Construction of a Cone by Using

Weak*- Total Families

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Abstract

This paper is devoted to construct a cone by using the so called weak*-total families in different spaces ( Banach space, Hilbert space, metric space,...etc.). We were advised some forms for weak*-total subsets to construct an exact forms for cones like ( Normal cones, Allows plastering cones, Solid cones,...etc.).

Keywords: cone; weak*-total subsets

1. Introduction

In the study of various problems in general system theory, mechanics , operator theory ... etc, an important rule is played by monotonicity of the corresponding " inputs" and "outputs", and by the positiveness or monotonicity of various linear and nonlinear mappings. The mathematical models of such objects lead in many cases to
operator equations in spaces, which partially ordered by some cone. The study of
ordered linear spaces was initiated in the last 1950's by Riesz. Many authors like M.
G. Krein, who worked on the geometry of cone in Banach space and has given proofs
of main theorems for cone construction in his famous paper [1], there are many
relations between cones, the class of positive definite functions, the class of negative
definite functions, the class of completely monotone functions and the class of
completely alternating functions [2-5]. In this paper, we will discuss the relations
between the construction of cones and the families of weak*-total subset. This paper
is devoted to give an exact method to construct cones using a weak*-total families.
Then we will search some types of weak*-total families that grants the
implementation of definite cones such as solid cone or allows plastering cone or
normal cone...etc.

2. Weak*-total subset

**Definition 2.1**: (Total subset) [7]
A subset $S$ of a topological vector space $E$ is said to be total in $E$ if each
continuous linear from on $E$ satisfying $f(x) = 0$ for all $x$ belonging to $S$
necessarily satisfies $f = 0$.

We suggest to call this notion of totality "weak totality" and parallel we give the
following definition of weak*-totality.

**Definition 2.2**: (weak*-total subset)
We will call a subset $F$ of the dual space $E^*$ of a topological vector space $E$
weak*-total
if for each element $\chi$ belonging to $E$ and satisfying $f(\chi) = 0$ for all $f$ in $F$
necessarily satisfies $\chi = 0$.

Using a weak*-total subset $F$ of the dual space $E^*$ of a Banach space $E$ we define
a partial ordering on $E$ associated with $F$ as follows $x \leq_F y$ if and only if
$f(x) \leq f(y)$ for all $f \in F$.

**Remark.** This partial ordering is well defined.
1- Reflexivity of the relation "$\leq_F$" is trivial.
2- The relation "$\leq_F$" is anti-symmetric (we remark that, the weak*- totality
guarantees the anti symmetricity of the above defined partial ordering). In fact, let
$x \leq_F y$ and $y \leq_F x$
Hence,
\[ f(x) \leq f(y) \text{ and } f(x) \leq f(y) \text{ for all } f \in F \]
So,
\[ f(x) = f(y) \text{ for all } f \in F \]
hence ,
\[ f(x - y) = 0 \]
Since the family \( F \) is weak*-total then,
\[ x - y = 0 \]
Hence \( x = y \).
3- The transitivity of the relation "\( \leq_F \)". In fact,
\[ x \leq_F y \text{ and } y \leq_F z \]
Implies that
\[ f(x) \leq f(y) \text{ and } f(y) \leq f(z) \text{ for all } f \in F \]
Hence,
\[ f(x) \leq f(z) \text{ for all } f \in F \]
So \( x \leq_F z \).

**Example 2.1**
The family \( F = \{e_n : n \in N\} \) is a weak*-total family in the space \( l^\infty \) of all bounded sequences which is the dual space of \( l_1 \) of all absolutely summable sequences. In fact,
\[ < x, e_n > = x_n = 0 \text{ for all } e_n \in l^\infty \]
implies that \( x = 0 \). We remarks that "The family \( F = \{e_n : n \in N\} \) is weak*-total in \( l^\infty \), but \( F \) is not abases for \( l^\infty \) (\( l^\infty \) being not separable i.e., \( l^\infty \) has no Schauder bases)". The partial ordering associated to \( F \) in this case is defined as follow
\[ x \leq_F y \text{ if and only if } x_i \leq y_i \text{ for all } i \in N \]
This is the well known ordering in \( l_1 \).

**Remark.**
On a Banach space \( E \) with a cone \( K \) a partial ordering is introduced in the following manner: \( x \leq_F y \) if and only if \( y - x \in K \).

