Hermite-Hadamard-like Type Integral Inequalities for Functions whose Derivatives of $n$-th Order are Preinvex

Jaekeun Park

Department of Mathematics
Hanseo University, Daegok-ri, Seosan-si
Choongchungnam-do, 356-706, Korea
jkpark@hanseo.ac.kr

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Abstract

In this article, we establish some estimates of Hermite-Hadamard-like type integral inequalities for functions whose $n$-times derivatives in absolute value at certain powers are preinvex.

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1 Introduction

The following definition is well known in the literature: Let $I$ be an interval in $R$. Then $f : I \to R$ is said to be convex on $I$ if

\[ f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \]

holds for all $x, y \in I$ and $t \in [0, 1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-hadamard inequality, due to its rich geometrical significance and applications, which is stated as follows: Let $f : I \subseteq R \to R$ be a
convex function and \(a, b \in I\) with \(a < b\). Then
\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_a^b f(x) \, dt \leq \frac{f(a) + f(b)}{2}.
\]

Hadamard’s inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found in [7, 9, 11, 12, 16] and references therein.

In recent years, several refinements and generalizations have been considered for classical convexity [16]. A significant generalization of convex functions is that of invex functions introduced by Hanson in [6].

Weir and Mond [17] introduced the concept of preinvex functions and applied it to the establishment of the sufficient optimality conditions and duality in nonlinear programming.


Noor [12] has established some Hermite-Hadamard type inequalities for preinvex functions. In recent papers, Noor and Barani et al. in [5, 12] presented some estimates of the right hand side of a Hermite-Hadamard type inequality in which some preinvex functions are involved.

**Definition 1.** A set \(K \subseteq R\) is said to be **invex** with respect to the map \(\eta : K \times K \to R\), if for any \(x, y \in K\) and \(t \in [0, 1]\), \(x + t\eta(y, x) \in K\).

It is obvious that every convex set is invex with respect to the map \(\eta(x, y) = y - x\), but there exist invex sets which are not convex [10].

**Definition 2.** Let \(K \subseteq R\) be an invex set with respect to the map \(\eta : K \times K \to R\). Then the function \(f : K \to R\) is said to be **preinvex** with respect to \(\eta\), if
\[
f(a + t\eta(b, a)) \leq (1 - t)f(a) + tf(b)
\]
for any \(a, b \in K\) and \(t \in [0, 1]\).

For the refinement and generalizations of preinvex functions, you may see [1, 8, 10, 14, 16, 18, 19]. Recently, Noor [3, 4, 12] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

**Theorem 1.1.** For an interval \([a, a + \eta(b, a)]\) on the real line \(R\), let \(f : [a, a + \eta(b, a)] \to R_+\) be a preinvex function on an interior \(K^0\) of the interval \(K\) and \(a, b \in K^0\) with \(a < a + \eta(b, a)\). Then the inequality holds:
\[
f\left(a + \frac{\eta(b, a)}{2}\right) \leq \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(t) \, dt \leq \frac{f(a) + f(a + \eta(b, a))}{2}.
\]
In [5], Barani et al. introduced some generalizations of Hermite-Hadamard type inequality for functions whose second derivatives absolute values are preinvex.

**Theorem 1.2.** Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $|f'|$ is preinvex on $K$, then for any $a, b \in K$ with $\eta(b, a) \neq 0$ the following inequality holds:

$$
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(t) dt \right|
\leq \frac{\eta(b, a)}{8} \left\{ |f'(a)| + |f'(a)| \right\}.
$$

(1)

**Theorem 1.3.** Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^\frac{p}{p-1}$ is preinvex on $K$, then for any $a, b \in K$ with $\eta(b, a) \neq 0$ the following inequality holds:

$$
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(t) dt \right|
\leq \frac{\eta(b, a)}{2(p + 1)^\frac{p}{p-1}} \left[ \frac{1}{2} \left\{ |f'(a)|^\frac{p}{p-1} + |f'(b)|^\frac{p}{p-1} \right\} \frac{p-1}{p} \right].
$$

