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Theory of Solvability of Boundary Value Problems for Third Order Operator-Differential Equations

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Abstract

In this paper, a nonlocal boundary value problem for third order operator-differential equation is considered. The equation and boundary condition are perturbed by some operators. Sufficient conditions providing well-posed solvability of the considered problem are obtained. All these conditions are expressed in terms of the coefficients of the equation and the operator involved in the boundary condition.

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Let H be a separable Hilbert space, A be a self-adjoint positive definite operator on H. It is known that the domain of definition $D(A^\gamma)$ of the operator A^γ becomes a Hilbert space H_γ with respect to the scalar product $(x,y)_\gamma = (A^\gamma_x,\ A^\gamma_y),\quad \gamma \geq 0$. For $\gamma = 0$, we consider that $H_0 = H$.

Let $L_2(R_+;H)$ be the Hilbert space of all vector functions $f(t)$ with values in H, defined on $R_+ = (0,\ \infty)$, with the norm

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^\infty \|f(t)\|^2\,dt\right)^{\frac{1}{2}}.$$

We define the following Hilbert space [1]

$$W_2^3(R_+;H) = \{u : u''' \in L_2(R_+;H),\ A^3u \in\ L_2(R_+;H)\},$$

with the norm

$$\|u\|_{W_2^3(R_+;H)} = \left(\|u'''\|^2_{L_2(R_+;H)} + \left\|A^3u\right\|^2_{L_2(R_+;H)}\right)^{\frac{1}{2}}.$$

By $L(X,Y)$ we denote the space of bounded operators acting from the space X into the space Y and assume that the operator $K \in L(W_2^3(R_+;H), H_{\frac{1}{2}})$. We define the following subspace of $W_2^3(R_+;H)$:

$$W_{2,K}^3(R_+;H) = \{u :\ u \in W_2^3(R_+;H),\ u(0) = 0,\ u''(0) = Ku\}.$$

It follows from the theorem on traces [1] that the space $W_{2,K}^3(R_+;H)$ is well-defined.

Consider the following boundary value problem in the Hilbert space H:

$$P(d/dt)u = u'''(t) - A^3u(t) + \sum_{j=0}^{2} A_{3-j}(t)\ u^{(j)}(t) = f(t),\ t \in R_+, \tag{1}$$

$$u(0) = 0, \quad u''(0) = Ku, \tag{2}$$

where $f(t),\ u(t) \in H$ almost everywhere in R_+ and the operator coefficients satisfy the conditions:
1) A is a self-adjoint positive definite operator on H;
2) $B_j = A_jA^{-j} \quad (j = \overline{0,\ 3})$ are bounded operators on H;
3) $K \in L(W_2^3(R_+;H), H_{\frac{1}{2}})$, and $\|K\|_{W_2^3(R_+;H)\to H_{\frac{1}{2}}} = k$.

Definition 1. If for any $f(t) \in L_2(R_+;H)$ there exists a vector function $u(t) \in W_2^3(R_+;H)$ satisfying equation (1) almost everywhere, the boundary conditions (2) are satisfied in the sense of convergence

$$\lim_{t\to+0} \|u(t)\|_{\frac{5}{2}} = 0, \lim_{t\to+0} \|u''(t) - Ku\|_{\frac{1}{2}} = 0$$

and the following estimate holds

$$\|u\|_{W_2^3(R_+;H)} \le const\,\|f\|_{L_2(R_+;H)},$$

then we say that $u(t)$ is a regular solution of problem (1), (2) and problem (1), (2) is said to be regularly solvable.

In this paper, we find conditions for regular solvability of boundary value problem (1), (2). Note that some of the similar problems are considered in [2-9].

