Dissemination of Epidemic for SIR Model

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Abstract

In this paper, we focus on a spatio-temporal distribution of healthy and infected populations, which allows to model the spread of an epidemic. This work is distributed in two parts: firstly we model the spread of an epidemic by adding a spatial component to the considered system, secondly we search for solutions in the form of progressive wave. To solve the considered system which models the spread of an epidemic, we use the finite difference approach based on explicit Euler schema. In addition we present the numerical results.

Mathematics Subject Classification: 92B05, 92D25, 92D30

Keywords: SIR model, Spread of epidemics, Numerical simulation

1 Introduction

An epidemic is an occurrence of a disease in excess of normal expectancy, while a disease is called endemic if it is habitually present; however, communicable disease models of all types are often referred to as epidemic models, and the study of disease occurrence is called epidemiology.

The mathematical epidemiology has grown considerably since the 1980. Indeed, a number of issues of great societal importance arise for which the mathematical epidemiology is the only way to have rational answers, it allows you to use all current knowledge and the tools used are those of dynamical systems.
We are interested to compartmental models which play a very important role in epidemiology, they possesses a long history in epidemiology from the famous model of Kermack and McKendrick (1927, 1932, and 1933) [12][13][14]. A detailed history of mathematical epidemiology and basics of SIR epidemic models may be found in the classical books of Bailey [2], Murray [19], and Anderson and May [1]. After Kermack-McKendrick model, different epidemic models have been proposed and studied in the literature (see Capasso and Serio [6], Hethcote and Tudor [10], Liu et al. [15][16], Hethcote et al. [9], Hethcote and van den Driessche [8], Derrick and van den Driessche [7], Beretta and Takeuchi [5][4], Beretta et al. [3], Ma et al. [17][18], Ruan and Wang [22], Song and Ma [23], Song et al. [24], D’Onofrio et al. [21], Xiao and Ruan [25].

The population is divided into three distinct classes: the susceptible \( S \), ”healthy individuals who can catch the disease”, the infected \( I \), ”those who have the disease and can transmit it”, and the removed \( R \), ”individuals who have had the disease and are now immune to the infection” (or removed from further propagation of the disease by some other means). Schematically, the individual goes through consecutive states \( S \rightarrow I \rightarrow R \). Such models are often called the SIR models.

In this work, we neglect the process of birth and natural death, it is assumed that the total population remains constant, i.e. that \( R(t) + S(t) + I(t) = P(t) \). The population is considered in its entirety (it does not take into account the age, nor sex, nor any other parameter of this type, only the characters ”healthy”, ”susceptible” and ”infected” are studied).

The remaining parts of this paper are organized as follows: section 2 presents the SIR model. In section 3 we study the behavior of the epidemic in each location of a territory, by adding a spatial component to the previous model. We present some efficient algorithms for solving the discretized problem by finite differences approach on Matlab In section 4. The five section to put spotlight on numerical results. The last section provides concluding remarks.

2 SIR model

The SIR epidemic model is considered by:

\[
\begin{align*}
\dot{S}(t) &= r_c S(t) \left( 1 - \frac{S(t)}{k} \right) - \frac{\alpha S(t) I(t)}{1 + a I(t)} \\
\dot{I}(t) &= \frac{\alpha S(t) I(t)}{1 + a I(t)} - \gamma I(t) \\
\dot{R}(t) &= \gamma I(t)
\end{align*}
\]

The model has a susceptible group designated by \( S \), an infected group \( I \), and
a recovered group $R$ with permanent immunity. $r_c$ is the intrinsic growth rate of susceptible, $k$ is the carrying capacity of the susceptible in the absence of infective, $\alpha$ is the maximum values of per capita reduction rate of $S$ due to $I$, $a$ is half saturation constants, $\gamma$ is the natural recover rate from infection.

3 Spread and dissemination of epidemic

We focus now on the spatial distribution of healthy and infected populations. So we add a spatial component to the system, to take into account population shifts.

There are several types of SIR model with dissemination: we can add a spatial component to $S$ and $I$ (both the healthy and infected populations may move), as is the case of Gaudart [11] to model the Black Death, to $S$ only (the sick, too weak, do not move) as in the case in the model of rabies proposed by Murray [20].

To model the spread of the epidemic for this sir model it is assumed here that the population moves: if the population is distributed in several cities, individuals can become infected within the same city, but more likely an individual can move in a city where it is an infection, then bring the disease in his hometown, or another infected individual can move in another city to spread the disease ...

It well modelled by the following system:

\[
\begin{align*}
\dot{S}(t) &= \nabla^2(S) + r_c S(t) \left(1 - \frac{S(t)}{k}\right) - \frac{\alpha S(t)I(t)}{1 + a I(t)} \\
\dot{I}(t) &= \nabla^2(I) + \frac{\alpha S(t)I(t)}{1 + a I(t)} - \gamma I(t)
\end{align*}
\]

First we study the stationary solutions (status of the population in big time when setting the values of $S$ and $I$ at the edges of an interval). From there, you can go to the study of solutions in the form of waves progressive. Indeed, it suffices to write on $S(x,t) = s(x-ct)$ and $I(x,t) = i(x-ct)$, where $c$ is the speed of propagation of the wave, and $(s, i)$ of a pair of stationary problem solutions. We share the initial system (2).

