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# Steady Flow of Reactive Power Law Fluid in a Cylindrical Pipe with an Isothermal Wall

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**Abstract**

An analysis is carried out to study the steady state solutions of a strongly exothermic reaction of a power law combustible fluid in a cylindrical pipe under Arrhenius kinetics. The problem is formulated in term of non-linear differential equations. Exact solution for velocity profile is constructed from non-linear momentum equation, while approximate solution for non-linear energy equation is obtained by using Homotopy perturbation method. In order to obtain a clear insight of the physical problem, the velocity and temperature behaviors are discussed for different dimensionless parameters involved in the governing equations. Graphical representations and discussions are also presented.

**Mathematics Subject Classification:** 76A05

**Keywords:** Reactive power-law, Isothermal wall, Homotopy perturbation method

## 1 Introduction

Non-Newtonain flows are often encountered in both science and engineering applications, such as oil recovering, soil remediution, or biological engineering, where the solution with polymeric additive, mud, blood, or colloids exhibits non-Newtonian features. These fluids have great importance in the food, pharmaceutical and chemical industries. The simplest and most common type of such a model is the power law fluid model. Zakia Hammouch [1] discussed the multiple solutions of steady MHD flow of power law fluid. In another paper Mohammed Guedda and Zakia Hammouch [2] discussed the similarity solutions of a non-Newtonian power law fluid model.

Reactive flows are the flows, in which there are interactions between chemical reactions and fluid dynamics. In fluid dynamics, different reactive flows have been generally modeled by extending the Navier-Stokes equations to include the appropriate chemical reactions. The theory of non-linear reaction diffusion equation is quite elaborate and its solutions in rectangular, cylindrical and spherical coordinates remain extremely important problems of practical relevance in the engineering sciences. The Frank-Kamenetskii theory [3] allowed for the temperature gradient to be taken into account. That is, there could be a considerable resistance to heat transfer in the reacting system, or the system has reactants with low thermal conductivity and highly conducting walls. Frank-Kamenetskii [3] successfully derived a criterion for stability in an ideal well-mixed reactor by defining a characteristic parameter $\delta$. In [3], he showed the existence of $\delta_c$, critical value of $\delta$, and concluded that for $\delta < \delta_c$, the stationary thermal state is achieved, but for $\delta > \delta_c$ a thermally stationary state is impossible. Adler [4] discussed temperature and radius of the hot gas bubble in an unsteady viscous, incompressible, chemically reactive flow stream, neglecting viscous dissipation and gravity effects. He also derived the criterion for the initiation of the thermal explosion using numerical techniques. In [5], Okoya studied the thermal stability for reactive third grade fluid in slab, he discussed three types of chemical reactions. That is, the Arrhenius reaction ($n = 0$), the bimolecular reaction ($n = 0.5$) and the sensitized reaction ($n = -2$). Makinde [6] discussed the steady flow of a reactive variable viscosity fluid in a cylindrical pipe with an isothermal wall. T. Haroon et al. [7] analysed the problem of reactive power law Poiseuille flow between parallel plates.

This paper runs as follows. In section 2, the governing equations are considered. In section 3, the physical problem is analyzed mathematically (we formulated the problem along with the relevant boundary conditions governing the power law fluid). The solution of the momentum equation is given in section 4. In section 5 the solution of energy equation using homotopy perturbation method (HPM) [8, 9] is provided. Graphs, discussions and some interesting conclusion are presented in section 6.

## 2 Governing Equations

The fundamental equations governing the steady motion of an incompressible power law fluid model are

$$\nabla \cdot \mathbf{V} = 0, \tag{1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = \nabla \cdot \boldsymbol{\tau}, \tag{2}$$

where $\mathbf{V}$ is the velocity vector, $\rho$ is density, $\frac{D}{Dt}$ is the material time derivative and $\boldsymbol{\tau}$ is the Cauchy stress tensor for the power law fluid defined as

$$\boldsymbol{\tau} = -p\mathbf{I} + \eta \left(tr\mathbf{A}_1^2\right)^m \mathbf{A}_1, \tag{3}$$

where $p$ is pressure, $\mathbf{I}$ is the unit tensor, $\eta$ is the flow consistency index, $m$ is the flow behavior index. For $m > 1$, the fluid is said to be dilatant or a shear thickening fluid, for $m < 1$, the fluid is called Pseudo-plastic or a shear thinning fluid and for $m = 0$, the fluid is simply the Newtonian fluid, and

$$\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^\top, \tag{4}$$

where $\mathbf{L} = \nabla\mathbf{V}$ and "$^\top$" denote the transpose of the tensor.
The energy equation with chemical reaction term is given by

$$\rho \frac{DT}{Dt} = K\nabla^2 T + \boldsymbol{\tau} \cdot \mathbf{L} + \wp_c, \tag{5}$$

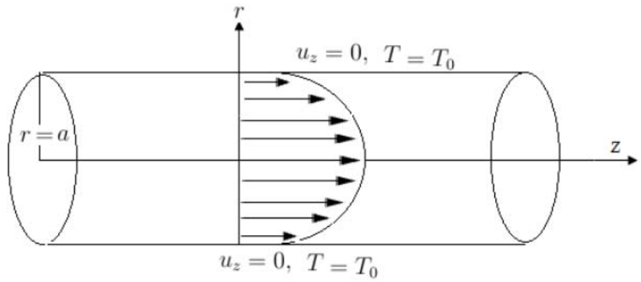
where $K$ is the thermal conductivity of material, $T$ is the temperature and $\wp_c = QC_0K_0(T)$ is the chemical reaction term [7], where $Q$ is the heat of reaction, $C_0$ is the initial concentration of reactant species and

$$K_0(T) = J\left(\frac{kT}{vh}\right)^n \exp\left(-\frac{E}{RT}\right), \tag{6}$$

where $J$ is the rate constant, $k$ is the Boltzmann's constant, $v$ is the vibration frequency, $h$ is the Planck's number, $E$ is the activation energy, $R$ is the universal gas constant and $n$ is a numerical exponent.

