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A Note on the Twisted λ-Daehee Polynomials

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Abstract. In this paper, we consider the twisted λ-Daehee polynomials which are derived from multivariate Volkenborn integral on $\mathbb{Z}_p$ and investigate several properties related to those polynomials.

1. Introduction

Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic numbers and the completion of algebraic closure of $\mathbb{Q}_p$. For $n \in \mathbb{Z}_{\geq 0}$, let $C_{p^n} = \left\{ \zeta \in \mathbb{C}_p | \, \zeta^{p^n} = 1 \right\}$. Then the space of locally constant functions is defined by $T_p = \lim_{n \to \infty} C_{p^n} = \bigcup_{n \geq 0} C_{p^n}$.

The twisted Bernoulli polynomials of order $r (\in \mathbb{N})$ are defined by the generating function to be

$$\left(\frac{t}{\zeta e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_{n,\zeta}^{(r)}(x) \frac{t^n}{n!}, \qquad (\zeta \in T_p). \tag{1}$$

When $x = 0$, $B_{n,\zeta}^{(r)} = B_{n,\zeta}^{(r)}(0)$ are called the twisted Bernoulli numbers of order r.

In particular, for $r = 1$, $B_{n,\zeta} = B_{n,\zeta}^{(1)}$ are called the twisted Bernoulli numbers, (see [3, 6, 10, 13, 18]).

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the Volkenborn integral on $\mathbb{Z}_p$ is defined by

$$I(f) = \int_{\mathbb{Z}_p} f(x) \, d\mu(x) = \lim_{n \to \infty} \frac{1}{p^n} \sum_{x=0}^{p^n - 1} f(x), \tag{2}$$

(see [7]).

From (2), we have

$$I(f_1) = I(f) + f'(0), \tag{3}$$

(see [6, 8, 10, 18]), where $f_1(x) = f(x+1)$.

The Stirling number of the first kind is defined by the falling factorial sequence to be

$$(x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^{n} S_1(n,l) \, x^l, \quad (n \geq 0). \tag{4}$$

The Stirling number of the second kind is defined by the generating function to be

$$\left(e^{t}-1\right)^{n}=n!\sum_{l=n}^{\infty}S_{2}\left(l,n\right)\frac{t^{l}}{l!},\tag{5}$$

(see [6]).

In this paper, we consider the twisted λ-Daehee polynomials which are derived from multivariate Volkenborn integral on $\mathbb{Z}_p$ and investigate several properties of those polynomials.

2. Twisted λ-Daehee polynomials

In this section, we assume that $\lambda\in\mathbb{Z}_p$. Let us consider the twisted λ-Daehee polynomials of order $k(\in\mathbb{N})$:

$$\left(\frac{\lambda\log\left(1+\zeta t\right)}{\left(1+\zeta t\right)^{\lambda}-1}\right)^{k}\left(1+\zeta t\right)^{x}=\sum_{n=0}^{\infty}D_{n,\zeta}^{(k)}\left(x|\lambda\right)\frac{t^{n}}{n!},\tag{6}$$

where $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$ and $\zeta\in T_p$.

When $x=0$, $D_{n,\zeta}^{(k)}\left(\lambda\right)=D_{n,\zeta}^{(k)}\left(0|\lambda\right)$ are called the twisted Daehee numbers of order k.

Let $f\left(x\right)=\left(1+\zeta t\right)^{\lambda x}$ where $\zeta\in T_p$, $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$. By (3), we get

$$\int_{\mathbb{Z}_p}\left(1+\zeta t\right)^{\lambda x}d\mu\left(x\right)=\frac{\lambda\log\left(1+\zeta t\right)}{\left(1+\zeta t\right)^{\lambda}-1}.\tag{7}$$