To move to the stationary problem, just look in infinite time, the system does not depend on time then we obtain:

\[
\begin{align*}
\frac{\partial^2 S}{\partial x^2} &= -r_c S \left(1 - \frac{S}{k}\right) + \frac{\alpha SI}{1 + a I} \\
\frac{\partial^2 I}{\partial x^2} &= -\frac{\alpha SI}{1 + a I} + \gamma I
\end{align*}
\]
To go in search of wave, performing the previous change of variable we obtain:

\[
\begin{align*}
\dot{S}(t) &= -\frac{1}{c} \left\{ \frac{\partial^2 S}{\partial x^2} + r_c S(t) \left( 1 - \frac{S(t)}{k} \right) - \frac{\alpha S(t) I(t)}{1 + a I(t)} \right\} \\
\dot{I}(t) &= -\frac{1}{c} \left\{ \frac{\partial^2 I}{\partial x^2} + \frac{\alpha S(t) I(t)}{1 + a I(t)} - \gamma I(t) \right\}
\end{align*}
\]

\[ (4) \]

4 Resolution by the finite difference method

In this part, we try to observe numerically the solutions of the system in the form of travelling waves to model an epidemic. For that, we approximate the solutions using a Finite difference scheme. To simplify the study it remains in one dimension, that is to say that people move on a straight line (digitally, it is restricted to a segment \([x_{\text{min}}, x_{\text{max}}]\)).

we consider the problem to the limits associated with the system.

\[
\begin{align*}
\dot{S}(t) &= -\frac{1}{c} \left\{ \frac{\partial^2 S}{\partial x^2} + r_c S(t) \left( 1 - \frac{S(t)}{k} \right) - \frac{\alpha S(t) I(t)}{1 + a I(t)} \right\} \\
\dot{I}(t) &= -\frac{1}{c} \left\{ \frac{\partial^2 I}{\partial x^2} + \frac{\alpha S(t) I(t)}{1 + a I(t)} - \gamma I(t) \right\}
\end{align*}
\]

We fixe \(M > 0\) the number of domestic points in segment \([x_{\text{min}}, x_{\text{max}}]\). We note the discretization in space \(\Delta x = h = \frac{x_{\text{max}} - x_{\text{min}}}{M + 1}\) and \(x_i = x_{\text{min}} + ih\). We divide the interval \([0, T]\) into \(N\) sub-intervals, defined by: \(0 = t_0 < t_1 < t_2 < \ldots < t_N = T\) with \(\Delta t = \frac{T}{N}\). We calculate an approximation of the solution, denoted by \((S^n_i, I^n_i)\).

The finite difference scheme leads to the following system:

\[
\begin{align*}
S^{n+1}_i &= -\frac{\Delta t}{c} \left[ S^n_i \left( -\frac{2}{\Delta x^2} + r_c \left( 1 - \frac{S^n_i}{k} \right) - \frac{\alpha I^n_i}{1 + a I^n_i} \right) + \frac{S^n_{i+1} + S^n_{i-1}}{\Delta x^2} \right] + S^n_i \\
I^{n+1}_i &= -\frac{\Delta t}{c} \left[ I^n_i \left( -\frac{2}{\Delta x^2} + \frac{\alpha S^n_i}{1 + a I^n_i} - \gamma \right) + \frac{I^n_{i+1} + I^n_{i-1}}{\Delta x^2} \right] + I^n_i
\end{align*}
\]

\[ (5) \]

5 Numerical simulation

For the numerical simulation we use the following data.

Table 1: Value of the parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>(S_0)</th>
<th>(I_0)</th>
<th>(a)</th>
<th>(\alpha)</th>
<th>(\gamma)</th>
<th>(k)</th>
<th>(r_c)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value</td>
<td>50</td>
<td>30</td>
<td>2.3</td>
<td>1.49</td>
<td>0.611</td>
<td>100</td>
<td>2.5</td>
</tr>
</tbody>
</table>
At time $t = 0$, the susceptibles are contaminated only where it was introduced the infected (in $x_0$). and then, at the instant $t=15$ in the figure below, in $x_0$, the contamination is amplified, and the infected population begins to disseminate (departure of two wave fronts on either side of $x_0$). In $x_0$, the infected population starts to decrease, since there is no longer enough individuals to contaminate. At $t = 30$, 60 and 100, the two fronts continue to move toward the non-infected area (where the population likely is still important). There is no longer a sick in the center. Finally, at $t = 500$, the two fronts are gone, the epidemic is finite. The infected population is constant equal to 0.

Figure 1: Solutions in the form of gradual wave.
In the stationary case we find the following figure:

![Figure 2: Stationary Solutions.](image)

6 Conclusion

We have studied an epidemiological model both theoretically and numerically. We have divided our approach into two steps: modeling epidemic dissemination using an SIR model and studying numerically the progressive solutions. This work has required an implementation of tools using Matlab, as well as the use of mathematical techniques from a variety of fields such as differential calculus, ordinary differential equations, and partial differential equations. In this work we found how varied the number of infected and susceptible individuals is during a broadcast of an epidemic.

References


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