On the Solutions of a Class of Initial Boundary Value Problem

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Abstract

In this paper, we consider the following initial boundary value problem

\[(p(t)u')' = c(t)p(t)\tilde{f}(u), \quad 0 \leq t < \infty,\]

where

\[
\tilde{f}(x) = \begin{cases} 
  f(M), & x \in (-\infty, M], \\
  f(x), & x \in [M, L], \\
  0, & x \in [L, +\infty), 
\end{cases}
\]

\(M\) is some constant and function \(f \in Lip((-\infty, L])\). By using shooting argument together with critical point theory, we precisely analyze the properties of certain solutions and improve the results in recent literatures.

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1 Introduction

In order to study the formation of microscopical bubbles in a nonhomogeneous fluid, the boundary value problem
\[ \gamma \Delta u = \mu(u) - \mu_0, \quad u'(0) = 0, \quad \lim_{t \to \infty} u(t) = u_l > 0, \tag{1.1} \]
has been used to describe the state of fluid in \( R^N \) when the motion of fluid is absent. Here \( u \) is the density of the medium, \( \mu(u) \) is the chemical potential of a nonhomogeneous fluid, \( \gamma \) and \( \mu_0 \) are suitable constants [1, 2].

Note that \( u(0) \) is the density of the gas at the center of the bubble. We are interested in the strictly increasing solution \( u \) to the boundary value problem (1.1) with \( 0 < u(0) < u_l \), since in this case \( u \) determines an increasing mass density profile [3]. To find this kind of solution on the half-line \([0, +\infty)\), problem (1.1) has been further transformed to
\[ (t^{N-1} u'(t))' = 4t^{N-1} \lambda^2 (u + 1) u(u - L), \quad u'(0) = 0, \quad u(+\infty) = L > 0, \tag{1.2} \]
under the classical assumption and through standard substitutions [3]. Here \( N = 2 \) or \( N = 3 \) denote the respective case of plane or spherical bubbles. From [4], we know that the alternative form of problem (1.2) arises in nonlinear field theory.

When considering a more general situation of problem (1.2), that is,
\[ (p(t)u')' = c(t)p(t)f(u), \quad 0 \leq t < \infty, \tag{1.3} \]
\[ u(0) = 0, \quad u(+\infty) = L > 0. \tag{1.4} \]
we find that the following initial value problem may be play an important role:
\[ (p(t)u')' = c(t)p(t)\tilde{f}(u), \quad 0 \leq t < \infty, \tag{1.5} \]
\[ u(0) = B \leq 0, \quad u'(0) = 0, \tag{1.6} \]
where
\[ \tilde{f}(x) = \begin{cases} 
  f(M), & x \in (-\infty, M], \\
  f(x), & x \in [M, L], \\
  0, & x \in [L, +\infty), 
\end{cases} \tag{1.7} \]
\( M \) is some constant defined in later essays and function \( f \in Lip((-\infty, L]) \).
We are interested in the strictly increasing solution of problem (1.3), (1.4) possessing just one zero in $(0, \infty)$ which refer to as a type of homoclinic solution [5, 6, 7]. For the simplest case $c(t) \equiv 1$, this problem has been investigated in [8, 9, 10] by means of differential and integral inequalities as well as upper and lower function approach. However, the same argument appears to be unavailable for the general case $c(t) \not\equiv 1$. Moreover, problem (1.3) and (1.4) can be transformed into finding the strictly decreasing and positive solution. This kind of solution has been discussed in [11, 12] in which $p(t) = t^k$, $k \in (1, \infty)$ and the shooting argument combining with variational method [13, 14] were successfully utilized.

Here it is a pity that we don’t solve the problem (1.3) with the singular boundary value (1.4), and we only obtain some results of the problem (1.5) with the initial boundary value (1.6) in this paper. To obtain the main results, we first introduce the following assumptions:

$(H_1)$ $c(t)$ is a bounded and continuous function in $[0, \infty)$ satisfying $c_1 \leq c(t) \leq c_2$, $t \in [0, \infty)$ for real numbers $c_2 \geq c_1 > 0$;

$(H_2)$ $f(x)$ is a continuous function in $(-\infty, L]$ possessing two zeros with $f(0) = f(L) = 0$, and $xf(x) < 0$ for $x \in (-\infty, L) \setminus \{0\}$, moreover, there exists $L_0 < 0$ such that $F(L_0) > F(L)$, where $F$ is given by

\[ F(x) = -\int_0^x f(z)dz, \quad x \in (-\infty, L]; \quad (1.8) \]