## 3 Problem Formulation

We consider the steady fully developed unidirectional flow of an incompressible reactive power law fluid inside a pipe of uniform cross-section in the $z$-direction, with a isothermal wall at $r = a$. Let $u_r, u_\theta$, and $u_z$ be the radial, tangential and axial velocity components, respectively. The flow is assumed due to the pressure gradient only. The boundary conditions become




$$\left.\begin{array}{lllllllll} \frac{du_z}{dr} & = & 0, & \frac{dT}{dr} & = & 0, & \text{at} & r = & 0, \\ u_z(r) & = & 0, & T(r) & = & T_0 & \text{at} & r = & a. \end{array}\right\} \tag{7}$$

We seek the velocity and temperature fields of the form

$$\mathbf{V} = \mathbf{V}[0, 0, u_z(r)], \quad T = T(r). \tag{8}$$

The continuity equation (1) is identically satisfied and using velocity profile (8) in momentum (2) and energy (5) equations, we obtain

$$2^m \eta \left( \frac{1}{r} \frac{d}{dr} \left( r \left( \frac{du_z}{dr} \right)^{2m+1} \right) \right) = \frac{dp}{dz}, \tag{9}$$

$$K \left( \frac{1}{r} \frac{d}{dr} \left( r \frac{dT}{dr} \right) \right) + 2^m \eta \frac{du_z}{dr} \left( \frac{du_z}{dr} \right)^{2m+1} + QC_0 K_0(T) = 0. \tag{10}$$

Introducing dimensionless parameters

$$\begin{array}{lll} r^* & = & \frac{r}{a}, \qquad w^* = \frac{u_z}{U}, \qquad \theta^* = \frac{T - T_0}{\beta T_0}, \qquad \beta^* = \frac{RT_0}{E}, \\ C^* & = & \frac{a}{\eta 2^m} \left( \frac{a}{U} \right)^{2m+1} \frac{dp}{dz}, \qquad \eta^* = \rho \frac{a^{2m+1}}{U^{2m-1}}, \qquad \Gamma^* = 2^m \eta \left( \frac{U}{a} \right)^{2m+2} \frac{a^2}{K\ \beta T_0}, \end{array}$$

where $U$ is the reference velocity, $T_0$ is the wall temperature, $\beta$, $\theta$, $C$ and $\Gamma$ are the dimensionless activation energy parameter, temperature, pressure gradient coefficient and viscous heating parameter, respectively. The system of governing equations (9) and (10), after dropping the asterisks, become

$$\frac{1}{r} \frac{d}{dr} \left( r \left( \frac{dw}{dr} \right)^{2m+1} \right) = C, \tag{11}$$

$$\frac{1}{r} \frac{d}{dr} \left( r \frac{d\theta}{dr} \right) + \Gamma \left( \frac{dw}{dr} \right)^{2m+2} + \delta f(\theta, \beta, n) = 0, \tag{12}$$

where $\delta$ is the reflection of the internal properties of a given system. So,

$$\delta = \frac{Q\ E\ y_0^2\ C_0\ k^n\ T_0^{n-2}}{\nu^n\ h^n\ R\ K}\ e^{-\frac{1}{\beta}}, \qquad f(\theta,\beta,n) = (1+\beta\theta)^n\ e^{\frac{\theta}{(1+\beta\theta)}},$$

and the corresponding boundary conditions (7) become

$$\left.\begin{array}{lllllllll} \frac{dw}{dr} & = & 0, & \frac{d\theta}{dr} & = & 0, & \text{at} & r & = & 0, \\ w(r) & = & 0, & \theta & = & 0 & \text{at} & r & = & 1. \end{array}\right\} \tag{13}$$

# 4 Solution of the Momentum equation

Momentum equation (11) subject to the boundary conditions (13) have the exact solution

$$w(r) = -\left(\frac{2m+1}{2m+2}\right)\left(\left|\frac{C}{2}\right|\right)^{\frac{1}{2m+1}}\left(1 - r^{\left(\frac{2m+2}{2m+1}\right)}\right). \tag{14}$$

For $m = 0$, we obtain the velocity profile for Newtonian fluid.

# 5 Solution of the Energy equation using HPM

Using the velocity profile (14) in energy equation (12), and dropping the absolute sign since $C/2$ is being raised to an even power, we have

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\theta}{dr}\right) + \Gamma\left(\left(\frac{C}{2}\right)^{\frac{1}{2m+1}} r^{\left(\frac{1}{2m+1}\right)}\right)^{2m+2} + \delta\left((1+\beta\theta)^n e^{\frac{\theta}{(1+\beta\theta)}}\right) = 0 \tag{15}$$

with the boundary conditions

$$\frac{d\theta}{dr} = 0 \quad \text{at} \quad r = 0 \qquad \text{and} \qquad \theta = 0 \quad \text{at} \quad r = 1. \tag{16}$$

