \tau^2}$, where $i = \sqrt{-1}$, hence ,the solution of V is

$$V(t) = C_1 + C_2 \cos(\sqrt{\kappa_1{}^2 + \kappa_2{}^2 + \tau^2})t + C_3 \sin(\sqrt{\kappa_1{}^2 + \kappa_2{}^2 + \tau^2})t \tag{15}$$

Now, as V is known , the following system has to be solved for T and U ,

$$T^{/} = \kappa_1 U + \kappa_2 V$$
$$U^{/} = -\kappa_1 T + \tau\ V \tag{16}$$

From (16)

$$U'' = -\kappa_1 T' + \tau V' \quad (17)$$

Substituting the first equation of (16) into (17) gives

$$U'' = -\kappa_1^2 U - \kappa_1\kappa_2 V + \tau V' \quad (18)$$

which is

$$U'' + \kappa_1^2 U = F(t) \quad (19)$$

where

$$\begin{aligned} F(t) &= -\kappa_1\kappa_2[C_1 + C_2\cos(\sqrt{\kappa_1^2+\kappa_2^2+\tau^2})t + C_3\sin(\sqrt{\kappa_1^2+\kappa_2^2+\tau^2})t] \\ &+ \tau\,[-C_2\sqrt{\kappa_1^2+\kappa_2^2+\tau^2}\sin(\sqrt{\kappa_1^2+\kappa_2^2+\tau^2})t + C_3\sqrt{\kappa_1^2+\kappa_2^2+\tau^2}\cos(\sqrt{\kappa_1^2+\kappa_2^2+\tau^2})t] \end{aligned} \quad (20)$$

If $\alpha = \sqrt{\kappa_1^2 + \kappa_1^2 + \tau^2}$ **,** then

$$F(t) = -\kappa_1\kappa_2 C_1 + [\tau\, C_3\alpha - \kappa_1\kappa_2 C_2]\cos(\alpha\, t) - [\tau\, C_2\alpha + \kappa_1\kappa_2 C_3]\sin(\alpha\, t) \quad (21)$$

Using the variation of parameters method, the solution of (19) is

$$U = \frac{-\kappa_2 C_1}{\kappa_1} + A_2\cos\kappa_1 t + A_3\sin\kappa_1 t + \frac{\tau\, C_2\alpha + \kappa_1\kappa_2 C_3}{\kappa_2^2+\tau^2}\sin(\alpha t) + \frac{\kappa_1\kappa_2 C_2 - \tau\, C_3\alpha}{\kappa_2^2+\tau^2}\cos(\alpha t) \quad (22)$$

Now, as U and V are known, finding T is obvious by solving the first equation of (16).

2.2 The four Dimensional System

Consider the following well-known four dimensional Frenet-Serret system [4] :

$$\begin{bmatrix} T' \\ U' \\ V' \\ W' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ U \\ V \\ W \end{bmatrix}. \quad (23)$$

It is clear that

$$U'' = -\kappa_1 T' + \kappa_2 V' \quad (24)$$

and

$$U^{(4)} = -\kappa_1 T''' + \kappa_2 V''' \quad (25)$$

So

$$U^{(4)} = -\kappa_1(\kappa_1 U'') + \kappa_2(-\kappa_2 U'' + \kappa_3 W'') \quad (26)$$

And hence,

$$U^{(4)} = -\kappa_1^2 U'' - \kappa_2^2 U'' + \kappa_2\kappa_3 W'' \quad (27)$$

But $W'' = -\kappa_3 V'$. Hence, from from (24),

$$V^{/} = \frac{1}{\kappa_2}[U^{//} + \kappa_1 T^{/}] \tag{28}$$

So

$$W^{//} = -\kappa_3[\frac{1}{\kappa_2}(U^{//} + \kappa_1 T^{/})] \tag{29}$$

But $T^{/} = \kappa_1 U$, so

$$W^{//} = -\frac{\kappa_3}{\kappa_2}[U^{//} + \kappa_1\kappa_1 U] \tag{30}$$

So

$$U^{(4)} = -\kappa_1{}^2 U^{//} - \kappa_2{}^2 U^{//} + \kappa_2\kappa_3(\frac{-\kappa_3}{\kappa_2}[U^{//} + \kappa_1{}^2 U]) \tag{31}$$

therefore

$$U^{(4)} + (\kappa_1{}^2 + \kappa_2{}^2 + \kappa_3{}^2)U^{//} + \kappa_1{}^2\kappa_3{}^2 U = 0 \tag{32}$$

Equation (32) is a homogeneous fourth order ordinary differential equations. Solving it for U makes the solution of system (23) obvious.

Conclusions and Future Perspectives

In this paper, the Frenet-Serret system (1) is efficiently reduced to a homogeneous third order ordinary differential equation which is solved for the binormal vector field. The normal vector field is obtained by solving a linear system of first order ordinary differential equations, while the tangent vector field can be found by solving a simple linear ordinary differential equation. A special case of four dimensional Frenet-Serret system when the torsion is *zero* has been analytically solved. As a next step, a circular motion bodies with non-constant velocities will be under consideration.

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