$(H_3)$ $p(t) \in C([0, \infty)) \cap C^1((0, \infty))$, $p(0) = 0$, $p'(t) > 0$ in $(0, \infty)$;

$(H_4)$ there exists $\alpha \in (0, 1)$ such that $p'(t)/p(t)$ is bounded as $t \to 0$;

$(H_5)$ $p''(t)/p(t)$ is bounded when $t$ is sufficiently large, and there exists a constant $\tilde{b} > 0$ such that

\[ \int_0^{\tilde{b}} p(t)dt > F_0 \int_{\tilde{b}}^{\tilde{b} - L_0 + L} p(t)dt, \]

where

\[ F_0 = \frac{1 + 2c_2F(L)}{2c_1(F(L_0) - F(L))}, \]

while $F$ and $c_2$ are given in (1.8) and $(H_1)$, respectively.

In addition, we need the following hypothesis

$(H_6)$ $\lim_{x \to -\infty} f(x)/|x| = 0$.

**Remark 1.1** (i) By $(H_2)$, there exists a constant $\tilde{B} \in (L_0, L)$ such that $F(\tilde{B}) = F(L)$.

(ii) It has been shown in [15] that the assumption $(H_3)$ together with the
simple formula
\[
\lim_{t \to \infty} \frac{p(t)}{p'(t)} = +\infty \tag{1.9}
\]
is a sufficient condition for \((H_5)\). There are many functions satisfying \((H_3) - (H_5)\) which have been listed in [15].

**Remark 1.2** We can choose \(\varepsilon > 0\) sufficiently small such that \(K_1 = c_2 \varepsilon (\bar{b} - L_0 + L)^2 < 1\). If \((H_6)\) holds, then according to the continuity of \(f\), there exists a constant \(M' > 0\) such that \(|f(x)| \leq M' + \varepsilon |x|, \forall x \in (-\infty, L]\). Set
\[
K_2 = c_2 M' (\bar{b} - L_0 + L)^2 \quad \text{and} \quad M < \min \left\{ L_0, -\frac{\bar{B} + K_2}{1 - K_1} \right\}. \tag{1.10}
\]

## 2 Main results

We first consider functions \(F\) given by (1.8) and \(\tilde{F}\) defined by
\[
\tilde{F}(u) = \int_0^u \tilde{f}(x + L) \, dx. \tag{2.1}
\]

Assumption \((H_2)\) immediately yields the following conclusions.

**Remark 2.1** The function \(F\) is continuous in \((-\infty, L]\), decreasing in \((-\infty, 0)\), increasing in \((0, L]\), \(F(B) > F(L)\) for \(B \in (-\infty, \bar{B})\) and \(F(B) < F(L)\) for \(B \in (B, L)\). Moreover, we have
\[
- \int_0^x \tilde{f}(z) \, dz \begin{cases} 
> F(M), & x \in (-\infty, M), \\
= F(x), & x \in [M, L], \\
= F(L), & x \in [L, \infty). 
\end{cases} \tag{2.2}
\]

**Remark 2.2** The function \(\tilde{F}\) is continuous in \((-\infty, +\infty)\) and \(\tilde{F}(x) = F(L) - F(x + L), x \in [L_0 - L, 0]\). Moreover, \(\tilde{F}\) increases on \([L_0 - L, -L]\), decreases on \([-L, 0]\) and \(\tilde{F}(-L) = F(L), \tilde{F}(L_0 - L) = F(L) - F(L_0)\). Furthermore, we have
\[
\tilde{F}(x) \begin{cases} 
< 0, & x \in (-\infty, \bar{B} - L), \\
> 0, & x \in (\bar{B} - L, 0), \\
= 0, & x \in [0, +\infty) \cup \{\bar{B} - L\}.
\end{cases}
\]

Now our task is to consider the initial value problem (1.5) and (1.6), the solution of which possesses the following properties.

**Theorem 2.1** If \(\tilde{f} \in \text{Lip}((-\infty, +\infty))\) and \((H_1) - (H_3), (H_6)\) and (1.7) are
satisfied, then we have

(i) The problem (1.5) and (1.6) subject to the initial value \( B \in (-\infty, 0] \) possesses a unique solution \( u \in C^1([0, \infty)) \cap C^2((0, \infty)) \) and \( u \equiv 0 \) for \( B = 0 \).