We observe from equation (15) that the chemical reaction term is causing the non-linearity. We will use homotopy perturbation method [9] to find the approximate solution. So, we construct a homotopy

$$v(r,q) : \Omega \times [0,1] \rightarrow R,$$

satisfying

$$H(v,q) = (1-q)\left[L(v) - L(\theta_0)\right] + q\left[L(v) + N(v) - g(r)\right] = 0, \tag{17}$$

where $L(v) = \frac{d}{dr}\left(r\frac{dv}{dr}\right)$ is the linear operator, $N(v) = \delta\left((1+\beta v)^n e^{\frac{v}{(1+\beta v)}}\right)$ is the non-linear operator, $g(r) = -\Gamma\left(\left(\frac{C}{2}\right)^{\frac{1}{2m+1}} r^{\left(\frac{1}{2m+1}\right)}\right)^{2m+2}$ and $q \in [0,1]$ is the embedding parameter. Rewriting equation (17),

$$L(v) - L(\theta_0) + qL(\theta_0) + q\left[r\delta\left((1+\beta v)^n e^{\frac{v}{(1+\beta v)}}\right) + \Gamma\left(\frac{C}{2}\right)^{\left(\frac{2m+2}{2m+1}\right)} r^{\left(\frac{4m+3}{2m+1}\right)}\right] = 0, \quad (18)$$

where $\theta_0$ is the initial approximation, which can be obtained by solving the linear part of equation (15). So,

$$\theta_0 = \Gamma\left(\frac{C}{2}\right)^{\left(\frac{2m+2}{2m+1}\right)}\left(\frac{2m+1}{6m+4}\right)^2\left(1 - r^{\left(\frac{6m+4}{2m+1}\right)}\right). \quad (19)$$

The non-linear terms in equation (18) can be simplified by expanding the binomial and Taylor series. Thus,

$$(1+\beta v)^n \exp\left(\frac{v}{1+\beta v}\right) = \left[1 + (1+n\beta)v + \left(\frac{1}{2} + \beta(n-1) + \frac{n(n-1)}{2!}\beta^2\right)v^2 + \cdots\right] \quad (20)$$

Using equation (20) in (18) we get

$$L(v) - L(\theta_0) + qL(\theta_0) + q\left[\delta r\left(1 + (1+n\beta)v + \left(\frac{1}{2} + \beta(n-1) + \frac{n(n-1)}{2!}\beta^2\right)v^2 + \cdots\right) - \Gamma\left(\frac{C}{2}\right)^{\left(\frac{2m+2}{2m+1}\right)} r^{\left(\frac{4m+3}{2m+1}\right)}\right] = 0. \quad (21)$$

We assume that the solution of equation (21) can be expressed as a power series in q. That is,

$$v = v_0 + qv_1 + \cdots \quad (22)$$

where $v_i$, $i = 0, 1, 2, 3\ldots$ are independent of $q$. Inserting series (22) in equation (21) and (16) then equating equal powers of $q$, we obtain the system of equations as follows:

## 5.1 Zeroth-Order Problem

The zeroth-order problem along with boundary conditions is

$$L(v_0) - L(\theta_0) = 0, \tag{23}$$

$$\frac{dv_0}{dr} = 0 \quad \text{at} \quad r = 0, \quad v_0(r) = 0 \quad \text{at} \quad r = 1. \tag{24}$$

Since $L$ is the linear operator $v_0 = \theta_0$. So the zeroth-order solution is (19)

$$v_0 = \Gamma \left(\frac{C}{2}\right)^{\left(\frac{2m+2}{2m+1}\right)} \left(\frac{2m+1}{6m+4}\right)^2 \left(1 - r^{\left(\frac{6m+4}{2m+1}\right)}\right). \tag{25}$$

## 5.2 First Order Problem

The first order problem and the boundary conditions are

$$L(v_1) + L(\theta_0) + \delta r \left\{1 + (1 + n\beta)v_0 + \left(\frac{1}{2} + \beta(n-1) + \frac{n(n-1)}{2!}\beta^2\right) v_0^2\right\} + \Gamma \left(\frac{C}{2}\right)^{\frac{2m+2}{2m+1}} \left(r^{\frac{4m+3}{2m+1}}\right) = 0, \tag{26}$$

$$\frac{dv_1}{dr} = 0 \quad \text{at} \quad r = 0, \quad v_1(r) = 0 \quad \text{at} \quad r = 1. \tag{27}$$

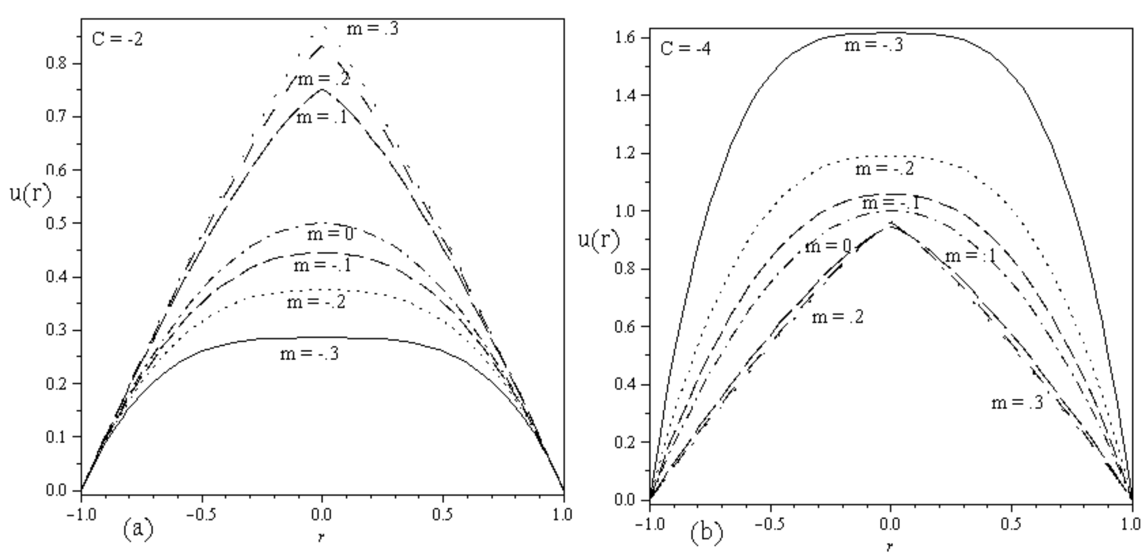
Using the zeroth-order solution in equation (26), then integrating two times and using the boundary conditions (27) we obtain the first order solution as