(2)

**Theorem 1.4.** Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^\frac{p}{p-1}$ is preinvex on $K$, then for any $a, b \in K$ with $\eta(b, a) \neq 0$ the following inequality holds:

$$
\left| \frac{f(a) + f(a + \eta(b, a))}{2} - \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(t) dt \right|
\leq \frac{\eta(b, a)}{2(p + 1)^\frac{p}{p-1}} \left[ \sup \left\{ |f'(a)|^\frac{p}{p-1}, |f'(b)|^\frac{p}{p-1} \right\} \right]^\frac{p-1}{p}.
$$

(3)

The main aim of this paper is to establish new generalized similar inequalities concerning Hermite-Hadamard-like and Simpson-like type inequality for the class of differentiable functions whose $n$-times derivatives at certain powers are preinvex functions.
2 Some new Hermite-Hadamard-type inequalities

To establish some new Hermite-Hadamard type inequalities for $s$-convex functions in the second sense, we need the following lemma.

**Lemma 1.** Let $K \subseteq R$ be an invex set with respect to the map $\eta : K \times K \to R$ and, $\eta(b,a) \neq 0$ with $0 \leq a \leq a + \eta(b,a) < \infty$ for all $a,b \in K$ with $a < b$. Suppose that $f : K \to R$ is an $n$-times differentiable function on the interior $K^0$ of $K$ such that $f^{(n)}(\cdot) \in L([a,a + \eta(b,a)])$. If $|f^{(n)}|$ is preinvex with respect to $\eta$ on $K$, then the following identity

$$R^n_a(f, \eta, n) = \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(u)du - \frac{1}{2} \left\{ f(r) + f(2a + \eta(b,a) - r) \right\}$$

$$+ \sum_{k=2}^n \frac{(-1)^{k+1}}{k!} \left\{ \frac{(r - a - \eta(b,a))^k - (r - a - \frac{1}{2}\eta(b,a))^k}{\eta(b,a)} \right\}$$

$$\times \left\{ f^{(k-1)}(r) + (-1)^{k+1} f^{(k-1)}(2a + \eta(b,a) - r) \right\}$$

$$= \frac{\eta^n(b,a)}{n!} \int_0^1 k_n(t)f^{(n)}(a + t\eta(b,a))dt,$$  \hspace{1cm} (4)

holds for any $r \in [a + \frac{1}{2}\eta(b,a), a + \eta(b,a)]$ with $\eta(b,a) > 0$, where

$$k_n(t) = \begin{cases} t^n, & 0 \leq t \leq \frac{a + \eta(b,a) - r}{\eta(b,a)} \\ (t - \frac{1}{2})^n, & \frac{a + \eta(b,a) - r}{\eta(b,a)} \leq t \leq \frac{r - a}{\eta(b,a)} \\ (t - 1)^n, & \frac{r - a}{\eta(b,a)} \leq t \leq 1. \end{cases}$$

**Proof.** By integration by parts, this equality (4) is proved by the mathematical induction. $\Box$

Now we turn our attention to establish inequalities of Hermit-Hadamard type for differentiable preinvex functions.

**Theorem 2.1.** Let $K \subseteq R$ be an invex set with respect to the map $\eta : K \times K \to R$ and, $\eta(b,a) \neq 0$ with $0 \leq a \leq a + \eta(b,a) < \infty$ for all $a,b \in K$ with $a < b$. Suppose that $f : K \to R$ is an $n$-times differentiable function on the interior $K^0$ of $K$ such that $f^{(n)}(\cdot) \in L([a,a + \eta(b,a)])$. If $|f^{(n)}|$ is preinvex with respect to $\eta$ on $K$, then the following inequality

$$\left| R^n_a(f, \eta, n) \right| \leq \frac{\eta^n(b,a)}{n!} \left\{ \mu_{11} + \mu_{12} + \mu_{13} + \mu_{14} \right\} \left\{ |f^{(n)}(a)| + |f^{(n)}(b)| \right\},$$  \hspace{1cm} (5)
holds for any \( r \in [a + \frac{1}{2} \eta(b, a), a + \eta(b, a)] \) with \( \eta(b, a) > 0 \), where