(ii) Suppose that \( u \) is the solution of problem (1.5) and (1.6) with the initial value \( B \in (B_0 - \delta, B_0 + \delta) \). For each \( b > 0, B_0 \in (-\infty, 0) \) and \( \delta \) small enough such that \( (B_0 - \delta, B_0 + \delta) \subseteq (-\infty, 0) \), there exists a constant \( M = \bar{M}(b, B_0, \delta) > 0 \) such that

\[
|u(t)| + |u'(t)| \leq \bar{M} \text{ for } t \in [0, b], \quad \text{and} \quad \int_0^b |u'(s)|p'(s)/p(s)ds \leq \bar{M}. \tag{2.3}
\]

(iii) Suppose that \( u_i, i = 1, 2 \) are unique solutions of problem (1.5) and (1.6) with the initial values \( B = B_i, i = 1, 2 \). For each \( b > 0, B_0 < 0 \) and \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( B_1, B_2 \in [B_0, 0) \), if \( |B_1 - B_2| < \delta \), we have

\[
|u_1(t) - u_2(t)| + |u_1'(t) - u_2'(t)| < \epsilon, \quad t \in [0, b]. \tag{2.4}
\]

**Proof.** It is obvious that \( \tilde{f} \) is Lipschitz and bounded in \((-\infty, +\infty) \) while \((H_1)\) implies the boundness of the function \( c(t) \). Then the proof of (i) is standard and is similar to that of [9](Lemma 4) by contraction mapping theorem. From the arguments of step 2 and step 3 in [8](Lemma 3), the result (ii) follows immediately. The proof of (iii) is similar to that of [8](Lemma 6) by taking advantage of Gronwall inequality. The proof is complete. \( \square \)

**Remark 2.3** For a given constant \( C \in (-\infty, +\infty) \), consider the initial value condition

\[
u(a) = C, \quad u'(a) = 0, \quad a \geq 0.
\tag{2.5}
\]

The proof of Theorem 2.1(i) shows that the initial value problem (1.5) and (2.5) has a unique solution \( u \) in \([a, +\infty)\). Especially, for \( C = 0 \) or \( C \geq L \), we have \( u \equiv C \).

**Theorem 2.2** Assume that \((H_1) - (H_3)\) and \((H_5)\) hold. Let \( u \in C^1([0, \infty)) \cap C^2((0, \infty)) \) be the solution of (1.5). If \( u \) increases in \((0, +\infty)\) with \( u(t) \in (-\infty, L) \) for \( t \in [0, \infty) \), then \( \lim_{t \to \infty} u(t) \) equals to 0 or \( L \). In addition, we have \( \lim_{t \to \infty} u(t) = L \) by further assuming that (1.9) is satisfied, \( f'(0) \) exists and is nonzero.

**Proof.** Firstly, we rewrite (1.5) in the equivalent form

\[
u''(t) + \frac{p'(t)}{p(t)} u'(t) = c(t) \tilde{f}(u(t)), \quad t \in (0, \infty).
\tag{2.6}
Notice that the assumption \((H_2)\) and \((1.7)\) enable us to find \(t_1 > 0\) such that for \(t \geq t_1\), \(\tilde{f}(u(t))\) will not change its sign anymore. Then by taking integration on both sides of \((2.6)\) from \(t_1\) to \(t > t_1\), we have

\[
\int_{t_1}^{t} c(s) \tilde{f}(u(s))ds = u'(t) - u'(t_1) + \int_{t_1}^{t} \frac{p'(s)}{p(s)} u'(s)ds, \quad t \geq t_1.
\]

Note that \(\lim_{t \to \infty} u'(t) = 0\), and assumptions \((H_3)\) and \((H_5)\) imply \(\int_{t_1}^{t} \frac{p'(s)}{p(s)} u'(s)ds\) is bounded for \(t \geq t_1\). We derive that \(\int_{t_1}^{+\infty} c(s) \tilde{f}(u(s))ds\) converges, which indicates that \(c(t) \tilde{f}(u(t)) \to 0\) as \(t \to +\infty\). By \((H_1), (H_2)\) and the assumptions on \(u\), we find that either \(\lim_{t \to \infty} u(t) = 0\) or \(\lim_{t \to \infty} u(t) = L\). The rest of the proof is similar to that of \([12]\)(Proposition 11) and so we omit it. The proof is complete.