First we will consider boundary value problems

$$P_0(d/dt)u(t) = u'''(t) - A^3u(t) = f(t), \quad t \in R_+ \tag{3}$$

$$u(0) = 0, \ \ u''(0) = Ku, \tag{4}$$

Let

$$P_0u = P_0(d/dt)u(t), P_1u = P_1(d/dt)u(t) = \sum_{j=0}^{3} A_{3-j}u^{(j)}$$

and

$$Pu = P_0u + P_1u, u \in W_{2,K}^3(R_+; H).$$

Then the following theorem holds.

Theorem 1. *Let conditions 1) and 3) be satisfied and $k < 1$. Then the operator $P_0 : W_{2,K}^3(R_+; H) \to L_2(R_+; H)$ is an isomorphism.*

Proof. First, we will show that the equation $P_0u = 0$ has only the trivial solution. Since the general solution of the equation $P_0(d/dt)u(t) = 0$ from the space $W_2^3(R_+; H)$ has the form

$$u_0(t) = e^{\omega_1 tA}x + e^{\omega_2 tA}y, \ \ x, y \in H_{\frac{5}{2}}, \ \ \omega_1 = -\frac{1}{2}(1 + i\sqrt{3}), \quad \omega_2 = -\frac{1}{2}(1 - i\sqrt{3}),$$

it follows from (4) that $y = -x$, $\omega_1^2A^2x + \omega_2^2A^2y = K(e^{\omega_1 tA}x + e^{\omega_2 tA}y)$. Therefore, $(\omega_1^2 - \omega_2^2)x = A^{-2}K(e^{\omega_1 tA}x - e^{\omega_2 tA}x)$. Thus, relative to the vector x we obtain

$$(x + Qx) = 0,$$

where

$$Qx = \frac{i}{\sqrt{3}}A^{-2}K(e^{\omega_1 tA}x - e^{\omega_2 tA}x), \quad x \in H_{\frac{5}{2}}. \tag{5}$$

It is obvious that

$$\|Qx\|_{\frac{5}{2}} = \frac{1}{\sqrt{3}}\left\|A^{\frac{1}{2}}K\left(e^{\omega_1 tA}x - e^{\omega_2 tA}x\right)\right\| = \frac{1}{\sqrt{3}}\left\|K\left(e^{\omega_1 tA}x - e^{\omega_2 tA}x\right)\right\|_{\frac{1}{2}} \leq$$

$$\leq \frac{k}{\sqrt{3}} \left\| K\left(e^{\omega_1 tA}x - e^{\omega_2 tA}x\right)\right\|_{W_2^3(R_+;H)}. \tag{6}$$

On the other hand, taking into account that $\omega_1^3 = \omega_2^3 = 1$ we obtain:

$$\left\|e^{\omega_1 tA}x - e^{\omega_2 tA}x\right\|^2_{W_2^3(R_+;H)} = \left\|\omega_1^3 A^3 e^{\omega_1 tA}x - \omega_2^3 A^3 e^{\omega_2 tA}x\right\|^2_{L_2(R_+;H)} +$$

$$+\left\|A^3\left(e^{\omega_1 tA}x - e^{\omega_2 tA}x\right)\right\|^2_{L_2(R_+;H)} = 2\left\|A^3\left(e^{\omega_1 tA}x - e^{\omega_2 tA}x\right)\right\|^2_{L_2(R_+;H)}. \tag{7}$$

Assume $A^{\frac{5}{2}}x = z$, we have:

$$\left\|A^3\left(e^{\omega_1 tA}x - e^{\omega_2 tA}x\right)\right\|^2_{L_2(R_+;H)} = \left\|A^{\frac{1}{2}}\left(e^{\omega_1 tA}z - e^{\omega_2 tA}z\right)\right\|^2_{L_2(R_+;H)} =$$

$$= \left\|A^{\frac{1}{2}} e^{\omega_1 tA}z\right\|^2_{L_2(R_+;H)} +$$

$$+\left\|A^{\frac{1}{2}} e^{\omega_2 tA}z\right\|^2_{L_2(R_+;H)} - 2Re\left(A^{\frac{1}{2}} e^{\omega_1 tA}z, A^{\frac{1}{2}} e^{\omega_2 tA}z\right)_{L_2(R_+;H)}. \tag{8}$$