\[
\mu_{11} = \frac{(a + \eta(b, a) - r)^{n+1}}{\eta^{n+2}(b, a)(n+1)(n+2)} \{ (n+1)(r-a) \},
\]

\[
\mu_{12} = \frac{(r - a - \frac{1}{2} \eta(b, a))^n}{\eta^{n+2}(b, a)(n+1)(n+2)} \{ \frac{1}{2}(2n+3) \eta(b, a) - (n+1)(r-a) \},
\]

\[
\mu_{13} = \frac{(r - a - \frac{1}{2} \eta(b, a))^n}{\eta^{n+2}(b, a)(n+1)(n+2)} \{ \frac{1}{2}(2n+3) \eta(b, a) - (n+1)(r-a) \},
\]

\[
\mu_{14} = \frac{a + \eta(b, a) - r}{\eta^{n+2}(b, a)(n+2)},
\]

(6)

**Proof.** Suppose that \( a, a + \eta(b, a) \in S \). Since \( S \) is an invex set with respect to \( \eta \), we have \( a + t \eta(b, a) \in S \) for any \( t \in [0, 1] \). By Lemma 1, we have

\[
|I_a^b(f, \eta, n)|
\]

\[
= \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(u)du - \frac{f(r) + f(2a + \eta(b, a) - r)}{2}
\]

\[
+ \frac{1}{\eta(b, a)} \sum_{k=2}^{n} (-1)^{k+1} \frac{k!}{k!} \left\{ (r - a - \eta(b, a))^k - (r - a - \frac{\eta(b, a)}{2})^k \right\}
\]

\[
\times \left\{ f^{(k-1)}(r) + (-1)^{k+1} f^{(k-1)}(2a + \eta(b, a) - r) \right\}
\]

\[
= \frac{\eta^n(b, a)}{n!} \int_0^1 \left| k_n(t) \right| \left| f^{(n)}(a + t \eta(b, a)) \right| dt
\]

\[
\leq \frac{\eta^n(b, a)}{n!} \left[ \int_0^{a+\eta(b,a)-r} \eta^{n-1}(b, a) t^n f^{(n)}(a + t \eta(b, a)) dt \right]
\]

\[
+ \int_{\frac{a+\eta(b,a)-r}{\eta(b,a)}}^{\frac{1}{2}} (t - \frac{1}{2})^n f^{(n)}(a + t \eta(b, a)) dt
\]

\[
+ \int_{\frac{1}{2}}^{\frac{a+\eta(b,a)}{\eta(b,a)}} (t - \frac{1}{2})^n f^{(n)}(a + t \eta(b, a)) dt
\]

\[
+ \int_{\frac{a+\eta(b,a)}{\eta(b,a)}}^{1} (1 - t)^n f^{(n)}(a + t \eta(b, a)) dt.
\]

(7)
Making use of the preinvexity of $|f^{(n)}|$ on $[a, a + \eta(b, a)]$ for $n \in N$, we get

\[(a) \int_0^{\frac{a + \eta(b, a) - r}{\eta(b, a)}} t^n \left| f^{(n)}(a + t\eta(b, a)) \right| dt \leq \int_0^{\frac{a + \eta(b, a) - r}{\eta(b, a)}} t^n \left\{ (1 - t)\left| f^{(n)}(a) \right| + t\left| f^{(n)}(b) \right| \right\} dt \]

\[= \mu_{11} |f^{(n)}(a)| + \nu_{11} |f^{(n)}(b)|. \tag{8} \]

\[(b) \int_{\frac{a + \eta(b, a) - r}{\eta(b, a)}}^{\frac{1}{2}} \left( \frac{1}{2} - t \right)^n \left| f^{(n)}(a + t\eta(b, a)) \right| dt \leq \mu_{12} |f^{(n)}(a)| + \nu_{12} |f^{(n)}(b)|. \tag{9} \]