The following result shows that under a certain initial condition \((1.6)\), the solution \(u\) of the problem \((1.5)\) and \((1.6)\) is unable to achieve \(L\). Some ideas of the discussion stem from \([15]\)(Proposition 3.3) as well as \([12]\)(Proposition 12) and \([8]\)(Theorem 13).

**Theorem 2.3** Let \((H_1) - (H_3)\) and \((H_5)\) hold. If the initial value \(B < 0\) given in \((1.6)\) is sufficiently close to 0, then the solution \(u\) of problem \((1.5)\) and \((1.6)\) either attains a local maximum \(u_{max} \in (0, L)\) at \(\bar{t} > 0\) with \(u\) increasing in \((0, \bar{t})\), or \(u\) increases in \((0, \infty)\) with \(\lim_{t \to \infty} u(t) = 0\).

**Proof.** We first claim that the values of local maximum are greater than 0. Let initial value \(B < 0\) sufficiently close to 0. Integrating \((1.5)\) from 0 to \(t\) gives

\[
u'(t) = \frac{1}{p(t)} \int_{0}^{t} c(s) p(s) \tilde{f}(u(s))ds, \quad t > 0. \tag{2.7}\]

This indicates that \(u'(t) > 0\) as long as \(u(t) < 0\) according to conditions \((H_1) - (H_3)\) and definition \((1.7)\), from which our claim follows immediately.

Suppose on the contrary that Theorem 2.3 does not hold. Then above discussion combining with Remark 2.3, Theorem 2.2, \((1.5)\) and \((H_2)\) indicate that there are only two possibilities: (i) \(u\) increases in \((0, \infty)\) and \(\lim_{t \to \infty} u(t) = L\); (ii) there exists \(b \in (0, \infty)\) such that \(u\) increases in \((0, b]\) and \(u(b) = L\) and \(u'(b) > 0\).

We first rule out the possibility (i). Suppose on contrary, then \(u\) has a unique zero \(\theta > 0\). For initial value \(B \in [L_0, 0]\), multiplying \((2.6)\) by \(u'\) and taking integration from 0 to \(\theta\) lead to

\[
\frac{u^2(\theta)}{2} + \int_{0}^{\theta} \frac{p'(s)}{p(s)} u^2(s)ds = \int_{0}^{\theta} c(s) \tilde{f}(u(s)) u'(s)ds.
\]
Furthermore, since \( c(s) \leq c_2, p'(s)/p(s) > 0 \) for \( s \in [0, \theta] \), and
\[
\int_0^\theta \tilde{f}(u(s))u'(s)ds = F(u(0)) = F(B),
\]
we have
\[
u'^2(\theta) \leq 2c_2F(B), \quad B \in [L_0, 0).
\]
(2.8)

Similarly, by integrating (2.6) again from \( \theta \) to \( t \), we obtain
\[
u'^2(t) - \nu'^2(\theta) + 2 \int_\theta^t \frac{p'(s)}{p(s)} u'^2(s)ds \leq -2c_1F(u(t)), \quad \text{for } t > \theta.
\]
(2.9)

Let \( t \to \infty \) in (2.9). We get that \( \nu'^2(\theta) \geq 2c_1F(L) \), which together with (2.8) implies \( c_1F(L) \leq c_2F(B) \). This is a contradiction since \( B \) is sufficiently close to 0.

If possibility (ii) holds, then there exists \( \theta > 0 \) such that \( u(\theta) = 0 \). Multiplying (2.6) by \( u' \) and then taking integration give
\[
\frac{u'^2(b)}{2} + \int_0^b \frac{p'(s)}{p(s)} u'^2(s)ds = \int_0^b c(s)\tilde{f}(u(s))u'(s)ds.
\]

Note that the left side of the equality is strictly larger than 0. Then splitting the region \([0, b]\) in the last integral into \([0, \theta]\) and \([\theta, b]\), and applying \((H_1), (H_2)\) and Remark 2.1 give
\[
0 < c_2\int_0^\theta \tilde{f}(u(s))u'(s)ds + c_1\int_\theta^b \tilde{f}(u(s))u'(s)ds = c_2F(B) - c_1F(L).
\]

Let \( B \to 0 \). Then we have \( 0 < c_2F(B) - c_1F(L) \leq 0 \), which is a contradiction. The proof is complete.

\[\square\]

References


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