Using the spectral decomposition of the operator A, we have:

$$\left\|A^{\frac{1}{2}} e^{\omega_1 tA}z\right\|^2_{L_2(R_+;H)} = \int_0^\infty\left(\int_{\mu_0}^\infty \mu e^{2Re\omega_1 t\mu}\left(dE_\mu z, z\right)\right)dt =$$

$$= \int_{\mu_0}^\infty \mu\left(dE_\mu z, z\right)\int_0^\infty e^{-t\mu}dt = \int_{\mu_0}^\infty \left(dE_\mu z, z\right) = \left\|A^{\frac{1}{2}}x\right\|^2 = \|x\|^2_{\frac{1}{2}}. \tag{9}$$

Similarly, we obtain that:

$$\left\|A^{\frac{1}{2}} e^{\omega_2 tA}x\right\|^2_{L_2(R_+;H)} = \|x\|^2_{\frac{5}{2}}. \tag{10}$$

Thus, we find that

$$Re(A^{\frac{1}{2}} e^{\omega_1 tA}z,\ A^{\frac{1}{2}} e^{\omega_2 tA}z)_{L_2(R_+;H)} = Re\int_0^\infty\left(\int_{\mu_0}^\infty \mu e^{2\omega_1 t\mu}(dE_\mu z, z)\right)dt =$$

$$= Re\int_{\mu_0}^\infty \mu(dE_\mu z, z)\int_0^\infty e^{2\omega_1 t}dt =$$

$$= Re\int_{\mu_0}^\infty -\frac{1}{2\omega_1}(dE_\mu z, z) = -Re\frac{1}{2\omega_1}\|z\|^2 = \frac{1}{4}\|z\|^2 = \frac{1}{4}\|x\|^2_{\frac{5}{2}}. \tag{11}$$

Taking into account equalities (9) – (11) in (8), we

$$\left\| A^3(e^{\omega_1 tA}x - e^{\omega_2 tA}x)\right\|^2_{L_2(R_+;H)} = \frac{3}{2}\left\|x\right\|^2_{\frac{5}{2}}, \tag{12}$$

and

$$\left\| A^3(e^{\omega_1 tA}x - e^{\omega_2 tA}x)\right\|^2_{W_2^3(R_+;H)} = 3\left\|x\right\|^2_{5/2}. \tag{13}$$

Thus, it follows from (6) that

$$\left\|Qx\right\|_{\frac{5}{2}} \le k\left\|x\right\|_{\frac{5}{2}}.$$

Since $k < 1$, then the operator $E + Q$ is invertible in $H_{\frac{5}{2}}$, and it follows from the equation $(x + Qx) = 0$ that $x = 0$. Then $y = 0$, therefore, $u_0(t) = 0$. Note that $\left\|(E+Q)^{-1}\right\|_{H_{\frac{5}{2}} \to H_{\frac{5}{2}}} < \frac{1}{1-k}$. Now we will prove that the equation $P_0 u = f$ has a solution for any $f \in L_2(R_+;H)$. It is obvious that the following vector function [4]

$$u_1(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}(i^3\varsigma^3 - A^3)^{-1}\int_0^{+\infty} f(s)e^{i\varsigma(t-s)}ds d\varsigma$$

belongs to the space $W_2^3(R;H)$ and satisfies the equation $P_0(d/dt))u(t) = f(t)$ almost everywhere in R_+. By $\alpha(t)$ we denote the restriction of the vector-function $u_1(t)$ on $[0,\ \infty)$. Then $\alpha(t) \in W_2^3(R_+;H)$ and it follows from the theorem on traces that $\alpha^{(j)}(0) \in H_{3-j-\frac{1}{2}},\ j = 0,1,2.$