From (7), we have

$$\begin{aligned}&\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k\text{ times}}\left(1+\zeta t\right)^{\lambda x_1+\cdots+\lambda x_k+x}d\mu\left(x_1\right)d\mu\left(x_2\right)\cdots d\mu\left(x_k\right)\\&=\left(\frac{\lambda\log\left(1+\zeta t\right)}{\left(1+\zeta t\right)^{\lambda}-1}\right)^{k}\left(1+\zeta t\right)^{x}\\&=\sum_{n=0}^{\infty}D_{n,\zeta}^{(k)}\left(x|\lambda\right)\frac{t^{n}}{n!},\end{aligned}\tag{8}$$

and

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (1+\zeta t)^{\lambda x_1+\cdots+\lambda x_k+x}\, d\mu(x_1)\, d\mu(x_2)\cdots d\mu(x_k) \\ =& \sum_{n=0}^{\infty} \zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1+\cdots+\lambda x_k+x)_n\, d\mu(x_1)\cdots d\mu(x_k)\, \frac{t^n}{n!}. \end{aligned} \tag{9}$$

By (8) and (9), we get

$$D_{n,\zeta}^{(k)}(x|\lambda) = \zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1+\cdots+\lambda x_k+x)_n\, d\mu(x_1)\cdots d\mu(x_k). \tag{10}$$

Now, we observe that

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} e^{(\lambda x_1+\cdots+\lambda x_k+x)t} d\mu(x_1)\cdots d\mu(x_k) \\ =& \left(\frac{\lambda t}{e^{\lambda t}-1}\right)^k e^{xt} \\ =& \sum_{n=0}^{\infty} \lambda^n B_n^{(k)}\left(\frac{x}{\lambda}\right) \frac{t^n}{n!}. \end{aligned} \tag{11}$$

From (11), we have

$$\begin{aligned} & \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1+\cdots+\lambda x_k+x)^n\, d\mu(x_1)\cdots d\mu(x_k) \\ =& \lambda^n B_n^{(k)}\left(\frac{x}{\lambda}\right). \end{aligned} \tag{12}$$

By (10) and (12), we get

$$\begin{aligned} D_{n,\zeta}^{(k)}(x|\lambda) &= \zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1+\cdots+\lambda x_k+x)_n\, d\mu(x_1)\cdots d\mu(x_k) \\ &= \zeta^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (\lambda x_1+\cdots+\lambda x_k+x)^l\, d\mu(x_1)\cdots d\mu(x_k) \\ &= \zeta^n \sum_{l=0}^{n} S_1(n,l)\, \lambda^l B_l^{(k)}\left(\frac{x}{\lambda}\right). \end{aligned} \tag{13}$$

In (6), by replacing t by $e^t - \frac{1}{\zeta}$, we get

$$\begin{aligned} & \left(\frac{\lambda t}{\zeta^{\lambda} e^{\lambda t} - 1}\right)^k \left(\zeta e^t\right)^x \qquad (14) \\ = & \sum_{n=0}^{\infty} D_{n,\zeta}^{(k)}(x|\lambda) \frac{1}{n!} \left(e^t - \frac{1}{\zeta}\right)^n \\ = & \sum_{n=0}^{\infty} D_{n,\zeta}^{(k)}(x|\lambda) \frac{1}{n!} \zeta^{-n} \left(\zeta e^t - 1\right)^n \\ = & \sum_{n=0}^{\infty} D_{n,\zeta}^{(k)}(x|\lambda) \frac{1}{n!} \zeta^{-n} n! \sum_{m=n}^{\infty} S_2(m,n) \frac{(\log \zeta + t)^m}{m!} \\ = & \sum_{m=0}^{\infty} \left(\sum_{n=0}^{m} D_{n,\zeta}^{(k)}(x|\lambda) \zeta^{-n} S_2(m,n) \right) \frac{t^m}{m!}, \end{aligned}$$

and

$$\left(\frac{\lambda t}{\zeta^{\lambda} e^{\lambda t} - 1}\right)^k \left(\zeta e^t\right)^x = \zeta^x \sum_{m=0}^{\infty} \lambda^m B_{m,\zeta^{\lambda}}^{(k)} \left(\frac{x}{\lambda}\right) \frac{t^m}{m!}. \qquad (15)$$

Thus, by (14) and (15), we get

$$\lambda^m B_{m,\zeta\lambda}^{(k)} \left(\frac{x}{\lambda}\right) = \sum_{n=0}^{m} D_{n,\zeta}^{(k)}(x|\lambda) \zeta^{-n-x} S_2(m,n). \qquad (16)$$

Therefore, by (13) and (16), we obtain the following theorem.