\[(c) \int_{\frac{1}{2}}^{\frac{1}{2} - \frac{r}{\eta(b, a)}} \left( t - \frac{1}{2} \right)^n \left| f^{(n)}(a + t\eta(b, a)) \right| dt \leq \mu_{13} |f^{(n)}(a)| + \nu_{13} |f^{(n)}(b)|. \tag{10} \]

\[(d) \int_{\frac{1}{2} - \frac{r}{\eta(b, a)}}^{1} (1 - t)^n \left| f^{(n)}(a + t\eta(b, a)) \right| dt \leq \mu_{14} |f^{(n)}(a)| + \nu_{14} |f^{(n)}(b)|. \tag{11} \]

By substituting (8)-(11) in (7), we get the desired result (5). \hfill \square

**Corollary 2.1.** Under the assumptions in Theorem 2.1 with $r = a + \eta(b, a)$, we have

\[
\left| I_a^b(f, \eta, n) \right| = \left| \frac{1}{\eta(b, a)} \int_a^{a + \eta(b, a)} f(u) du - \frac{f(a) + f(a + \eta(b, a))}{2} \right|

+ \sum_{k=2}^{n} \frac{(-1)^k \eta^{k-1}(b, a)}{2^k k!} \left\{ f^{(k-1)}(a + \eta(b, a)) + (-1)^k f^{(k-1)}(a) \right\}

\leq \frac{\eta^n(b, a)}{2^{n+1} (n+1)!} \left\{ \left| f^{(n)}(a) \right| + \left| f^{(n)}(b) \right| \right\}.
\]
Theorem 2.2. Let $K \subseteq R$ be an invex set with respect to the map $\eta : K \times K \to R$ and, $\eta(b, a) \neq 0$ with $0 \leq a \leq a + \eta(b, a) < \infty$ for all $a, b \in S$ with $a < b$. Suppose that $f : K \to R$ is an $n$-times differentiable function on the interior $K^0$ of $K$ such that $f^{(n)} \in L([a, a + \eta(b, a)])$. Assume $q \in R$ with $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $| f^{(n)} |^q$ is preinvex with respect to $\eta$ on $K$, then the following inequality holds for any $r \in [a + \frac{1}{2}\eta(b, a), a + \eta(b, a)]$ with $\eta(b, a) > 0$, where

$$
| I_a^b(f, \eta, n) | \leq \frac{\eta^n(b, a)}{n!} \left[ M_{11}^{\frac{1}{p}} \left( \mu_{21} f^{(n)}(a) + \mu_{24} f^{(n)}(b) \right)^q + M_{12}^{\frac{1}{p}} \left( \mu_{22} f^{(n)}(a) + \mu_{23} f^{(n)}(b) \right)^q \right]^\frac{1}{q},
$$

where

$$
\mu_{21} = \frac{\eta^2(b, a) - (r - a)^2}{2\eta^2(b, a)},
$$

$$
\mu_{22} = \frac{4(r - a)^2 - \eta^2(b, a)}{8\eta^2(b, a)},
$$

$$
\mu_{23} = \frac{\eta^2(b, a) - 4(r - (a + \eta(b, a)))^2}{8\eta^2(b, a)},
$$

$$
\mu_{24} = \frac{(a + \eta(b, a) - r)^2}{2\eta^2(b, a)},
$$

and

$$
M_{11} = \frac{1}{np + 1} \left( \frac{a + \eta(b, a) - r}{\eta(b, a)} \right)^{np+1},
$$

$$
M_{12} = \frac{1}{np + 1} \left( \frac{r - (a + \frac{1}{2}\eta(b, a))}{\eta(b, a)} \right)^{np+1}.
$$