Now we will find a solution of the equation $P_0 u = f$ in the form

$$u(t) = \alpha(t) + e^{\omega_1 tA}x + e^{\omega_2 tA}y,$$

where $x, y \in H_{\frac{5}{2}}$ are the unknown vectors belonging to the determination. It follows from (3) that $y = -x - \alpha(0)$ and

$$(\omega_1^2 - \omega_2^2)x + A^{-2}\alpha''(0) = A^{-2}K(\alpha(t)) + A^{-2}K(e^{\omega_1 tA}x - e^{\omega_2 tA}x)$$

or

$$(E+Q)\,x = \psi,$$

where $\psi = -\frac{iA^{-2}}{\sqrt{3}}\left(K\left(\alpha\left(t\right) - A^{-2}\alpha''\left(0\right)\right)\right) \in H_{\frac{5}{2}}$. Hence, we find that

$$x = (E+Q)^{-1}\psi,\ y = -(E+Q^{-1})\psi - \alpha\left(0\right) \in H_{\frac{5}{2}}.$$

Thus, $u(t) \in W_{2,K}^3(R_+;H)$. From the inequality $\left\|P_0 u\ \right\|^2_{L_2(R_+;H)} \le 2\left\|u\right\|^2_{W_2^3(R_+;H)}$ and from the Banach theorem on the inverse operator follows that the operator P_0^{-1} exists and it is bounded. The theorem is proved.

For $K = 0$ we have the following assertion

Corollary 1. *Let condition 1) be satisfied. Then problem*

$$u''' - A^3 u = f(t), \tag{14}$$

$$u(0) = u''(0) = 0 \tag{15}$$

is regularly solvable.

Using the procedure offered in [10], we will prove the following important lemma.

Lemma. *The regular solution $\varsigma(t)$ of problem (14), (15) satisfies the following estimates*

$$\|\varsigma(t)\|_{W_2^3(R_+;H)} \leq \frac{2\sqrt{3}}{3} \|P_0\varsigma\|_{L_2(R_+;H)}, \tag{16}$$

$$\left\|A^2\varsigma'\right\|_{L_2(R_+;H)} \leq \frac{1}{\sqrt[3]{2}} \|P_0\varsigma\|_{L_2(R_+;H)}, \tag{17}$$

$$\|A\varsigma''\|_{L_2(R_+;H)} \leq \frac{1}{\sqrt[3]{2}} \|P_0\varsigma\|_{L_2(R_+;H)}. \tag{18}$$

Proof. We denote by

$$W_{2,0}^3(R_+;H) = \left\{\varsigma : \ \varsigma \in W_2^3(R_+;H), \ \varsigma(0) = \varsigma''(0) = 0\right\}.$$

Then, it is easy to see that for $\varsigma \in W_{2,0}^3(R_+;H)$

$$\|P_0\varsigma\ \|^2_{L_2(R_+;H)} = \|\varsigma\|^2_{W_2^3(R_+;H)} - \|\varsigma'(0)\|^2_{3/2}. \tag{19}$$

Consider a polynomial pencil for $\beta \in (0,\ 1)$

$$P(\lambda;\beta : A) = (1-\beta)(-\lambda E^6 + A^6) = \Phi(\lambda;\beta : A)\ \Phi(-\lambda;\beta : A),$$

where

$$\Phi(\lambda;\beta : A) = \sqrt{1-\beta}(\lambda^3 E + 2\lambda^2 A + 2\lambda A^2 + A^3) =$$

$$= \sqrt{1-\beta}(\lambda E + A)(\lambda E - \omega_1 A)(\lambda E - \omega_2 A), \quad \omega_1 = \overline{\omega_2} = -\frac{1}{2}(1+\sqrt{3}i).$$