**Definition 2.3:**
Using a weak*-total subset \( F \) we can define a corresponding cone \( K_F \) as follow:
\[ K_F = \{z \in X : f(z) \geq 0 \text{ for all } f \in F\} \]
We will call \( K_F \) the cone generated by the weak*-total subset \( F \).
Theorem 2.1. Let $F$ be a bounded, closed and convex subset of a normed space $E$, which does not contain the zero of $E$. Then the set

$$K(F) = \{ x \in E : \quad x = tz \quad \text{where} \quad z \in F \quad \text{and} \quad t \geq 0 \}$$

is a cone.

Proof. First we will prove that $K(F)$ is closed. Actually, let $(u_n)_{n=1}^\infty \in K(F)$ and $\|u_n - v\| \to 0$ (without loss of generality let $v \neq 0$). Every element $u_n$ has the representation $u_n = t_n z_n$ where $z_n \in F$ and $t_n \geq 0$ for all $n \in \mathbb{N}$. Since $F$ is bounded, then there exist $m, M \geq 0$ such that

$$m \leq \|z_n\| \leq M \quad \text{for all} \quad n \in \mathbb{N}$$

since $(u_n)_{n=1}^\infty$ is convergent then it is bounded, so we have

$$\|u_n\| = t_n \|z_n\| \leq C \quad \text{for some} \quad C > 0$$

since $(u_n)_{n=1}^\infty$ is bounded, so $(t_n)_{n=1}^\infty$ is bounded, hence there exists a convergent subsequence $(t_{n_j}) \subseteq (t_n)$ and $t_0 > 0$ such that:

$$t_{n_j} \xrightarrow{j \to \infty} t_0$$

We contract a subsequence $(z_{n_j})_{j=1}^\infty$ of the sequence $(z_n)_{n=1}^\infty$ such that:

$$\|z_{n_j} - \frac{v}{t_0}\| = \|z_{n_j} - \frac{t_{n_j}}{t_0}z_{n_j} + \frac{t_{n_j}}{t_0}z_{n_j} - \frac{v}{t_0}\| \leq \frac{1}{t_0} \{ \|z_{n_j}\| \|t_0 - t_{n_j}\| + \|t_{n_j}z_{n_j} - v\}$$

Hence,

$$\|z_{n_j} - \frac{v}{t_0}\| \xrightarrow{n \to \infty} 0$$

Consequently,

$$z_0 = \lim_{n \to \infty} z_{n_j} = \frac{v}{t_0}$$

Since $z_{n_j} \in F$ for all $n_j \in \mathbb{N}$, then

$$v = t_0 z_0, \quad t_0 > 0 \quad \text{and} \quad z_0 \in F$$

Hence $v \in K(F)$, so $K(F)$ is closed.

Secondly, we will prove that $K(F)$ is a convex set. Actually, let $u, v \in K(F)$ and $\alpha, \beta \geq 0$, hence there exists $t_1, t_2 \geq 0$ and $z_1, z_2 \in F$ such that:

$$u = t_1 z_1 \quad \text{and} \quad v = t_2 z_2$$
then

\[ \lambda = \frac{\alpha t_1}{\alpha t_1 + \beta t_2}, \] so \( \alpha u + \beta v \in K(F) \) i.e., \( K(F) \) is a convex set.

Finally, we will prove that \( K(F) \cap -K(F) = 0 \). Let \( \omega, -\omega \in K(F) \), hence there exists \( z_0 \in F \) such that:

\[ -t_0z_0 \in K(F), \quad \text{for some } t_0 > 0. \]

Consequently

\[ -t_0z_0 = t_1z_1 \in K(F), \quad \text{for some } t_1 > 0 \quad \text{and} \quad z_1 \in F. \]

Therefore

\[ \theta = \frac{t_0}{t_0 + t_1}z_0 + \frac{t_1}{t_0 + t_1}z_1 \]

which gives a contradiction.

**Definition 2.3**: Let \( E \) be a topological vector space, a cone \( K \subseteq E \) allows the plastering \( K_1 \) (\( K_1 \) is another cone in \( E \)), if every point \( x_0 \in K \) lies in the cone \( K_1 \) together with a spherical neighborhood \( \{ x : \| x - x_0 \| \leq b \| x_0 \| \} \), where \( b > 0 \) does not depend on \( x_0 \); the cone \( K_1 \) is called the plastering of \( K \).

**Theorem 2.2**: Let \( E \) be a normed space. A necessary and sufficient condition for a uniformly positive linear functional \( f \) on a cone \( K \subseteq E \) is that the cone \( K \) allows plastering.