Proof. Suppose that $a, a + \eta(b, a) \in K$. Since $K$ is invex with respect to $\eta$, for any $t \in [0, 1]$, we have $a + t\eta(b, a) \in S$. By Lemma 1 and Hölder integral inequality for $q > 1$, we have

$$
\frac{n!}{\eta^n(b, a)} | I_a^b(f, \eta, n) |
$$
\[\begin{align*}
&\leq \int_0^a \left( \frac{a + \eta(b,a) - r}{\eta(b,a)} \right)^r t^n \left| f^{(n)}(a + t\eta(b,a)) \right| dt \\
&+ \int_0^{\frac{1}{2}} \left( \frac{1}{2} - t \right)^n \left| f^{(n)}(a + t\eta(b,a)) \right| dt \\
&+ \int_{\frac{1}{2}}^1 (t - 1)^n \left| f^{(n)}(a + t\eta(b,a)) \right| dt \\
&+ \int_{\frac{1}{2}}^1 (1 - t)^n \left| f^{(n)}(a + t\eta(b,a)) \right| dt \\
&\leq \left( \int_0^a \left( \frac{a + \eta(b,a) - r}{\eta(b,a)} \right)^r t^n \left| f^{(n)}(a + t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}} \\
&+ \left( \int_0^{\frac{1}{2}} \left( \frac{1}{2} - t \right)^n \left| f^{(n)}(a + t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}} \\
&+ \left( \int_{\frac{1}{2}}^1 (t - 1)^n \left| f^{(n)}(a + t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}} \\
&+ \left( \int_{\frac{1}{2}}^1 (1 - t)^n \left| f^{(n)}(a + t\eta(b,a)) \right|^q dt \right)^{\frac{1}{q}}. \tag{14}
\end{align*}\]

Making use of the preinvexity of \( f^{(n)} \) \( q \)-on \([a, a + \eta(b,a)]\), for any \( t \in [0, 1] \) we know that
\[
\left| f^{(n)}(a + t\eta(b,a)) \right|^q \leq (1 - t) \left| f^{(n)}(a) \right|^q + t \left| f^{(n)}(b) \right|^q,
\]
which implies that
\[
(a) \quad \int_0^a \left( \frac{a + \eta(b,a) - r}{\eta(b,a)} \right)^r \left| f^{(n)}(a + t\eta(b,a)) \right|^q dt \\
\leq \mu_{21} \left| f^{(n)}(a) \right|^q + \mu_{24} \left| f^{(n)}(b) \right|^q \tag{15}
\]
\[
(b) \quad \int_0^{\frac{1}{2}} \left( \frac{1}{2} - t \right)^n \left| f^{(n)}(a + t\eta(b,a)) \right|^q dt \\
\leq \mu_{22} \left| f^{(n)}(a) \right|^q + \mu_{23} \left| f^{(n)}(b) \right|^q \tag{16}
\]
\[ (c) \quad \int_{a}^{r} \left| f^{(n)}(a + t\eta(b, a)) \right|^q \, dt \]
\[ \leq \mu_{23} \ | f^{(n)}(a) |^q + \mu_{22} \ | f^{(n)}(b) |^q \quad (17) \]

\[ (d) \quad \int_{\frac{r-a}{\eta(b,a)}}^{1} \left| f^{(n)}(a + t\eta(b, a)) \right|^q \, dt \]
\[ \leq \mu_{24} \ | f^{(n)}(a) |^q + \mu_{21} \ | f^{(n)}(b) |^q . \quad (18) \]

By the simple calculations, we have

\[ (i) \quad \int_{0}^{\frac{a+\eta(b,a)-r}{\eta(b,a)}} t^{np} \, dt = \int_{\frac{r-a}{\eta(b,a)}}^{1} (1-t)^{np} \, dt = M_{11}, \quad (19) \]

\[ (ii) \quad \int_{\frac{a+\eta(b,a)-r}{\eta(b,a)}}^{\frac{1}{2}} (\frac{1}{2} - t)^{np} \, dt = \int_{\frac{1}{2}}^{\frac{r-a}{\eta(b,a)}} (t - \frac{1}{2})^{np} \, dt = M_{12}. \quad (20) \]