Further, for all $\varsigma \in W_{2,0}^3(R_+;H)$ we have:

$$\left\|\Phi\left(\frac{d}{dt};\beta : A\right)\varsigma\right\|^2_{L_2(R_+;H)} = (1-\beta)\left(\left\|\varsigma'''\right\|^2_{L_2(R_+;H)} + 4\left\|A\varsigma''\right\|^2_{L_2(R_+;H)} + \right.$$

$$+4\left\|A^2\varsigma'\right\|^2_{L_2(R_+;H)}+\left\|A^3\varsigma\right\|^2_{L_2(R_+;H)}+4Re\left(\varsigma''',A\varsigma''\right)_{L_2(R_+;H)}+$$

$$+4Re\left(\varsigma''',A^2\varsigma'\right)_{L_2(R_+;H)}+2Re\left(\varsigma''',A^3\varsigma\right)_{L_2(R_+;H)}+8Re\left(A\varsigma'',A^2\varsigma'\right)_{L_2(R_+;H)}+$$

$$\left.+4Re\left(A\varsigma'',A^3\varsigma\right)_{L_2(R_+;H)}\right). \quad (20)$$

Integrating by parts, we obtain:

$$Re(\varsigma''',A\varsigma'')_{L_2(R_+;H)}=-\frac{1}{2}\left\|\varsigma''(0)\right\|^2_{1/2}, \quad (21)$$

$$Re(\varsigma''',A^2\varsigma')=-Re(A^{\frac{1}{2}}\varsigma''(0),A^{\frac{3}{2}}\varsigma'(0))-\left\|A\varsigma''\right\|^2_{L_2(R_+;H)}=-\left\|A\varsigma''\right\|^2_{L_2(R_+;H)}, \quad (22)$$

$$Re(A\xi'',A^2\varsigma')_{L_2(R_+;H)}=-\frac{1}{2}\left\|\varsigma'(0)\right\|^2_{\frac{3}{2}}, \quad (23)$$

$$Re(A\varsigma'',\ A^3\varsigma)_{L_2(R_+;H)}=-\left\|A\varsigma'\right\|^2_{L_2(R_+;H)}, \quad (24)$$

$$Re(A^2\varsigma',A^3\varsigma)_{L_2(R_+;H)}=-\frac{1}{2}\left\|\varsigma(0)\right\|^2_{\frac{5}{2}}=0. \quad (25)$$

Taking into account equalities (21)-(25) in (20), considering equality (19), we obtain:

$$\left\|\Phi\left(\frac{d}{dt};\beta;A\right)\right\|^2_{L_2(R_+;H)}=(1-\beta)\left\|\varsigma\right\|^2_{W_2^3(R_+;H)}-4\left(1-\beta\left\|\varsigma'(0)\right\|^2_{\frac{3}{2}}\right)=$$

$$=\left(\left\|\varsigma\right\|^2_{W_2^3(R_+;H)}-\left\|\xi'(0)\right\|^2_{\frac{3}{2}}\right)-\beta\left\|\varsigma\right\|^2_{W_2^3(R_+;H)}-(3-4\beta)\left\|\varsigma'(0)\right\|^2_{\frac{3}{2}},$$

or

$$\left\|\Phi(d|dt;\beta:A)\varsigma\right\|^2_{L_2(R_+;H)}+(3-4\beta)\left\|\varsigma'(0)\right\|^2_{\frac{3}{2}}=\left\|P_0\varsigma\right\|^2_{L_2(R_+;H)}-\beta\left\|\varsigma\right\|^2_{W_2^3(R_+;H)}.$$