Theorem 1. *For $m \in \mathbb{Z}_{\geq 0}$, we have*

$$\zeta^{-m} D_{m,\zeta}^{(k)}(x|\lambda) = \sum_{l=0}^{m} S_1(m,l) \lambda^l B_l \left(\frac{x}{\lambda}\right),$$

and

$$\lambda^m B_{m,\zeta^{\lambda}}^{(k)} \left(\frac{x}{\lambda}\right) = \sum_{n=0}^{m} D_{n,\zeta}^{(k)}(x|\lambda) \zeta^{-n-x} S_2(m,n).$$

The rising factorial sequence is defined by

$$\begin{aligned} x^{(n)} &= x(x+1)\cdots(x+n-1) = (-1)^n (-x)_n \\ &= \sum_{l=0}^{n} (-1)^{n-l} S_1(n,l) x^l, \end{aligned}$$

where $n \in \mathbb{Z}_{\geq 0}$.

Let us define the twisted λ-Daehee polynomials of the second kind with order $k (\in \mathbb{N})$:

$$\left(\frac{\lambda\log\left(1+\zeta t\right)\left(1+\zeta t\right)^{\lambda}}{\left(1+\zeta t\right)^{\lambda}-1}\right)^{k}\left(1+\zeta t\right)^{x}=\sum_{n=0}^{\infty}\hat{D}_{n,\zeta}^{(k)}\left(x|\lambda\right)\frac{t^{n}}{n!}, \tag{17}$$

where $t\in\mathbb{C}_p$ with $|t|_p<p^{-\frac{1}{p-1}}$ and $\zeta\in T_p$.

Let us take $f\left(x\right)=\frac{1}{(1+\zeta t)^{\lambda x}}$. By (3), we get

$$\int_{\mathbb{Z}_p}\left(1+\zeta t\right)^{-\lambda x}d\mu\left(x\right)=\frac{\left(1+\zeta t\right)^{\lambda}}{\left(1+\zeta t\right)^{\lambda}-1}\left(\lambda\log\left(1+\zeta t\right)\right). \tag{18}$$

By (18), we get

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\zeta t\right)^{-(\lambda x_1+\cdots+\lambda x_k)+x}d\mu\left(x_1\right)\cdots d\mu\left(x_k\right)\\&=\left(\frac{\lambda\log\left(1+\zeta t\right)}{\left(\zeta t+1\right)^{\lambda}-1}\left(\zeta t+1\right)^{\lambda}\right)^{k}\left(1+\zeta t\right)^{x}\\&=\sum_{n=0}^{\infty}\hat{D}_{n,\zeta}^{(k)}\left(x|\lambda\right)\frac{t^{n}}{n!},\end{aligned} \tag{19}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(1+\zeta t\right)^{-(\lambda x_1+\cdots+\lambda x_k)+x}d\mu\left(x_1\right)\cdots d\mu\left(x_k\right)\\&=\sum_{n=0}^{\infty}\zeta^{n}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(-\left(\lambda x_1+\cdots+\lambda x_k\right)+x\right)_{n}d\mu\left(x_1\right)\cdots d\mu\left(x_k\right)\frac{t^{n}}{n!}.\end{aligned} \tag{20}$$

Thus, by (19) and (20), we get

$$\begin{aligned}&\zeta^{n}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(-\lambda\left(x_1+\cdots+x_k\right)+x\right)_{n}d\mu\left(x_1\right)\cdots d\mu\left(x_k\right)\\&=\hat{D}_{n,\zeta}^{(k)}\left(x|\lambda\right),\quad\left(n\in\mathbb{Z}_{\geq 0}\right).\end{aligned} \tag{21}$$