**Proof**. Let \( f \) be a uniformly positive linear functional on \( K \), hence

\[ f(x) \geq a \| x \| \quad \text{for all } x \in K \]

Let

\[ N = \{ x \in E; \quad f(x) = 1 \} \]

It is clear that \( M = K \cap N \) is a closed convex set. Denote by \( F \) the set

\[ F = \{ x \in N; \quad \| x - x^* \| \leq 2e \} \]

where \( x^* \) is a fixed element of \( M \) and \( e \) is a positive number satisfies the inequality:

\[ \| x - x_1 \| \leq e \]
where \( x \in N \) and \( x_1 \in M \). Construct a cone \( K_1 \) such that \( K_1 = K(F) \). We will show that the cone \( K_1 \) is the plastering of \( K \). Indeed, let \( x_0 \in K \) and

\[
\| h \| \leq \frac{e a^2}{2 \| f \| + e a \| f \|} \| x_0 \|. \]

To prove \( x_0 + h \in K_1 \) it is suffices to show that

\[
\left\| \frac{x_0 + h}{f(x_0 + h)} - \frac{x_0}{f(x_0)} \right\| \leq e
\]

Since \( \frac{x_0 + h}{f(x_0 + h)}, \frac{x_0}{f(x_0)} \in N \) and \( \frac{x_0}{f(x_0)} \in M \). Hence

\[
\| f(x_0)h - x_0 f(h) \| \leq 2 \| f \| \| x_0 \| \frac{e a^2}{2 \| f \| + e a \| f \|} \| x_0 \|
\]

But

\[
e f(x_0) \{ f(x_0) + f(h) \} \geq e a \| x_0 \| \{ a \| x_0 \| + f(h) \}
\geq e a \| x_0 \| \{ a \| x_0 \| - \| f \| \| h \| \}
\geq e a \| x_0 \|^2 \{ a - \frac{e a^2}{2 + e a} \}
\geq \frac{2 e a^2}{(2 + e a) \| f \|} \| x_0 \|^2
\]

Therefore,

\[
\left\| \frac{x_0 + h}{f(x_0 + h)} - \frac{x_0}{f(x_0)} \right\| \leq e
\]

So, \( \frac{x_0 + h}{f(x_0 + h)} \in F \), this implies \( x_0 + h = f(x_0 + h) \frac{x_0 + h}{f(x_0 + h)} \in K(F) \), hence \( x_0 \in K \) is an interior point of \( K_1 \).

Conversely. Suppose the cone \( K \) allows the plastering \( K_1 \). Let \( x \in K \), \( x \neq 0 \) then there exists \( b > 0 \) such that:

\[
N(x, b \| x \|) \subseteq K_1
\]

If \( y \in N(x, b \| x \|) \subseteq K_1 \) then for all \( z \in E \) such that \( f(z) > 0 \) we have

\[
y = x - b \| x \| \frac{z}{\| z \|}
\]

Let \( f : E \to R \) be a positive functional on \( K_1 \), so

\[
f(y) = f(x - b \| x \| \frac{z}{\| z \|}) \geq 0
\]
hence, \( f(x) \geq b \| x \| \frac{f(z)}{\| z \|} \). Putting \( a = b \frac{f(z)}{\| z \|} \) implies \( K(F) \) allows the plastering.

**Theorem 2.3:**
Let \( F \) be a bounded, closed and convex subset of a normed space \( E \) which does not containing the zero of the space \( E \). then the cone \( K(F) \) allows plastering.

**Proof.** Construct a hyperplane \( f(x) = C \), separating the set \( F \) and the sphere \( \| x \| \leq e \) which is disjoint from \( F \), where \( e \) is a fixed positive number. since \( F \) is bounded then for every \( y \in F \) there exist \( \alpha_1, \alpha_2 \geq 0 \) such that:
\[
\alpha_1 \leq \| y \| \leq \alpha_2
\]

From the definition of \( K(F) \), every element \( x \in K(F) \) can be written as \( x = ty \) where \( y \in F \) and \( t \geq 0 \). Since for all \( x \in K(F) \) and \( t \geq 0 \) we have
\[
x = \frac{\| x \|}{t} \frac{tx}{\| x \|} \in F.
\]
For every \( y \in F \) we have \( f(y) \geq C \), so for all \( x \in K(F) \) we have \( f(x) \geq C \), hence for all \( x \in K(F) \) we have \( f(x) \geq \frac{C}{t} \| x \| \).