It follows from the results of [10] that for all $\varsigma\in W^3_{2,0}(R_+;H)$, the exact inequality holds

$$\left\|\varsigma\right\|^2_{W_2^3(R_+;H)}\le\frac{4}{3}\left\|P_0\varsigma\right\|^2_{L_2(R_+;H)},$$

since the equation $3-4\beta=0$ has the solution $\beta=\frac{3}{4}\in(0,1)$, i.e. inequality (14) is proved. For obtaining inequality (17) we will consider the following operator pencil for $\beta\in\left(0,\ \frac{3}{2^{2/3}}\right)$

$$P_1(\lambda;\beta:A)=(-\lambda^6E+A^6)-\beta(i\lambda)^2A^4.$$

For $\beta\in\left(0,\ \frac{3}{2^{2/3}}\right)$, $P_1\left(\lambda;\beta:A\right)$ does not have a spectrum on the imaginary axis and it is represented in the form

$$P_1(\lambda;\beta:A)=\Phi_1(\lambda;\beta:A)\Phi_1(-\lambda;\beta:A), \tag{26}$$

where

$$\Phi_1(\lambda;\beta:A)=\lambda^3E+a_2(\beta)\lambda^2A+\alpha_1(\beta)\lambda A^2+A^3=\prod_{j=1}^{3}(\lambda E-\eta_j(\beta)A),$$

where $Re\eta_j(\beta)<0,\ \ a_j(\beta)>0$.
From equality (26) follows that the coefficients $\alpha_j(\beta)$ satisfy the equalities

$$a_1^2(\beta)-2\alpha_2(\beta)=-\beta,\quad a_2^2(\beta)=2\alpha_1(\beta). \tag{27}$$

Using equalities (21) – (25), easily we find that

$$\|\Phi_1(d|dt:\beta:A)\varsigma\|^2_{L_2(R_+;H)}+(\alpha_1(\beta)\alpha_2(\beta)-2)\left\|\varsigma'(0)\right\|^2_{\frac{3}{2}}=\left\|P_0u\right\|^2-\beta\left\|A^2u'\right\|^2_{L_2(R_+;H)}.$$

Then it follows from the results of [10] if the equation $\alpha_1(\beta)\alpha_2(\beta)=2$ has solution in the interval $\left(0,\ 3/2^{2/3}\right)$, β_0 is the smallest of them, then the following exact inequality holds:

$$\left\|A^2\varsigma'\right\|^2_{L_2(R_+;H)}\le\frac{1}{\beta_0^2}\left\|P_0\varsigma\right\|^2_{L_2(R_+;H)}.$$

But by taking into account equation (27), the equation $\alpha_1(\beta)\alpha_2(\beta)=2$ has the root $\beta_0=\sqrt[3]{4}$. Really, taking into account $\alpha_1(\beta)=\frac{2}{\alpha_2(\beta)}$ in the second equation, we obtain from (21) that $\alpha_2(\beta)=\sqrt[3]{4}$. Then $\alpha_1(\beta)=\sqrt[3]{2}$, and $\beta=2\alpha_2(\beta)-\alpha_1^2(\beta)=\sqrt[3]{4}$. Then $\beta_0=\sqrt[3]{4}$ and

$$\left\|A^2\varsigma'\right\|_{L_2(R_+;H)}\le\frac{1}{\sqrt[3]{2}}\left\|P_0\varsigma\right\|_{L_2(R_+;H)}.$$

Similarly, inequality (18) is proved also. The lemma is proved.

Now we will prove the following inequality.

Theorem 2. *For all $u\in W^2_{2,K}(R_+;H)$, the following inequality*

$$\left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)}\le c_j(k)\left\|P_0u\right\|_{L_2(R_+;H)},\ j=\overline{0,3}, \tag{28}$$

where

$$c_0(k) = c_3(k) = \frac{2\sqrt{3}}{3}\left(1+\sqrt{3}\frac{k}{1-k}\right),\ c_1(k) = \frac{1}{\sqrt[3]{2}} + \frac{\sqrt{2}k}{1-k},\ c_2(k) = \frac{1}{\sqrt[3]{2}} + \frac{2k}{1-k}.$$