From (4), we have

$$\zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(-\lambda\left(x_1+\cdots+x_k\right)+x\right)_n d\mu\left(x_1\right)\cdots d\mu\left(x_k\right) \tag{22}$$

$$=\zeta^n \sum_{l=0}^{n} S_1(n,l) \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(-\lambda\left(x_1+\cdots+x_k\right)+x\right)^l d\mu\left(x_1\right)\cdots d\mu\left(x_k\right)$$

$$=\zeta^n \sum_{l=0}^{n} S_1(n,l)(-1)^l \lambda^l \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(x_1+\cdots+x_k-\frac{x}{\lambda}\right)^l d\mu\left(x_1\right)\cdots d\mu\left(x_k\right)$$

$$=\zeta^n \sum_{l=0}^{n} S_1(n,l)(-1)^l \lambda^l B_l^{(k)}\left(-\frac{x}{\lambda}\right).$$

We observe that

$$\left(\frac{t}{e^t-1}\right)^k e^{(k-x)t} = \left(\frac{-t}{e^{-t}-1}\right)^k e^{-xt} \tag{23}$$

$$= \sum_{n=0}^{\infty} (-1)^n B_n^{(k)}(x)\frac{t^n}{n!}.$$

Thus, by (23), we get

$$B_n^{(k)}(k-x) = (-1)^n B_n^{(k)}(x). \tag{24}$$

From (22) and (24), we have

$$\zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left(-\lambda\left(x_1+\cdots+x_k\right)+x\right)_n d\mu\left(x_1\right)\cdots d\mu\left(x_k\right) \tag{25}$$

$$=\zeta^n \sum_{l=0}^{n} S_1(n,l)\lambda^l B_l^{(k)}\left(k+\frac{x}{\lambda}\right).$$

By (21) and (24), we get

$$\hat{D}_{n,\zeta}^{(k)}(x|\lambda) = \zeta^n \sum_{l=0}^{n} S_1(n,l)\lambda^l B_l^{(k)}\left(k+\frac{x}{\lambda}\right). \tag{26}$$

In (17), by replacing t by $e^t-\frac{1}{\zeta}$, we get

$$\left(\frac{\lambda t\zeta^{\lambda}e^{\lambda t}}{\zeta^{\lambda}e^{\lambda t}-1}\right)^k \left(\zeta e^t\right)^x = \sum_{n=0}^{\infty} \hat{D}_{n,\zeta}^{(k)}(x|\lambda)\frac{1}{n!}\left(e^t-\frac{1}{\zeta}\right)^n \tag{27}$$

$$= \sum_{n=0}^{\infty} \hat{D}_{n,\zeta}^{(k)}(x|\lambda)\zeta^{-n}\sum_{m=n}^{\infty} S_2(m,n)\frac{(\log\zeta+t)^m}{m!}$$

$$= \sum_{m=0}^{\infty}\left(\sum_{n=0}^{m} \hat{D}_{n,\zeta}^{(k)}(x|\lambda)S_2(m,n)\zeta^{-n}\right)\frac{t^m}{m!},$$

and

$$\begin{aligned}\left(\frac{\lambda t\zeta^{\lambda}e^{\lambda t}}{\zeta^{\lambda}e^{\lambda t}-1}\right)^{k}\left(\zeta e^{t}\right)^{x} &= \left(\frac{\lambda t}{\zeta^{\lambda}e^{\lambda t}-1}\right)^{k}e^{(\lambda k+x)t}\zeta^{\lambda k+x} \\ &= \zeta^{\lambda k+x}\sum_{m=0}^{\infty}\lambda^{m}B_{m,\zeta\lambda}^{(k)}\left(k+\frac{x}{\lambda}\right)\frac{t^{m}}{m!}.\end{aligned} \tag{28}$$

By (27) and (28), we get

$$\lambda^{m}B_{m,\zeta\lambda}^{(k)}\left(k+\frac{x}{\lambda}\right)=\sum_{n=0}^{m}\hat{D}_{n,\zeta}^{(k)}\left(x|\lambda\right)S_{2}\left(m,n\right)\zeta^{-n-\lambda k-x}. \tag{29}$$

Therefore, by (26) and (29), we obtain the following theorem.