$$v_1(r) = \delta \left[\frac{1-r^2}{4} + \Gamma(1+n\beta) \left(\frac{2m+1}{6m+4}\right)^2 \left(\frac{C}{2}\right)^{\frac{2m+2}{2m+1}} \left(\left(\frac{2m+1}{10m+6}\right)^2 \left(1 - r^{\frac{10m+6}{2m+1}}\right) + \frac{1-r^2}{4}\right) + \left(\frac{1}{2} + \beta(n-1) + \frac{n(n-1)}{2!}\beta^2\right) \left(\Gamma \left(\frac{2m+1}{6m+4}\right)^2 \left(\frac{C}{2}\right)^{\frac{2m+2}{2m+1}}\right)^2 \left(\left(\frac{2m+1}{16m+10}\right)^2 \left(1 - r^{\frac{16m+10}{2m+1}}\right) + 2\left(\frac{2m+1}{10m+6}\right)^2 \left(1 - r^{\frac{10m+6}{2m+1}}\right) + \frac{1-r^2}{4}\right)\right]. \tag{28}$$

By letting $q \to 1$, the finial solution up to first order becomes

$$\theta(y) = -\Gamma\left(\frac{2m+1}{6m+4}\right)^2\left(\frac{C}{2}\right)^{\frac{2m+2}{2m+1}}\left(1-r^{\frac{6m+4}{2m+1}}\right)+\delta\Bigg[\frac{1-r^2}{4}+(1+n\beta)$$
$$\Gamma\left(\frac{2m+1}{6m+4}\right)^2\left(\frac{C}{2}\right)^{\frac{2m+2}{2m+1}}\left(\left(\frac{2m+1}{10m+6}\right)^2\left(1-r^{\frac{10m+6}{2m+1}}\right)+\frac{1-r^2}{4}\right)$$
$$+\left(\frac{1}{2}+\beta(n-1)+\frac{n(n-1)}{2!}\beta^2\right)\left(\Gamma\left(\frac{2m+1}{6m+4}\right)^2\left(\frac{C}{2}\right)^{\frac{2m+2}{2m+1}}\right)^2$$
$$\left(\left(\frac{2m+1}{16m+10}\right)^2\left(1-r^{\frac{16m+10}{2m+1}}\right)+2\left(\frac{2m+1}{10m+6}\right)^2\left(1-r^{\frac{10m+6}{2m+1}}\right)\right.$$
$$\left.+\frac{1-r^2}{4}\right)\Bigg]. \tag{29}$$

Equation (29) is the solution of the energy equation (15) for power law fluid in pipe. For $\Gamma = 0$, we will obtain the temperature distribution without the viscous dissipation. For $\delta = 0$ the temperature profile for the power law fluid without the chemical reaction term is retrieved, which is the same as the exact solution of the energy equation for the power law fluid without any source term.