Proof. $u \in W^3_{2,K}(R_+;H)$ can be represented in the form

$$u(t) = \varsigma(t) + e^{\omega_1 tA}x + e^{\omega_2 tA}y,$$

where $\varsigma(t) \in W^3_{2,0}(R_+;H)$. Then $y = -x$,

$$u''(0) = \omega_1^2 A^2 x + \omega_2^2 A^2 y = K(\varsigma(t)) + K(e^{\omega_1 tA}x + e^{\omega_2 tA}y),$$

or $(E+Q)\,x = A^{-2}K\left(\varsigma(t)\right)$. It is obvious, that $A^{-2}K(\varsigma(t)) \in H_{\frac{5}{2}}$. Then $x = (E+Q)^{-1}\left(A^{-2}K\left(\varsigma(t)\right)\right)$. Hence we get that $\|x\|_{\frac{5}{2}} \le \frac{k}{1-k}\|\varsigma\|_{W_2^3(R_+;H)}$. Thus, from inequality (13) and (16) we obtain:

$$\|u\|_{W_2^3(R_+;H)} \le \|\varsigma\|_{W_2^3(R_+;H)} + \left\|\left(e^{\omega_1 tA} - e^{\omega_2 tA}\right)x\right\|_{W_2^3(R_+;H)} \le \frac{2\sqrt{3}}{3}\|P_0\varsigma\|_{L_2(R_+;H)} + \sqrt{3}\|x\|_{\frac{5}{2}} =$$

$$= \frac{2\sqrt{3}}{3}\|P_0 u\|_{L_2(R_+;H)} + \frac{\sqrt{3}k}{1-k}\|\varsigma\|_{W_2^3(R_+;H)} = \frac{2\sqrt{3}}{3}\left(1+\frac{\sqrt{3}k}{1-k}\right)\|P_0 u\|_{L_2(R_+;H)}.$$

Here we considered that $P_0 u = P_0\varsigma$. Hence, it follows that

$$\left\|A^3 u\right\|_{L_2(R_+;H)} \le \frac{2\sqrt{3}}{3}\left(1+\frac{\sqrt{3}k}{1-k}\right)\|P_0 u\|_{L_2(R_+;H)},$$

$$\|u'''\|_{L_2(R_+;H)} \le \frac{2\sqrt{3}}{3}\left(1+\sqrt{3}\frac{k}{1-k}\right)\|P_0 u\|_{L_2(R_+;H)}.$$

Obviously, that

$$\left\|A^2 u'\right\|_{L_2(R_+;H)} \le \left\|A^2\varsigma'\right\|_{L_2(R_+;H)} + \left\|A^3\omega_1 e^{\omega_1 tA}x - A^3\omega_2 e^{\omega_2 tA}\right\|_{L_2(R_+;H)} \quad (29)$$

Also assume $A^{\frac{5}{2}}x = z$, we have:

$$\left\|A^{\frac{1}{2}}\omega_1 e^{\omega_1 tA}z - A^{\frac{1}{2}}\omega_2 e^{\omega_2 tA}z\right\|^2_{L_2(R_+;H)} \le \left\|A^{\frac{1}{2}}\omega_1 e^{\omega_1 tA}z\right\|^2_{L_2(R_+;H)} + \left\|A^{\frac{1}{2}}\omega_2 e^{\omega_2 tA}z\right\|^2_{L_2(R_+;H)} -$$

$$-2Re\left(\omega_1^2 A e^{2Re\omega_1 tA}z, z\right)_{L_2(R_+;H)} = 2\|z\|^2 - 2Re\omega_1^2\int_0^\infty\int_{\mu_0}^\infty \mu e^{-2t\omega_1\mu}\left(dE_\mu z, z\right)dt =$$

$$= 2\|z\|^2 + 2Re\frac{1}{2\omega_1}\|z\|^2 = \frac{3}{2}\|x\|^2_{\frac{5}{2}} \leq$$

$$\leq \frac{3}{2}\left(\frac{k}{1-k}\right)^2 \|\varsigma\|^2_{W_2^3(R_+;H)} \leq \frac{3}{2}\left(\frac{k}{1-k}\right)^2 \frac{4}{3}\|P_0\varsigma\|^2 = 2\left(\frac{k}{1-k}\right)^2 \|P_0u\|^2 .$$