The boundedness of \( F \) implies that the number \( t \) is uniformly bounded by some positive number \( \gamma \) i.e., for all \( x \in K(F) \) we have \( f(x) \geq \frac{C}{\gamma} \| x \| \). This mean that \( f \) is uniformly positive linear functional on the cone \( K(F) \) and therefore \( K(F) \) allows plastering.

**Remark.** As pointed out from [ ] "every allows plastering cone is normal" so we directly obtains the following:

**Corollary 2.1.** Let \( F \) be a bounded, closed and convex subset of a normed space \( E \) which does not containing the zero of the space \( E \). then the cone \( K(F) \) is normal.

**Definition 2.4:**
We denote by the smallest cone containing the weak*-total subset \( F \) i.e.,
\[
C(F) = \left\{ \sum_{i=1}^{n} \lambda_i f_i : f_i \in F, \lambda_i \geq 0 \text{ and } n \in \mathbb{N} \right\}
\]
Theorem 2.4

The conjugate cone $K_F^*$ of the cone $K_F$ contains the cone $C(F)$, but not necessarily coincide with $C(F)$.

**Proof.** Since $C(F)$ is the smallest cone containing the weak*-total subset $F$, then to prove that $C(F) \subseteq K_F^*$ it is sufficient to prove that $F \subseteq K_F^*$. Indeed, let $g \in F$ and $z \in K_F$, hence $g(z) \geq 0$, so for all $\alpha \geq 0$ we have $\alpha g(z) \geq 0$ consequently, for all we have $\alpha g(z) \in K_F^*$. Hence, $F \subseteq K_F^*$.

Finally, we will give an example for a weak*-total family $F$, for which $C(F) \neq K_F^*$. Let us choose for the separable space $l_1$ and for which we will take the weak*-total family $F = \{e_n\}_{n=1}^{\infty} \subset l^\infty$, so

$$K_F = K_{\{e_n\}_{n=1}^{\infty}} = \{z \in l_1 : e_n(z) \geq 0 \text{ for all } e_n \in F\}$$

$$= \{z \in l_1 : z_n \geq 0 \text{ for all integer } n \in \mathbb{N}\}$$

and

$$C(F) = \{\sum_{i=1}^{m} \lambda_i e_i : e_i \in F, \lambda_i \geq 0 \text{ and } m \in \mathbb{N}\} = C_0^+$$

Clearly, the element $y_0 = (1,1,1,...,1,...)$ satisfies that $y_0 \in K_F^*$ but $y_0 \notin C(F)$ so, $C(F) \neq K_F^*$.

**Theorem (Weirestrass Polynomial Approximation):**[8]

Suppose that $f(t)$ be a real valued continuous functions on the closed interval $[0, 1]$. Then there exists a sequence of polynomials $p_n(t)$ which converges to $f(t)$ as $n \to \infty$ on and

$$p_n(t) = \sum_{i=0}^{n} C_i f(i/n) t^i (1-t)^{n-i}$$

**Example 2.2:**

For the space $C_{[0,1]}$ of all continuous functions on the interval $[0, 1]$, the family $F_n(t) = \{t^n ; 0 \leq t \leq 1\}$ is total in this space but not basis in this space.
Proof.

Using the above theorem by putting \( f(t) = t^n \) we find a sequence of polynomials

\[
p_n(t) = \sum_{i=0}^{n} C_i \ f(i/n) \ t^i \ (1-t)^{n-i}
\]

converges uniformly to \( f(t) = t^n \) as \( n \to \infty \).

This implies that the family \( F_n(t) = \{t^n; 0 \leq t \leq 1\} \) is total in the space \( C_{[0,1]} \). On the other hand, let be the set \( \{1, t, t^2, \ldots\} \) be a basis in the space \( C_{[0,1]} \) and so we can write the element \( \sqrt{t} \) in the form

\[
\sqrt{t} = a_0 + a_1 t + a_2 t^2 + \ldots, \quad a_i \in C, i = 1, 2, 3, \ldots
\]

Replacing \( t \) by \( t^2 \) implies that:

\[
t = a_0 + a_1 t^2 + a_2 t^4 + \ldots, \quad a_i \in C, i = 1, 2, 3, \ldots
\]

This gives a contradiction, so the family \( F_n(t) = \{t^n; 0 \leq t \leq 1\} \) is not basis in this space \( C_{[0,1]} \).

3. Conclusion

In this paper we were gave an exact method to construct cones using a weak*-total families. also, we were searched some types of weak*-total families that grantees the implementation of definite cones such as solid cone or allows plastering cone or normal cone...etc.

References


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