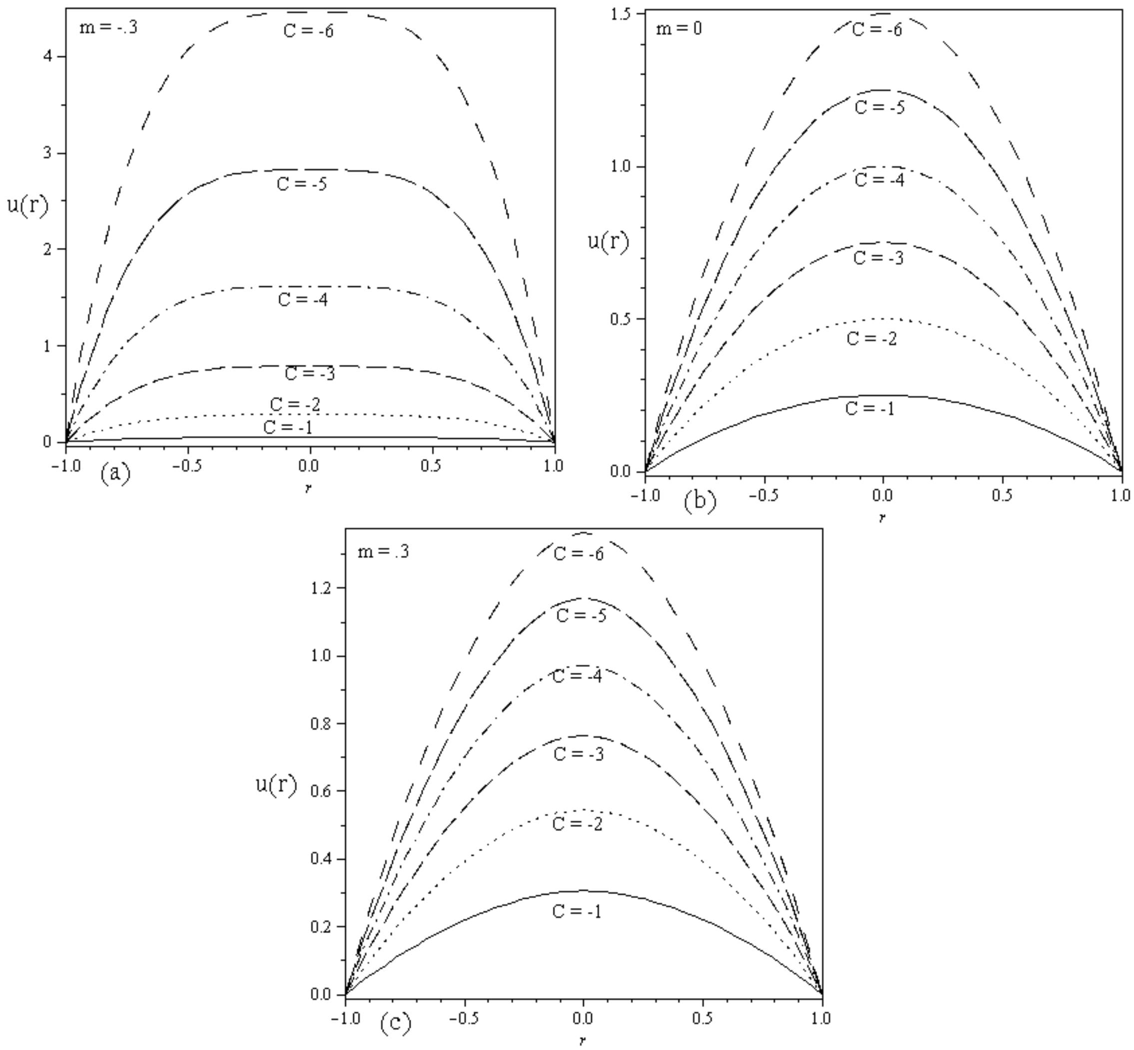

Figs. (1) Effect of power law index $m$ on velocity profile.

# 6 Discussion and Results

In this article steady flow of a fully developed incompressible reactive power law fluid in pipe at temperature, $T_0$ is studied. The chemical reaction is assumed to be strongly exothermic under Arrhenius Kinetics. We obtained exact solution for velocity profile (16) and approximate solution for the nonlinear energy equation (28) by using Homotopy Perturbation method. Velocity and temperature behaviors are discussed for different dimensionless parameters involved in the governing equations. An interesting phenomenon has been observed in Fig. (1a) where for a low pressure gradient coefficient $C = -2$, the shear thinning/thickening behavior is in contradiction with the literature while in Fig. (1b), where the pressure gradient coefficient is $C = -4$, the shear thinning/thickening behavior is in agreement with the literature. We conclude from these figures that in Poiseuille flow the behavior of the fluid depends not only on the power law index, but also on the pressure gradient and the gap between the walls.
We observed:

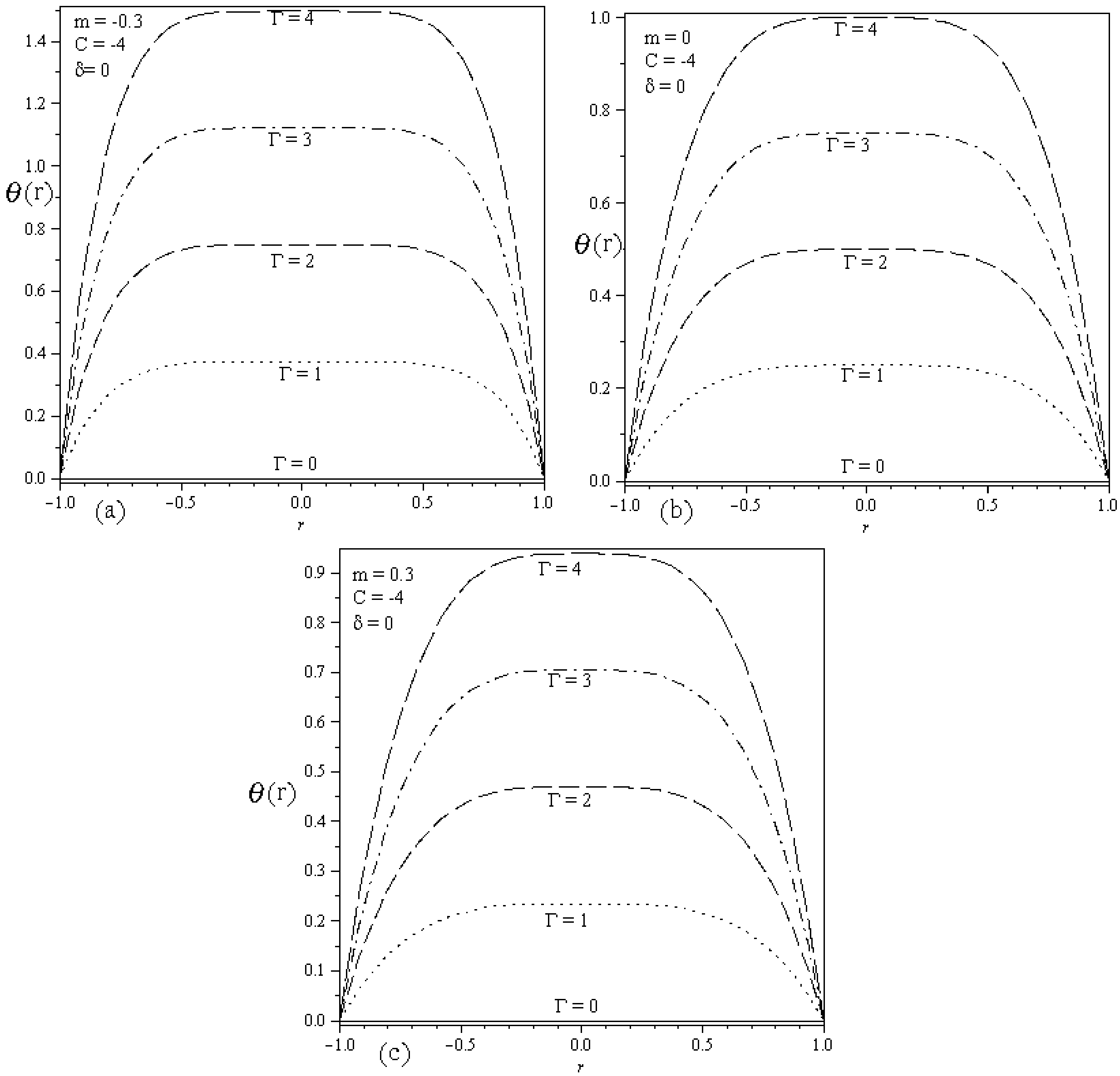
- For both the shear thinning fluid $m = -0.3$ and the viscous fluid $m = 0$, same maximum velocity is achieved at $C = -2.57$ which is the critical value for the pressure gradient coefficient. Above this value of $C$, the shear thinning as quoted in [10], is achieved but below this critical value the behavior is reversed.