Using this inequality and inequality (17) in (28), we obtain that

$$\left\|A^2u'\right\|_{L_2(R_+;H)} \leq \left(\frac{1}{\sqrt[3]{2}} + \frac{\sqrt{2}k}{1-k}\right)\|P_0u\|_{L_2(R_+;H)} .$$

Similarly, we have:

$$\|Au''\|_{L_2(R_+;H)} \leq \|A\varsigma''\|_{L_2(R_+;H)} + \left\|\omega_1^2A^3e^{\omega_1tA}x - \omega_2^2e^{\omega_2tx}\right\|_{L_2(R_+;H)} \leq$$

$$\leq \frac{1}{\sqrt[3]{2}}\|P_0u\|_{L_2(R_+;H)} + \left\|\omega_1^2A^{\frac{1}{2}}e^{\omega_1tA}z - \omega_2^2e^{\omega_2tA}z\right\|_{L_2(R_+;H)} .$$

Since

$$\left\|\omega_1^2e^{\omega_1tA}z - \omega_2^2e^{\omega_2tA}z\right\|^2_{L_2(R_+;H)} = 2\|z\|^2 - 2Re\left(A\omega_1e^{\omega_1tA}z, z\right)_{L_2(R_+;H)} =$$

$$= 2\|z\|^2 - 2Re\omega_1\left(e^{\omega_1tA}z,\ z\right)_{L_2(R_+;H)} = 2\|z\|^2 - 2Re\omega_1\left(-\frac{1}{2\omega_1}\right)\|z\|^2 =$$

$$= 3\|z\|^2 = 3\|x\|^2_{\frac{5}{2}} \leq 3\left(\frac{k}{1-k}\right)^2 \|\varsigma\|^2_{W_2^3(R_+;H)} \leq 4\left(\frac{k}{1-k}\right)^2 \|P_0\varsigma\|^2 .$$

Thus,

$$\|Au''\|_{L_2(R_+;H)} \leq \left(\frac{1}{\sqrt[3]{2}} + \frac{2k}{1-k}\right)\|P_0u\|_{L_2(R_+;H)} .$$

The theorem is proved.

Now we will prove the main theorem.

Theorem 3. *Let the conditions 1) - 3) be satisfied, $k < 1$ and*

$$\alpha(k) = \sum_{j=0}^{3} c_j(k)\|B_{3-j}\| < 1,$$

where the coefficients $c_j(k)$ are defined from Theorem 2. Then problem (1), (2) *is regularly solvable.*

Proof. Problem (1), (2) can be written in the form of the operator equation $Pu = P_0u + P_1u = f$, where $f \in L_2(R_+; H)$, $u \in W^3_{2,K}(R_+; H)$. After the replacement $\nu = P_0u$, we obtain the equation $v + P_1P_0^{-1}v = f$ in the space $L_2(R_+; H)$. Using Theorem 2, we have:

$$\|P_1P_0v\|_{L_2(R_+;H)} = \|P_1u\|_{L_2(R_+;H)} \leq \sum_{j=0}^{3} \|B_{3-j}\| \left\|A^{3-j}u^{(j)}\right\|_{L_2(R_+;H)} \leq$$

$$\leq \left(\sum_{j=0}^{3} c_j(k) \|B_{3-j}\|\right) \|P_0u\|_{L_2(R_+;H)} = \alpha(k) \|v\|_{L_2(R_+;H)} .$$

It follows from the condition $\alpha(k) < 1$ that the operator $E + P_1P_0^{-1}$ is invertible in $L_2(R_+; H)$, therefore $u = P_0^{-1}(E + P_1P_0^{-1})^{-1}f$ and $\|u\|_{W_2^2(R_+;H)} \leq const \|f\|_{L_2(R_+;H)}$.

The theorem is proved.

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