By substituting (15)-(20) in (14), we get the desired result.

\[ \Box \]

**Corollary 2.2.** Under the assumptions in Theorem 2.2 with \( r = a + \eta(b, a) \), we have

\[ \left| I_a^b(f, \eta, n) \right| = \left| \frac{1}{\eta(b,a)} \int_0^{a+\eta(b,a)} f(u) \, du - \frac{f(a) + f(a + \eta(b,a))}{2} + \sum_{k=2}^{n} \frac{(-1)^k \eta^{k-1}(b,a)}{2^k k!} \left\{ f^{(k-1)}(a + \eta(b,a)) + (-1)^k f^{(k-1)}(a) \right\} \right| \]
\[ \leq \frac{\eta^n(b,a)}{n!} \left( \frac{1}{2^{np+1}(np+1)} \right)^\frac{1}{p} \left( \frac{1}{8} \right)^\frac{1}{q} \left\{ | f^{(n)}(a) | + | f^{(n)}(b) | \right\}. \]

**Corollary 2.3.** Under the assumptions in Theorem 2.2 with \( r = a + \frac{1}{2} \eta(b, a) \),
we have
\[
\left| I_a^b(f, \eta, n) \right| = \left| \frac{1}{\eta(b, a)} \int_a^{a+\eta(b,a)} f(u) du - f(a + \frac{1}{2} \eta(b, a)) \right| \\
- \Sigma_{k=2}^{n} \frac{\eta^{k-1}(b, a)}{2^k k!} f^{(k-1)}(a + \frac{1}{2} \eta(b, a)) \right| \\
\leq \frac{\eta^n(b, a)}{n!} \left( \frac{1}{2^{np+1}(np+1)} \right)^{\frac{1}{q}} \\
\times \left\{ \left( \frac{3}{8} f^{(n)}(a) \right)^q + \frac{1}{8} \left| f^{(n)}(b) \right|^q \right\}^{\frac{1}{q}} \\
+ \left( \frac{1}{8} f^{(n)}(a) \right)^q + \frac{3}{8} \left| f^{(n)}(b) \right|^q \right\}^{\frac{1}{q}}. 
\]

**Theorem 2.3.** Let \( K \subseteq R \) be an invex set with respect to the map \( \eta : K \times K \to R \) and, \( \eta(b, a) \neq 0 \) with \( 0 \leq a \leq a + \eta(b, a) < \infty \) for all \( a, b \in K \) with \( a < b \). Suppose that \( f : K \to R \) is a differentiable function on the interior \( K^0 \) of \( K \) such that \( f' \in L([a, a + \eta(b, a)]) \). Assume \( q \in R \) with \( q > 1 \) and \( \frac{1}{p} + \frac{1}{q} = 1 \). If \( | f' |^q \) is preinvex with respect to \( \eta \) on \( K \), then the following inequality
\[
\left| I_a^b(f, \eta, n) \right| \\
\leq \frac{\eta^n(b, a)}{n!} \left[ M_{31}^{\frac{1}{q}} \left( \mu_{31} \left| f^{(n)}(a) \right|^q + \mu_{34} \left| f^{(n)}(b) \right|^q \right) \right]^{\frac{1}{q}} \\
+ M_{32}^{\frac{1}{q}} \left( \mu_{32} \left| f^{(n)}(a) \right|^q + \mu_{33} \left| f^{(n)}(b) \right|^q \right)^{\frac{1}{q}} \\
+ M_{33}^{\frac{1}{q}} \left( \mu_{33} \left| f^{(n)}(a) \right|^q + \mu_{32} \left| f^{(n)}(b) \right|^q \right)^{\frac{1}{q}} \\
+ M_{34}^{\frac{1}{q}} \left( \mu_{34} \left| f^{(n)}(a) \right|^q + \mu_{31} \left| f^{(n)}(b) \right|^q \right)^{\frac{1}{q}}
\]
holds, where
\[
\mu_{31} = \left( a + \eta(b, a) - r \right)^{n+1} \left\{ \frac{1}{n+1} \left( \frac{a + \eta(b, a) - r}{\eta(b, a)} \right) \right\}, \\
\mu_{32} = \left( r - a - \frac{1}{2} \eta(b, a) \right)^{n+1} \left\{ \left( n+1 \right) \left( \frac{a + \eta(b, a) - (2n+3)a}{2(n+2)\eta^{n+2}(b, a)} \right) \right\}, \\
\mu_{33} = \left( r - a - \frac{1}{2} \eta(b, a) \right)^{n+1} \left\{ \left( \frac{1}{2} (2n+3)(a + \eta(b, a)) - a \right) - (n+1) \frac{a + \eta(b, a) - (2n+3)a}{2(n+2)\eta^{n+2}(b, a)} \right\}, \\
\mu_{34} = \left( a + \eta(b, a) - r \right)^{n+2} \left( \frac{a + \eta(b, a) - r}{\eta(b, a)} \right) \\
\]
and
\[
M_{31} = \frac{1}{n+1} \left( \frac{a + \eta(b,a) - r}{\eta(b,a)} \right)^{n+1},
\]
\[
M_{32} = \frac{1}{n+1} \left( \frac{r - (a + \frac{1}{2} \eta(b,a))}{\eta(b,a)} \right)^{n+1}.
\]