- For both shear thickening fluid $m = 0.3$ and for viscous fluid $m = 0$, we obtain the critical value for pressure gradient coefficient $C = -2.93$. Above this value of $C$, the the velocity behavior is the same as quoted in [10], but below this critical values the behavior is reversed.



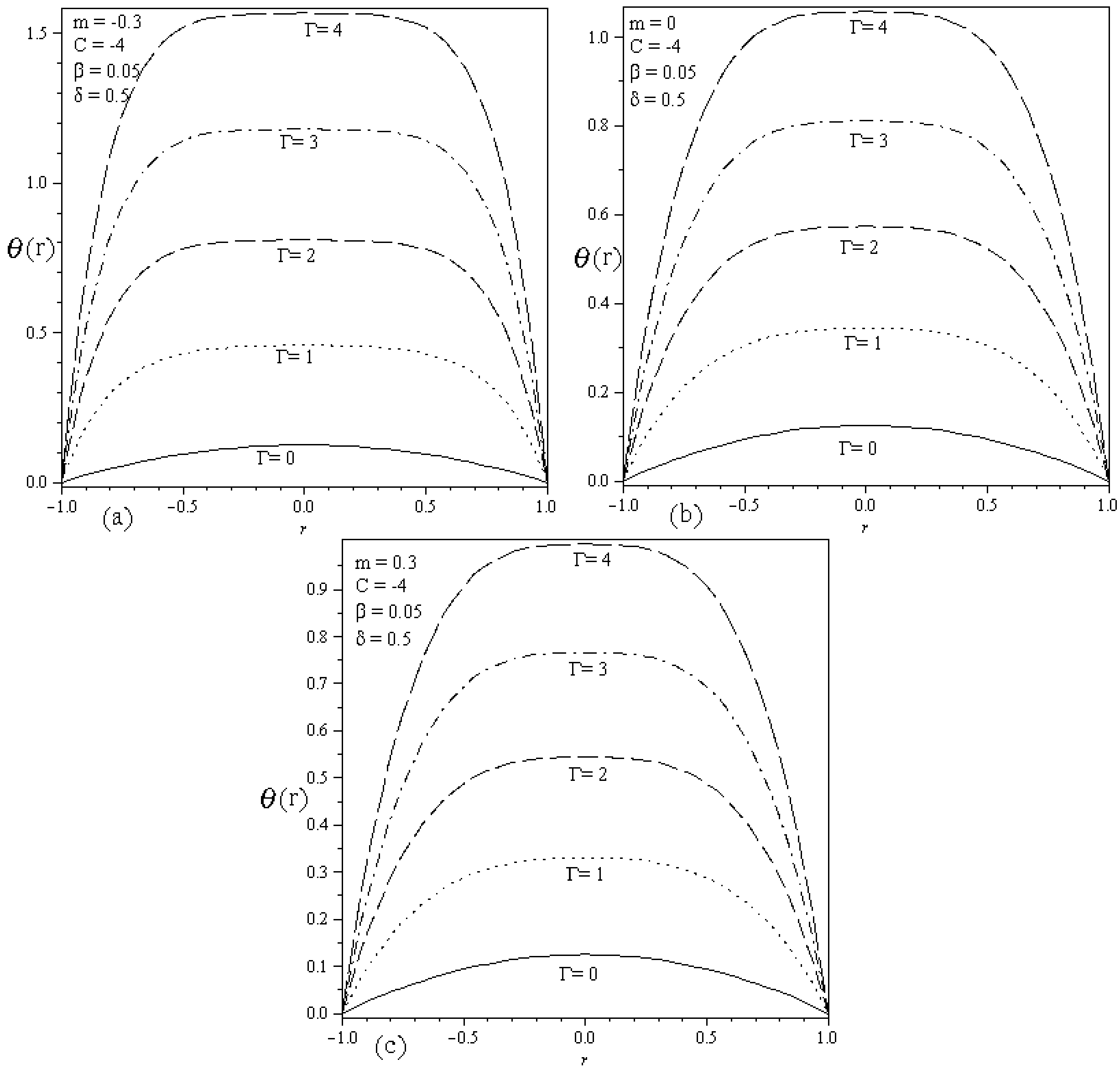

Figs. (2) Effect of pressure gradient $C$ on velocity profile.