Theorem 2. *For $m\in\mathbb{Z}_{\geq0}$, we have*

$$\zeta^{-m}\hat{D}_{m,\zeta}^{(k)}\left(x|\lambda\right)=\sum_{l=0}^{m}S_{1}\left(m,l\right)\lambda^{l}B_{l}^{(k)}\left(k+\frac{x}{\lambda}\right),$$

and

$$\lambda^{m}B_{m,\zeta\lambda}^{(k)}\left(k+\frac{x}{\lambda}\right)=\sum_{n=0}^{m}\hat{D}_{n,\zeta}^{(k)}\left(x|\lambda\right)S_{2}\left(m,n\right)\zeta^{-n-\lambda k-x}.$$

Now, we observe that

$$\begin{aligned}&\left(-1\right)^{n}\frac{D_{n,\zeta}^{(k)}\left(x|\lambda\right)}{n!} \\ =&\zeta^{n}\left(-1\right)^{n}\int_{\mathbb{Z}_{p}}\cdots\int_{\mathbb{Z}_{p}}\binom{x+\lambda x_{1}+\cdots+\lambda x_{k}}{n}d\mu\left(x_{1}\right)\cdots d\mu\left(x_{k}\right) \\ =&\zeta^{n}\int_{\mathbb{Z}_{p}}\cdots\int_{\mathbb{Z}_{p}}\binom{-\left(\lambda x_{1}+\cdots+\lambda x_{k}\right)-x+n-1}{n}d\mu\left(x_{1}\right)\cdots d\mu\left(x_{k}\right) \\ =&\zeta^{n}\sum_{m=0}^{n}\binom{n-1}{n-m}\int_{\mathbb{Z}_{p}}\cdots\int_{\mathbb{Z}_{p}}\binom{-\left(\lambda x_{1}+\cdots+\lambda x_{k}\right)-x}{m}d\mu\left(x_{1}\right)\cdots d\mu\left(x_{k}\right) \\ =&\zeta^{n}\sum_{m=0}^{n}\frac{\binom{n-1}{n-m}}{m!}m!\int_{\mathbb{Z}_{p}}\cdots\int_{\mathbb{Z}_{p}}\binom{-\left(\lambda x_{1}+\cdots+\lambda x_{k}\right)-x}{m}d\mu\left(x_{1}\right)\cdots d\mu\left(x_{k}\right) \\ =&\sum_{m=1}^{n}\zeta^{n-m}\binom{n-1}{m-1}\frac{\hat{D}_{m,\zeta}^{(k)}\left(-x|\lambda\right)}{m!}\end{aligned} \tag{30}$$

and

$$(-1)^n \frac{\hat{D}_{n,\zeta}^{(k)}(x|\lambda)}{n!} \tag{31}$$
$$= (-1)^n \zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{-(\lambda x_1 + \cdots + \lambda x_k) + x}{n} d\mu(x_1) \cdots d\mu(x_k)$$
$$= \zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\lambda x_1 + \cdots + \lambda x_k - x + n - 1}{n} d\mu(x_1) \cdots d\mu(x_k)$$
$$= \sum_{m=0}^{n-1} \binom{n-1}{n-m} \zeta^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{\lambda x_1 + \cdots + \lambda x_k - x}{m} d\mu(x_1) \cdots d\mu(x_k)$$
$$= \sum_{m=1}^{n} \zeta^{n-m} \binom{n-1}{m-1} \frac{D_{m,\zeta}^{(k)}(-x|\lambda)}{m!}.$$

Therefore, by (30) and (31), we obtain the following theorem.

References

[1] For $n \in \mathbb{N}$, we have

$$(-1)^n \frac{D_{n,\zeta}^{(k)}(x|\lambda)}{n!} = \sum_{m=1}^{n} \zeta^{n-m} \binom{n-1}{m-1} \frac{\hat{D}_{m,\zeta}^{(k)}(-x|\lambda)}{m!},$$

and

$$(-1)^n \frac{\hat{D}_{n,\zeta}^{(k)}(x|\lambda)}{n!} = \sum_{m=1}^{n} \zeta^{n-m} \binom{n-1}{m-1} \frac{D_{m,\zeta}^{(k)}(-x|\lambda)}{m!}.$$

Remark. Recently, several authors have studied twisted special polynomials related to Bernoulli, Euler and Genocchi polynomials (see [1-18]).

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