Proof. Suppose that \( a, a + \eta(b,a) \in K \). Since \( K \) is invex with respect to \( \eta \), for any \( t \in [0,1] \), we have \( a + t\eta(b,a) \in K \). Making use of the preinvexity of \( |f^{(n)}|^q \) on \([a, a + \eta(b,a)]\), Lemma 1 and Hölder’s inequality, we get
\[
\frac{n!}{\eta^n(b,a)} |I_2^\beta(f, \eta, n)|
\leq \left( \int_0^{\frac{a + \eta(b,a) - r}{\eta(b,a)}} t^n \, dt \right)^{\frac{1}{p}} \left( \int_0^{\frac{a + \eta(b,a) - r}{\eta(b,a)}} t^n |f^{(n)}(a + t\eta(b,a))| \, dt \right)^{\frac{1}{q}}
\]
\[
+ \left( \int_{\frac{a + \eta(b,a) - r}{\eta(b,a)}}^{\frac{1}{2}} \frac{1}{2} - t^n \, dt \right)^{\frac{1}{p}} \left( \int_{\frac{a + \eta(b,a) - r}{\eta(b,a)}}^{\frac{1}{2}} \frac{1}{2} - t^n |f^{(n)}(a + t\eta(b,a))| \, dt \right)^{\frac{1}{q}}
\]
\[
+ \left( \int_{\frac{a + \eta(b,a) - r}{\eta(b,a)}}^{1} (1 - t)^n \, dt \right)^{\frac{1}{p}} \left( \int_{\frac{a + \eta(b,a) - r}{\eta(b,a)}}^{1} (1 - t)^n |f^{(n)}(a + t\eta(b,a))| \, dt \right)^{\frac{1}{q}}. \tag{21}
\]
By the simple calculations, we have
\[
(i) \int_0^{\frac{a + \eta(b,a) - r}{\eta(b,a)}} t^n \, dt = \int_{\frac{a + \eta(b,a) - r}{\eta(b,a)}}^{1} (1 - t)^n \, dt = M_{31}, \tag{22}
\]
\[
(ii) \int_{\frac{a + \eta(b,a) - r}{\eta(b,a)}}^{\frac{1}{2}} \frac{1}{2} - t^n \, dt = \int_{\frac{a + \eta(b,a) - r}{\eta(b,a)}}^{\frac{1}{2}} (t - \frac{1}{2})^n \, dt = M_{32}. \tag{23}
\]
Making use of the preinvexity of \( |f^{(n)}|^q \) on \([a, a + \eta(b,a)]\), for any \( t \in [0,1] \) we know that
\[
|f^{(n)}(a + t\eta(b,a))|^q \leq (1 - t) \, |f^{(n)}(a)|^q + t \, |f^{(n)}(b)|^q,
\]
which implies that
\[
(a) \int_0^{\frac{a + \eta(b,a) - r}{\eta(b,a)}} t^n |f^{(n)}(a + t\eta(b,a))|^q \, dt
\]
\[
\leq \mu_{31} |f^{(n)}(a)|^q + \mu_{34} |f^{(n)}(b)|^q \tag{24}
\]
\[ (b) \int_{\frac{a + \eta(b, a) - r}{\eta(b, a)}}^{\frac{1}{2}} (1 - t)^n \left| f^{(n)}(a + t\eta(b, a)) \right|^q \, dt \]