Figs. 2(a,b,c) shows the effect of the pressure gradient coefficient on the power law fluid when $m = -0.3$, $m = 0$ and $m = 0.3$. We obtain the parabolic velocity profiles for all these three cases. We find that the magnitude of velocity between the walls for shear thinning case is very large as compared to the viscous and shear thickening cases. That is, the magnitude of velocity decreases with the increase in $m$.



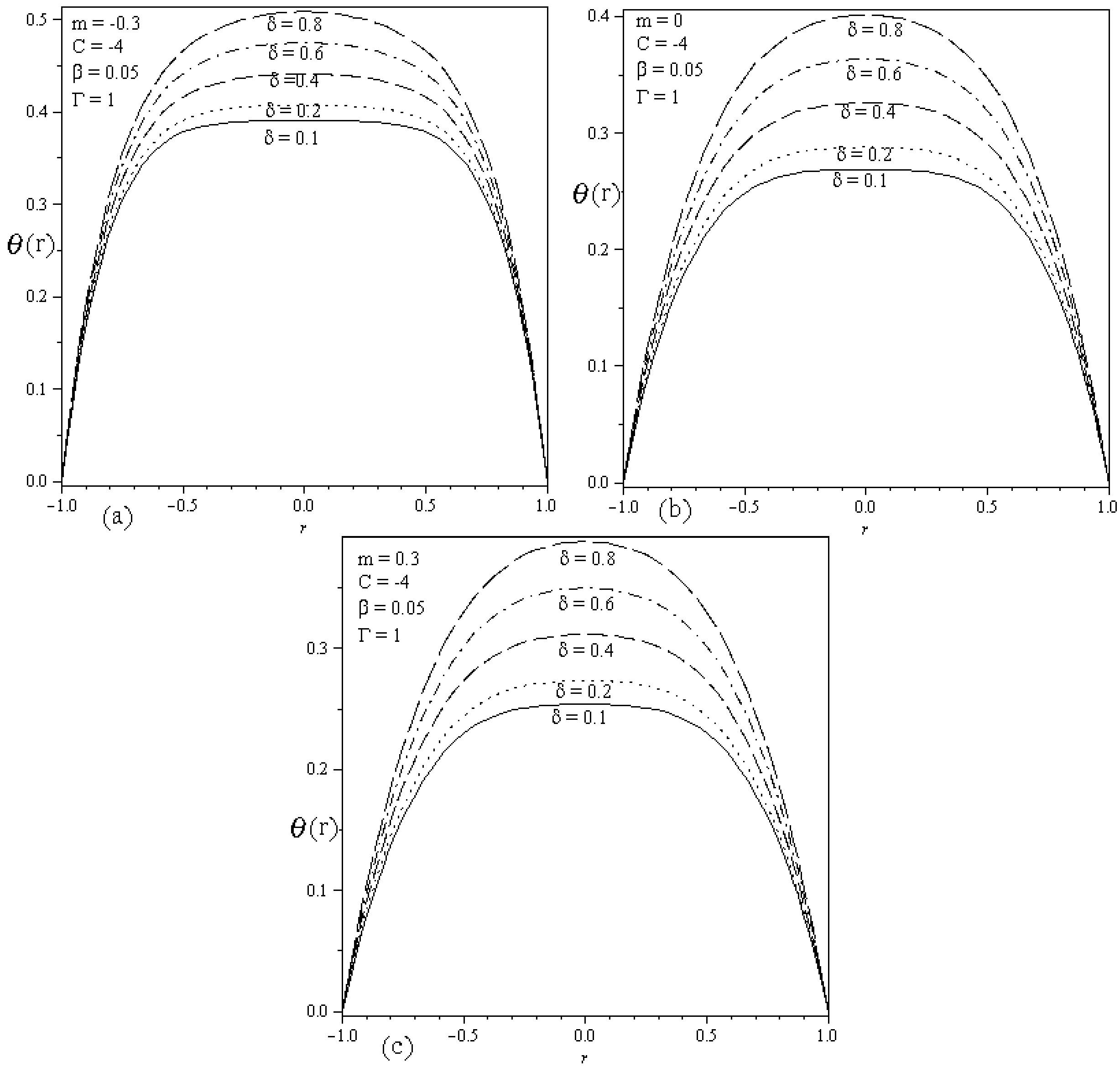

Figs. (3) Effect of viscous heating parameter Γ on temperature profile without source term.

Figs. 3(a,b,c) show the effect of viscous heating parameter, $\Gamma$, on the power law fluid in the absence of any chemical reaction. Increase in the magnitude of temperature is very high for shear thinning fluid (Fig. 3(a)) as compared to the viscous fluid (Fig. 3(b)) and shear thickening power law fluid (Fig. 3(c)). The temperature increases with the increases in $\Gamma$ for all values of $m$.




Figs. (4) Effect of viscous heating parameter $\Gamma$ on temperature distribution for Arrhenius ($n = 0$), bimolecular ($n = 0.5$) and sensitized ($n = -2$) reactions.

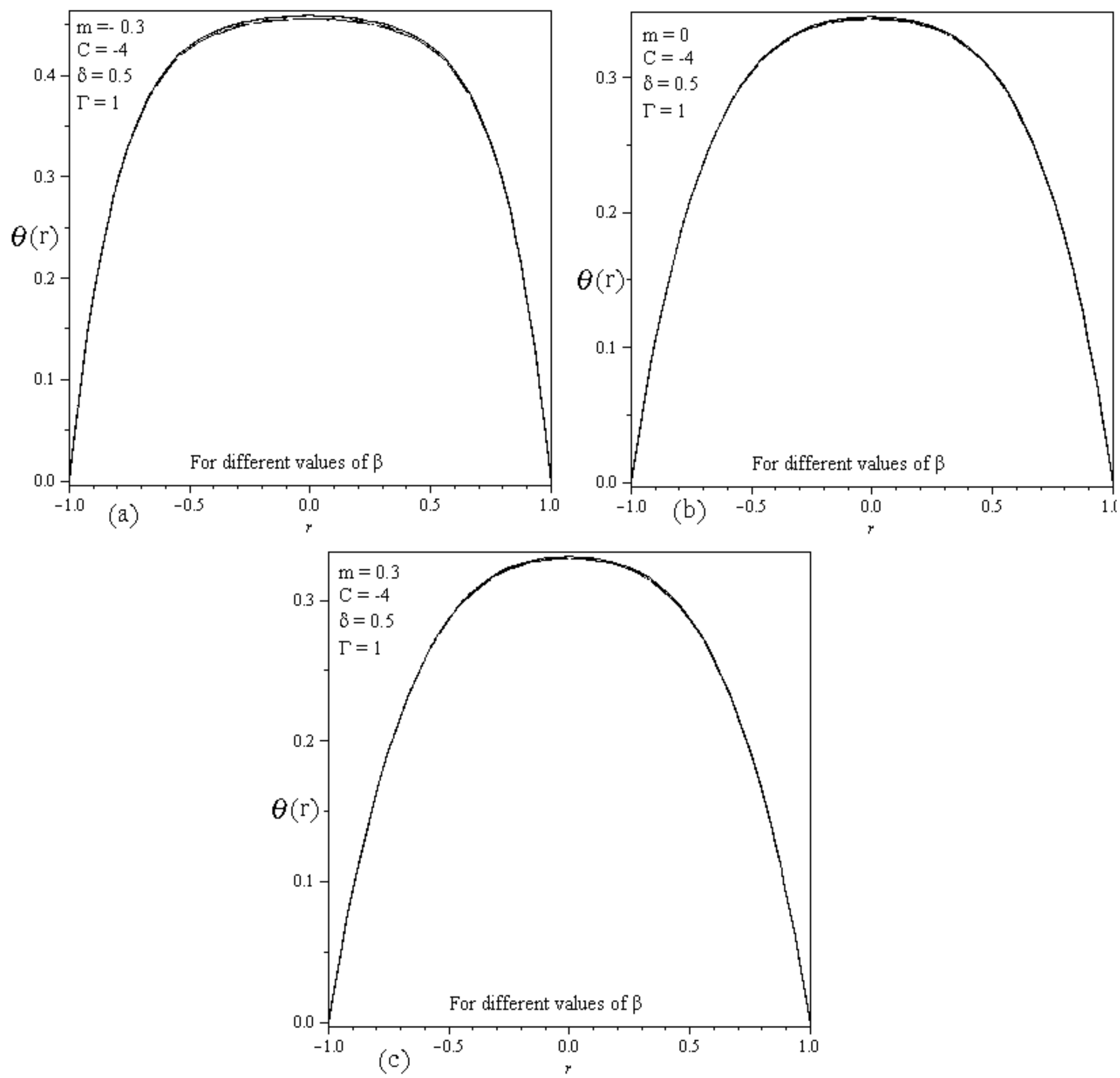
Figs. 4(a,b,c) show the effect of viscous heating parameter $\Gamma$ in the presence of chemical reaction term with $\delta = 0.5$, $\beta = 0.05$ for three chemical reactions. That is, Arrhenius ($n = 0$), bimolecular ($n = 0.5$) and sensitized ($n = -2$) reactions, for shear thinning/thickening and viscous fluid. Chemical reaction terms (below critical values) play an important role to change temperature distribution in the pipe. The temperature increases with increases in $\Gamma$. The temperature increases for shear thinning fluid as compared to viscous and shear thickening cases.




Figs. (5) Effect of Frank-Kamenetskii parameter $\delta$ on temperature distribution for Arrhenius ($n = 0$), bimolecular ($n = 0.5$) and sensitized ($n = -2$) reactions.

The effect of reflection of the internal properties of the given system, $\delta$, for the Arrhenius, bimolecular and sensitized reactions, when $\Gamma = 1$ and $\beta = 0.05$ can be seen in Figs. 6(a,b,c). We observe that temperature distribution increases with increases $\delta$,temperature increases for shear thinning fluid as compare to viscous and shear thickening cases.

When activation energy $\beta << 1$, we do not observe any change in the temperature profile for ($n = -2, 0, 0.5$) cases. With the increase in $\beta$ the temperature increases near the walls and decreases at the center for the shear thinning case.




Figs. (6) Effect of activation energy parameter $\beta$ on temperature distribution for Arrhenius (n=0), bimolecular (n=0.5) and sensitized (n=-2) reactions.

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