\[ \leq \mu_{32} \left| f^{(n)}(a) \right|^q + \mu_{33} \left| f^{(n)}(b) \right|^q \quad (25) \]

\[ (c) \int_{\frac{a}{\eta(b, a)}}^{1} (t - \frac{1}{2})^n \left| f^{(n)}(a + t\eta(b, a)) \right|^q \, dt \]

\[ \leq \mu_{33} \left| f^{(n)}(a) \right|^q + \mu_{32} \left| f^{(n)}(b) \right|^q \quad (26) \]

\[ (d) \int_{\frac{a}{\eta(b, a)}}^{1} (1 - t)^n \left| f^{(n)}(a + t\eta(b, a)) \right|^q \, dt \]

\[ \leq \mu_{34} \left| f^{(n)}(a) \right|^q + \mu_{31} \left| f^{(n)}(b) \right|^q . \quad (27) \]

By substituting (22)-(27) in (21), we get the desired result. \[ \square \]

**Corollary 2.4.** Under the assumptions in Theorem 2.3 with \( r = a + \eta(b, a) \), we have

\[
\left| I_a^b(f, \eta, n) \right| \\
= \frac{1}{\eta(b, a)} \int_{a}^{a+\eta(b, a)} f(u)du - \frac{f(a) + f(a + \eta(b, a))}{2} \]

\[ + \sum_{k=2}^{n} (-1)^k \eta^{k-1}(b, a) \left\{ f^{(k-1)}(a + \eta(b, a)) + (-1)^k f^{(k-1)}(a) \right\} \]

\[ \leq \eta^n(b, a) \left( \frac{1}{n!} \right)^{1-\frac{1}{q}} \left( \frac{1}{2^{n+2}(n+1)(n+2)} \right)^{\frac{1}{q}} \]

\[ \times \left[ \left\{ (2n+3) \left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right\}^{\frac{1}{q}} \right] \]

\[ + \left\{ \left| f^{(n)}(a) \right|^q + (2n+3) \left| f^{(n)}(b) \right|^q \right\}^{\frac{1}{q}} . \]

**Corollary 2.5.** Under the assumptions in Theorem 2.3 with \( r = a + \frac{1}{2} \eta(b, a) \),
we have

\[
\left| I^b_a(f, \eta, n) \right| = \left| \frac{1}{\eta(b, a)} \int_a^{a+\eta(b, a)} f(u)du - f(a + \frac{1}{2}\eta(b, a)) \right| \\
- \sum_{k=2}^{n} \frac{\eta^{k-1}(b, a)}{2k^k} \left| f^{(k-1)}(a + \frac{1}{2}\eta(b, a)) \right| \\
\leq \frac{\eta^n(b, a)}{n!} \left( \frac{1}{2^{n+1}(n+1)} \right)^{1-\frac{q}{n}} \left( \frac{1}{2^{n+2}(n+1)(n+2)} \right)^{\frac{1}{2}} \\
\times \left\{ \left| f^{(n)}(a) \right|^q + (n+1) \left| f^{(n)}(b) \right|^q \right\}^{\frac{1}{q}} \\
+ \left\{ (n+1) \left| f^{(n)}(a) \right|^q + \left| f^{(n)}(b) \right|^q \right\}^{\frac{1}{q}}.
\]

References


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